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Induction of representations of the generalized Bondi-Metzner group

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Fisica matematica. — *Induction of representations of the generalized Bondi–Metzner group* (*). Nota (**) di VITTORIO CANTONI, presentata dal Corrisp. C. CATTANEO.

RIASSUNTO. — Si costruisce una classe di rappresentazioni del gruppo di Bondi–Metzner mediante una generalizzazione del metodo di Wigner [8] per la costruzione delle rappresentazioni unitarie del gruppo di Poincaré. Si dimostra che le rappresentazioni ottenute sono equivalenti, a meno di un sistema di moltiplicatori, alle rappresentazioni unitarie ottenute precedentemente dall'autore [2] con procedimento diverso.

1. INTRODUCTION.—It is possible to characterize a class of "asymptotically flat" space-times, which can be expected to include plausible models for physical situations corresponding to bounded systems emitting gravitational radiation, by assuming, with Bondi-Metzner-van der Burg [1] and Sachs [6], that outside some region with finite spatial cross-section the space-time manifold can be covered by a single system of "generalized polar coordinates" (u, r, θ, φ) such that:

a) the metric takes the form

$$ds^2 = -A\, du^2 - 2\, B\, du\, dr + r^2 \left(C\, d\theta^2 + 2\, D\, d\theta\, d\varphi + \frac{\sin^2\theta + D^2}{C}\, d\varphi^2 + E\, du\, d\theta + F\, du\, d\varphi\right), \tag{1}$$

where A, B, C, D, E, F are functions of the coordinates, the ranges of the coordinates being $-\infty < u < \infty$, $r_0 \leq r < \infty$, $0 \leq \theta \leq \pi$, $0 \leq \varphi \leq 2\pi$ and the points $P(u, r, \theta, 0)$ and $P(u, r, \theta, 2\pi)$ being coincident;

b) for sufficiently large values of r the functions A, B, C, D, E, F can be expanded in power series of r^{-1}, and for $r \to \infty$ (u, θ, φ constant), the limiting form of the metric is

$$ds^2 = -du^2 - 2\, du\, dr + r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2), \tag{2}$$

i.e. the flat-space metric expressed in polar coordinates and in terms of the retarded time u.

It can be shown [6] [7] that to each coordinate transformation preserving the conditions a) and b) there corresponds a well-defined "asymptotic transformation", which can be regarded as a mapping of the cartesian product R×S of the real line R with the two-dimensional sphere S onto itself. The set of such asymptotic transformations forms a group, which will be denoted

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by $\mathcal{G}$ and called, with Sachs, the generalized Bondi–Metzner group (GBM group). If (u, θ, φ) are regarded as coordinates in $R \times S$, the generic GBM transformation has the form

$$(3) \qquad u' = \frac{u + \alpha(\theta, \varphi)}{K(\theta, \varphi)} \quad , \quad \theta' = H(\theta, \varphi) \quad , \quad \varphi' = I(\theta, \varphi),$$

where $\alpha(\theta, \varphi)$ is an arbitrary C^2 function on the two-sphere, while H , I and K are subjected to the condition

$$K^2(H, I) \cdot (dH^2 + \sin^2 H \cdot dI^2) = d\theta^2 + \sin^2\theta \cdot d\varphi^2,$$

so that H and I represent a conformal mapping of the two-sphere onto itself, and K is given by

$$(4) \qquad K(\theta, \varphi) = \sin^{1/2}\theta \cdot \sin^{-1/2} H \cdot [\partial(H, I)/\partial(\theta, \varphi)]^{-1/2}.$$

It can be shown [7] that the subgroup $\mathcal{L}$ of $\mathcal{G}$ obtained by putting $\alpha = 0$ is isomorphic, with the homogeneous orthochronous Lorentz group, while the subgroup $\mathcal{S}$ of $\mathcal{G}$ obtained by restricting α to the form

$$(5) \qquad \alpha(\theta, \varphi) = -a^0 + a^1 \sin\theta \cos\varphi + a^2 \sin\theta \sin\varphi + a^3 \cos\theta,$$

(a^0, a^1, a^2, a^3 constants) is isomorphic with the inhomogeneous orthochronous Lorentz group (Poincaré group). The transformations obtained by putting $H = \theta$, $I = \varphi$ are called "supertranslations" and constitute an abelian normal subgroup $\{s\}$ of $\mathcal{G}$; the translations are special supertranslations satisfying condition (5), and constitute a four-parameter abelian normal subgroup $\{t\}$ of G. The generic GBM transformation (3) can be regarded as the product of the supertranslation defined by the function α, denoted by (I, α), followed by the "homogeneous Lorentz transformation" defined by H and I, and denoted by $(\Lambda, 0)$: it will be denoted by (Λ, α), so that $(\Lambda, \alpha) = (\Lambda, 0)(I, \alpha)$.

It has been shown in a previous paper [2] that with any element $(\Lambda, \alpha) \in \mathcal{G}$ and any pair of values (θ, φ) of the angular coordinates one can associate a well defined element $(\Lambda, a_{\theta\varphi}) \in \mathcal{S}$, referred to as "the inhomogeneous Lorentz transformation asymptotically tangent to (Λ, α) in the ray direction (θ, φ)", (briefly AT (θ, φ) to (Λ, α)); an element of $\mathcal{G}$ is completely determined by the set of its AT transformations, and the latter can be exploited to construct, from each unitary representation of $\mathcal{S}$, a unitary representation of $\mathcal{G}$.

In this paper it is shown that the same representations of $\mathcal{G}$ can be obtained by an alternative procedure which is a generalization of Wigner's method for the construction of the unitary representations of $\mathcal{S}$, [4] [8]. The representations of $\mathcal{S}$ and $\mathcal{G}$, respectively, associated with a given "momentum vector" and a given representation of the corresponding "little group", are shown to be such that the latter can be obtained from the former, up to a system of multipliers, by the procedure described in ref. [2].

2. INDUCED REPRESENTATIONS [4].—Let G be a group of transformations, assumed to act transitively on a space M. Denote by $\overset{\circ}{p}$ a fixed element of M, and by G_0 the isotropy group of G at $\overset{\circ}{p}$, i.e. the subgroup of G constituted of all the elements h such that $h\overset{\circ}{p} = \overset{\circ}{p}$. It is well-known that M can be identified with the homogeneous space G/G_0: in fact, the generic left coset $\bar{g}\,G_0$, $(\bar{g} \in G)$, can be associated with the element $p = g'\,\overset{\circ}{p}$, $(g' \in \bar{g}\,G_0)$, which is clearly independent from the particular choice of g' in the coset; conversely, on account of the assumed transitivity of G on M, each element p of M can be obtained from $\overset{\circ}{p}$ by acting on it with some element $\bar{g} \in G$, or with any other element of the coset $\bar{g}\,G_0$, so that one has a one-to-one correspondence between elements of M and elements of G/G_0; moreover, such a correspondence is preserved by the action of G, i.e. $g_2\,(g_1\,G_0) \overset{\text{def}}{=} (g_2\,g_1)\,G_0$ corresponds to $g_2\,p$ whenever $g_1\,G_0$ corresponds to p.

Consider a linear representation σ of G_0: denote by U the representation space, and by $\sigma(h)$ the operator corresponding to the element h of G_0. The aim is to construct, starting from σ, a linear representation ρ of G acting on a suitably defined representation space $\hat{U}$.

Define $\hat{U}$ as the set of all maps $\hat{\Phi} : g \to \hat{\Phi}(g)$ of G into U which satisfy the condition

$$\hat{\Phi}(gh) = \sigma(h^{-1})\,\hat{\Phi}(g) \tag{6}$$

for all $g \in G$ and all $h \in G_0$. In order to assign one of such maps it is sufficient to assign arbitrarily the vector $\hat{\Phi}(e) \in U$ corresponding to the identity of G, (then (6) determines $\hat{\Phi}(h)$ for all $h \in G_0$), and to assign, for each left coset $p = \bar{g}\,G_0$, an arbitrary vector $\hat{\Phi}(\bar{g}) \in U$ corresponding to a fixed representative $\bar{g}$ of the coset, (then (6) determines $\hat{\Phi}(g)$ for any other element $g \in \bar{g}\,G_0$). The space $\hat{U}$ can be regarded as a linear space with the following definition of addition and multiplication with scalars:

$$a\hat{\Phi} + b\hat{\psi} : g \to a\hat{\Phi}(g) + b\hat{\psi}(g),$$

and it is immediately verified that $a\hat{\Phi} + b\hat{\psi}$ actually belongs to $\hat{U}$, (i.e. satisfies (6)), whenever $\hat{\Phi}$ and $\hat{\psi}$ belong to $\hat{U}$.

The linear representation ρ of $\mathcal{G}$, acting on the vector space $\hat{U}$, can be defined as follows:

$$\rho(g_0)\,\hat{\Phi} : g \to \hat{\Phi}(g_0^{-1}g) \quad , \quad (g_0\,,\,g \in G\,;\,\hat{\Phi} \in \hat{U}). \tag{7}$$

The mapping $\hat{\psi} \equiv \rho(g_0)\,\hat{\Phi}$ of G into U satisfies (6) and therefore belongs to $\hat{U}$, since one has:

$$\hat{\psi}(gh) = \hat{\Phi}(g_0^{-1}gh) = \sigma(h^{-1})\,\hat{\Phi}(g_0^{-1}g) = \sigma(h^{-1})\,\hat{\psi}(g).$$

It is also easily verified that the transformations ρ are linear, and that $\rho(g_1\,g_0) = \rho(g_1)\,\rho(g_0)$.

ρ is called the representation of G induced by σ.

Since the assignment of a mapping of G into U satisfying (6) is equivalent to the assignment of an arbitrary mapping of a set $\{\bar{g}\}$ of representatives of the left cosets of G_0 into U, for a fixed choice of the set $\{\bar{g}\}$ one can associate with each element $\hat{\Phi} \in \hat{U}$ the mapping of M into U defined by

$$\Phi : p \rightarrow \hat{\Phi}(\bar{g}) \quad , \quad (p = \bar{g}\, G_0).$$

With the help of the mappings Φ one can establish a one-to-one correspondence between the elements of $\hat{U}$ and the cross-sections of the product bundle $M \times U$ of M with U, (M being the base space and U the fibre). In fact, with the element $\hat{\Phi} \in \hat{U}$ one can associate the cross-section $\hat{\Phi}^*$ defined by

$$\hat{\Phi}^* : p \rightarrow \hat{\Phi}^*(p) \stackrel{\text{def}}{=} p \times \Phi(p) \equiv p \times \hat{\Phi}(\bar{g}). \tag{8}$$

Since U is a vector space, $M \times U$ is a vector bundle, so that the space $\hat{U}^*$ of its cross-sections has a natural linear structure, and it is an immediate consequence of (8) that the correspondence just defined between $\hat{U}$ and $\hat{U}^*$ is an isomorfism between vector spaces. Therefore one can define a representation ρ^* of G, equivalent to the representation ρ and acting on the representation space $\hat{U}^*$. Since the action of $\rho(g_0)$ on an element $\hat{\Phi} \in \hat{U}$ is given by (7), $\rho^*(g_0)$ must transform the generic element $\hat{\Phi}^* \in \hat{U}^*$ into the element $\hat{\psi}^* \in \hat{U}^*$ such that

$$\begin{aligned} \hat{\psi}^*(p) &\equiv p \times \hat{\psi}(\bar{g}) = p \times [\rho(g_0)\, \hat{\Phi}](\bar{g}) = p \times \hat{\Phi}(g_0^{-1}\bar{g}) = \\ &= p \times \hat{\Phi}(\overline{g_0^{-1} g}\, h_{g_0,p}^{-1}) = p \times \sigma(h_{g_0,p})\, \hat{\Phi}(\overline{g_0^{-1} g}) = \\ &= p \times \sigma(h_{g_0,p})\, \Phi(g_0^{-1} p) \end{aligned} \tag{9}$$

where $\overline{g_0^{-1} g}$ is the representative of the coset $g_0^{-1}\bar{g}\, G_0$, $h_{g_0,p}$ is the element of G_0 such that $g_0^{-1}\bar{g} = \overline{g_0^{-1} g}\, h_{g_0,p}^{-1}$, and (6) has been taken into account.

Whenever two representations ρ_1^* and ρ_2^* of G act on the same representation space $\hat{U}^*$, (the space of cross-sections of the bundle $M \times U$), and there exists a set $\{m_{g,p}\}$ of linear transformations whose generic element depends on g and p, acting on the fibre over p, and such that

$$(\rho_2^*(g)\, \hat{\Phi}^*)(p) = m_{g,p}\, (\rho_1^*(g)\, \hat{\Phi}^*)(p)$$

for all $g \in G$, $p \in M$ and $\hat{\Phi}^* \in \hat{U}^*$, the two representations ρ_1^* and ρ_2^* will be said to coincide *up to the system of multipliers* $\{m_{g,p}\}$. (In particular, it is not hard to see that two distinct choices of the set $\{\bar{g}\}$ of representatives of the left cosets lead to two representations which coincide up to a system of multipliers).

3. THE REPRESENTATIONS OF THE GROUPS $\mathcal{G}$ AND $\mathcal{G}$ ASSOCIATED WITH A GIVEN MOMENTUM VECTOR AND A GIVEN REPRESENTATION OF THE "LITTLE GROUP".—The application of the method just described to the construction

of a representation of any specific group G requires:

a) the interpretation of G as a transitive group of transformations on some space M;

b) the determination of a representation of the isotropy group G_0 of G at any point $\mathring{p}$ of M; (on account of the transitivity of action of G on M, the isotropy group is the same at all points of M, up to isomorphisms).

In the case of the inhomogeneous orthochronous Lorentz group, ($G \equiv \mathcal{S}$), M is defined as the set of all vectors $\boldsymbol{p} = (p_0, p_1, p_2, p_3)$ in Minkowski space which can be obtained from a given *momentum vector* $\mathring{\boldsymbol{p}}$ by means of an homogeneous orthochronous Lorentz transformation (thus, all the vectors $\boldsymbol{p}$ of M have the same norm, and the same orientation in time whenever they are timelike or null). The homogeneous orthochronous Lorentz group $\mathfrak{L}$ acts transitively on M, the action $\boldsymbol{p} \to \Lambda \boldsymbol{p}$ of an element $\Lambda \in \mathfrak{L}$ on $\boldsymbol{p} \in M$ being defined in the obvious way. If the action of any translation on M is taken to be the identity transformation, then $\mathcal{S}$ also acts transitively on M, and the action of the generic element $(\Lambda, a) \in \mathcal{S}$, (i.e. the translation defined by the four-vector $\boldsymbol{a}$ followed by the homogeneous transformation Λ), is the same as the action of its homogeneous part alone. The isotropy subgroup $\mathcal{S}_0$ of $\mathcal{S}$ at $\mathring{\boldsymbol{p}}$ is the product of the isotropy subgroup $\mathfrak{L}_0$ of $\mathfrak{L}$ at $\mathring{\boldsymbol{p}}$ (the "little group" corresponding to the momentum vector $\mathring{\boldsymbol{p}}$), with the subgroup $\{t\}$ of translations.

If $\bar{\sigma}$ denotes a linear representation of the little group, acting on a complex vector space U, then the mapping

$$\sigma : (\Lambda_0, a) \to \exp(i\boldsymbol{a} \cdot \mathring{\boldsymbol{p}}) \cdot \bar{\sigma}(\Lambda_0), \tag{10}$$

($\Lambda_0 \in L_0$; $\boldsymbol{a} \cdot \mathring{\boldsymbol{p}} \equiv a^i \mathring{p}_i$), is a linear representation of $\mathcal{S}_0$, acting on U, and the corresponding induced representation ρ (or ρ^*) of $\mathcal{S}$ can be constructed. The corresponding representation space will be denoted by $\hat{U}_{\mathcal{S}}$ (or $\hat{U}^*_{\mathcal{S}}$). It can be shown that with this procedure all the irreducible unitary representations of $\mathcal{S}$ can be obtained, starting from all possible momentum vectors and all possible unitary representations of the corresponding little groups.

Without changing the definition of M, consider now the GBM group ($G \equiv \mathcal{G}$), and define the action of any supertranslation (I, α) as the identity transformation. Then $\mathcal{G}$ acts transitively on M, the action of its generic element (Λ, α) being the same as the action of the transformation $\Lambda \in \mathfrak{L}$ alone. Denote by $\mathcal{G}_0$ the product $\mathfrak{L}_0 \{s\}$ of the little group $\mathfrak{L}$: with the supertranslation subgroup $\{s\}$, so that $\mathcal{G}_0$ is the isotropy group of $\mathcal{G}$ at $\mathring{\boldsymbol{p}}$. If $\bar{\sigma}$ is a representation of $\mathfrak{L}_0$, the mapping

$$\tilde{\sigma} : h \equiv (\Lambda_0, \alpha) \to \tilde{\sigma}(h) \equiv \exp(i\boldsymbol{a}_{\Lambda_0^{-1} p} \cdot \mathring{\boldsymbol{p}})\, \sigma(\Lambda_0), \tag{11}$$

($\Lambda_0 \in \mathfrak{L}$, $\alpha \in \{s\}$), where $\boldsymbol{a}_P$ denotes the vector of the translation AT (θ, φ) to $(1, \alpha)$, (see ref. [2]), defines a linear representation of $\mathcal{G}_0$ acting on the tensor product $V = F \otimes U$ of the space F of all C^2 functions on the sphere with the representation space U of $\bar{\sigma}$, the action of $\tilde{\sigma}(h)$ on V being such that

$$\tilde{\sigma}(h) \cdot f(\theta, \varphi) \otimes \varphi = \exp(i\, \boldsymbol{a}_{\Lambda_0^{-1} p} \cdot \mathring{\boldsymbol{p}})\, f(\Lambda_0^{-1} \theta, \Lambda_0^{-1} \varphi) \otimes \bar{\sigma}(\Lambda_0)\, \varphi, \tag{12}$$

($f \in F$, $\varphi \in U$, $h \equiv (\Lambda_0, \alpha) \in \mathcal{G}_0$). Hence one can construct the representation $\tilde{\rho}$ (or $\tilde{\rho}^*$) of $\mathcal{G}$ induced by $\tilde{\sigma}$. The corresponding representation space will be denoted by $\hat{U}_{\mathcal{G}}$, (or $\hat{U}^*_{\mathcal{G}}$).

4. RELATION BETWEEN THE REPRESENTATIONS ρ^* AND $\tilde{\rho}^*$.—It will now be shown that *whenever the representations* ρ^* *and* $\tilde{\rho}^*$ *of* $\mathcal{S}$ *and* $\mathcal{G}$ *respectively are constructed from the same momentum vector* $\mathring{\boldsymbol{p}}$ *and the same representation* $\bar{\sigma}$ *of the corresponding little group* $\mathfrak{L}_0$, $\tilde{\rho}^*$ *coincides, up to a system of multipliers, with the representation of G obtained from* ρ^* *by the procedure described in reference* [2].

Consider, first, the representation ρ^*. Denote by $\{\varphi_\zeta\}$ a basis in U; (it is convenient to regard ζ as a discrete index, as in fact it is whenever $\mathring{\boldsymbol{p}}$ is timelike, in which case the little group is the three-dimensional rotation group. However the assumption that ζ be discrete is not essential). With respect to this basis the generic element of U is described by a real-valued function $f(\zeta)$; a cross-section $\Phi^*_{\mathcal{S}}$ of the product bundle $M \times U$ is therefore described by a function $f(p, \zeta)$ of ζ and the three independent components (p_1, p_2, p_3) of $\boldsymbol{p}$. The operator $\rho^*(a)$ corresponding to the translation a defined by the vector $\boldsymbol{a}$ transforms the cross-section $\hat{\Phi}^*_{\mathcal{S}} \in \hat{U}^*_{\mathcal{S}}$, according to (9) and (10), into the cross-section $\hat{\psi}^*_{\mathcal{S}}$ such that

$$\begin{aligned} \hat{\psi}^*_{\mathcal{S}}(\boldsymbol{p}) &= \boldsymbol{p} \times \hat{\Phi}_{\mathcal{S}}(a^{-1} \bar{g}) = \boldsymbol{p} \times \hat{\Phi}_{\mathcal{S}}(\bar{g}\, \bar{g}^{-1} a^{-1} \bar{g}) = \\ &= \boldsymbol{p} \times \sigma(\bar{g}^{-1} a \bar{g})\, \hat{\Phi}_{\mathcal{S}}(\bar{g}) = \boldsymbol{p} \times \exp(i \boldsymbol{\Lambda}^{-1} \boldsymbol{a} \cdot \mathring{\boldsymbol{p}})\, \hat{\Phi}_{\mathcal{S}}(\bar{g}) = \\ &= \boldsymbol{p} \times \exp(i \boldsymbol{a} \cdot \boldsymbol{p})\, \hat{\Phi}_{\mathcal{S}}(\bar{g}), \end{aligned} \tag{13}$$

where Λ denotes the homogeneous part of the inhomogeneous transformation $\bar{g}$, and the fact that $\bar{g}^{-1} a \bar{g}$ is the translation represented by the vector $\boldsymbol{\Lambda}^{-1} \boldsymbol{a}$ has been taken into account. Clearly the function f' which represents $\hat{\psi}^*_{\mathcal{S}}$ is given by $f'(p, \zeta) = \exp(i \boldsymbol{a} \cdot \boldsymbol{p})\, f(p, \zeta)$. Similarly, the operator $\rho^*(\Lambda)$ corresponding to a homogeneous Lorentz transformation Λ transforms $\hat{\Phi}^*_{\mathcal{S}}$, according to (9), into the cross-section $\hat{\psi}^*_{\mathcal{S}}$ such that

$$\hat{\psi}^*_{\mathcal{S}}(\boldsymbol{p}) = \boldsymbol{p} \times Q(\Lambda, p)\, \Phi_{\mathcal{S}}(\boldsymbol{\Lambda}^{-1} \boldsymbol{p}),$$

where $Q(\Lambda, p)$ is a linear transformation of U depending on Λ and $\boldsymbol{p}$, so that the function f' associated with $\hat{\psi}^*_{\mathcal{S}}$ has the form

$$f'(p, \zeta) = \sum_\eta Q(\Lambda, p)_{\zeta\eta}\, f(\Lambda^{-1} p, \eta), \tag{14}$$

where $Q(\Lambda, \boldsymbol{p})_{\zeta\eta}$ is the matrix of the transformation $Q(\Lambda, p)$ in the basis $\{\varphi_\zeta\}$.

Consider now the representation $\tilde{\rho}^*$. The representation space V of $\tilde{\sigma}$ admits the basis $\{\mathbf{Y}_{lm} \otimes \varphi_\zeta\}$, where the Y_{lm}'s denote the associated Legendre polynomials, and the generic element of V is described by a function $f(\zeta, l, m)$. A cross-section $\hat{\Phi}^*_{\mathcal{G}} \in \hat{U}^*_{\mathcal{G}}$ can be associated with a function $f(p, \zeta, l, m)$, or, equivalently, with the function $f(p, \zeta; \theta, \varphi) \equiv \sum_{l,m} f(p, \zeta, l, m)\, \mathbf{Y}_{lm}(\theta, \varphi)$ which can be regarded as defining a continuous mapping

$$(\theta, \varphi) \rightarrow \hat{\Phi}^*_{\mathcal{G}}(\theta, \varphi) \quad , \quad (\hat{\Phi}^*_{\mathcal{G}}(\theta, \varphi) \in \hat{U}^*_{\mathcal{G}}),$$

of the sphere into the space of cross–sections of the bundle $M \times U$: this shows that the representation space of $\tilde{\rho}^*$ is identical with the representation space $\tilde{\mathcal{H}}$ described in ref. [2], provided that the representation $\mathcal{T}$ of $\mathcal{G}$ on which the construction of $\tilde{\mathcal{T}}$ is based is taken to be identical wich ρ^*.

According to (9) and (11), taking the linearity of $\tilde{\sigma}$ into account, one has, for a supertranslation $\zeta \equiv (1, \alpha)$:

$$(\rho^*(s)\, \hat{\Phi}^*_{\mathcal{G}})(\boldsymbol{p}) = \boldsymbol{p} \times \hat{\Phi}_{\mathcal{G}}(s^{-1}\bar{g}) = \boldsymbol{p} \times \hat{\Phi}_{\mathcal{G}}(\bar{g}\,\bar{g}^{-1} s^{-1} \bar{g}) =$$

$$= \boldsymbol{p} \times \tilde{\sigma}(\bar{g}^{-1} s \bar{g})\, \hat{\Phi}_{\mathcal{G}}(\bar{g}) = \boldsymbol{p} \times \sum_{\zeta} \{[\exp(i\, \boldsymbol{\Lambda}^{-1} \boldsymbol{a}_{\mathrm{P}} \cdot \mathring{\boldsymbol{p}}) \cdot$$

$$\cdot f(p, \zeta; \theta, \varphi)] \otimes \varphi_\zeta\} = \boldsymbol{p} \times \sum_{\zeta} [\exp(i\, \boldsymbol{a}_{\mathrm{P}} \cdot \boldsymbol{p}) f(p, \zeta; \theta, \varphi) \otimes \varphi_\zeta]$$

where Λ denotes the homogeneous part of the GBM transformation $\bar{g}$ representing the left coset associated with $\boldsymbol{p}$, and the fact that $\bar{g}^{-1} a_{\mathrm{P}}\, \bar{g}$ is the translation represented by the vector $\boldsymbol{\Lambda}^{-1} \boldsymbol{a}_{\mathrm{P}}$ has been taken into account. Regarding $\tilde{\rho}^*(s)\, \hat{\Phi}^*_{\mathcal{G}}$ as a map of the sphere into $\hat{U}^*_{\mathcal{G}}$, the last result can be read as follows:

$$\tilde{\rho}^*(s)\, \hat{\Phi}^*_{\mathcal{G}} : (\theta, \varphi) \rightarrow \exp(i\, \boldsymbol{a}_{\mathrm{P}} \cdot \boldsymbol{p})\, \hat{\Phi}^*_{\mathcal{G}}(\theta, \varphi) \equiv \rho^*(\boldsymbol{a}_{\mathrm{P}})\, \hat{\Phi}^*_{\mathcal{G}}(\theta, \varphi).$$

This shows that, for supertranslations, the operators of the representation $\tilde{\mathcal{T}}$ described in ref. [2] are identical with the corresponding operators of $\tilde{\rho}^*$, provided that $\mathcal{T}$ is taken to be identical with ρ^*.

Finally, if $\Sigma \in \mathcal{L}$ is a homogeneous Lorentz transformation, one has, from (9)

$$(\tilde{\rho}^*(\Sigma)\, \hat{\Phi}^*_{\mathcal{G}})(\boldsymbol{p}) = \boldsymbol{p} \times \tilde{\sigma}(h_{\Sigma, \boldsymbol{p}})\, \Phi_{\mathcal{G}}(\Sigma^{-1} \boldsymbol{p}),$$

where $h_{\Sigma, p}$ is an element of G_0 depending on Σ and $\boldsymbol{p}$, for a given choice of the representatives of the cosets. If the choice of the representatives is the same as for the representation ρ^* of $\mathcal{G}$, one can write, taking (9), (12) and (14) into account:

$$(\tilde{\rho}^*(\Sigma)\, \hat{\Phi}^*_{\mathcal{G}})(\boldsymbol{p}) = \boldsymbol{p} \times \Big[\sum_{\zeta} f(\Sigma^{-1} p, \zeta; \Lambda_0^{-1} \theta, \Lambda_0^{-1} \varphi) \otimes \bar{\sigma}(h_{\Sigma, p}) \cdot \varphi_\zeta\Big] =$$

$$= \boldsymbol{p} \times \Big[\sum_{\zeta\eta} Q(\Sigma, p)_{\zeta\eta}\, R_{\Sigma, p}\, f(\Sigma^{-1} p, \eta; \Sigma^{-1} \theta, \Sigma^{-1} \varphi) \otimes \varphi_\zeta\Big],$$

where Λ_0 denotes the homogeneous part of the transformation $h_{\Sigma, p}$, and $R_{\Sigma, p}$ is a linear transformation on the space F, depending on Σ and $\boldsymbol{p}$. Regarding the elements of $\hat{U}^*_{\mathfrak{S}}$ as mappings of the sphere into $\hat{U}^*_{\mathfrak{S}}$, the last result can be read as follows:

$$(\tilde{\rho}^* (\Sigma) \hat{\Phi}^*_{\mathfrak{S}}) (\boldsymbol{p}) : (\theta , \varphi) \rightarrow \boldsymbol{p} \times R_{\Sigma,\boldsymbol{p}} \rho^* (\Sigma) \hat{\Phi}^*_{\mathfrak{S}} (\Lambda^{-1} \theta , \Lambda^{-1} \varphi).$$

By comparison with the definition of the operators of $\tilde{\mathfrak{T}}$, (ref. [2]), it is immediately seen that the restrictions of $\tilde{\rho}^*$ and $\tilde{\mathfrak{T}}$ to $\mathfrak{L}$ coincide, up to a system of multipliers, whenever the representation $\mathfrak{T}$ of $\mathfrak{S}$ is taken to be identical with ρ^*. On account of the previous results, the same is true for the whole representations $\tilde{\rho}^*$ and $\tilde{\mathfrak{T}}$ of $\mathfrak{S}$.

Indirectly, this shows that the linear representations of $\mathfrak{S}$ obtained from *unitary* representations of the little group can always be transformed, by suitable choices of multipliers, into *unitary* representations of $\mathfrak{S}$.

The irreducible components of the latter representations are determined in reference [3].

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