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Vector valued functions on semigroups with almost periodic differences

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Matematica. — *Vector valued functions on semigroups with almost periodic differences.* Nota di H. GÜNZLER, presentata (*) dal Corrisp. L. AMERIO.

RIASSUNTO. — Si approfondisce l'analisi delle funzioni q.p. su semigruppi, ottenendo, tra l'altro, estensioni di risultati di Amerio e di Doss.

Even before H. Bohr defined almost periodicity, in connection with the asymptotic behavior of solutions of differential equations Bohl considered and solved the following question: Given a ' quasi-periodic ' function g on the reals R, when is the indefinite integral $f(x) := \int_0^x g(t)\,dt$ again quasiperiodic? His answer 'f is quasi-periodic if and only if it is bounded ' and the applications to differential equations he gave have been extended in many directions and by many authors, especially Bohr, Favard, Bochner, Brauers to mention a few.

In recent years, interest in these questions has been revived due to the penetrating work of Amerio on a.p. solutions of the wave equation. Motivated by problems which arose there, Amerio and his coworkers, Bochner and others obtained many new and important results on the almost periodicity of indefinite integrals of vector valued a.p. functions or (partly with the aid of these results) more generally of solutions of differential equations—see the survey of Amerio [4]; Bochner's almost automorphic functions were also introduced in this context.

The underlying group till then was always the real line R. Now if $f(x) = \int_0^x g(t)\,dt$ with a.p. g, for fixed s the *difference* function f_s,

$$f_s(t) := f(t+s) - f(t)$$

is a.p. in $t \in \mathrm{R}$, and this has a meaning for arbitrary groups. It was R. Doss who had this simple but extremely fruitful idea, he showed [8] that bounded complex valued f with all f_s a.p. are a.p. too.

In this note we will indicate how the results of Amerio and Doss can be combined, giving conditions on the range (space) so that totally bounded respectively bounded f with all differences f_s a.p. are a.p.; instead of groups arbitrary semigroups are admitted. However, there still remain many open questions, for example: Is the theorem of Amerio-Doss valid for arbitrary reflexive Banach spaces, or for L^1? (See after theorem D below).

The proofs, more details and literature will appear in [10], [11]. Our terminology is that of Köthe [12].

(*) Nella seduta del 21 giugno 1967.

A *semigroup* S is a non-empty set with a binary associative operation $s \cdot t$. If S has a unit u, $us = su = s$ for $s \in S$, then $S_u := S$, else S_u denotes the semigroup obtained from S by formally adjoining a unit. A *topological semigroup* S is a semigroup and a topological space such that $(s, t) \to st$ is separately continuous in s respectively t. A function f defined on such a S and with values in a uniform space will be called *almost periodic* (a.p.) if the following is true (Maak [14], Günzler [9]): f is continuous, and to every neighborhood N of the uniformity there exist finitely many subsets $P_1, \cdots, P_m \subset S$ with $S = \bigcup_{\mu=1}^{m} P_\mu$ such that whenever for some $c', d', x, y \in S_u$ the elements $c'xd'$ and $c'yd'$ are in the same P_μ, then $(f(cxd), f(cyd)) \in N$ for all $c, d \in S_u$ with cxd and $cyd \in S$.

If S is a group, this coincides with the definitions of Bochner, von Neumann and Maak. If $S = R$, the additive group of the reals in the usual topology, and the range space is C = complex numbers or any complex Banach space, the definition above is equivalent with Bohr's definition; this is true also for convex subsemigroups of R^n with non-empty interior.

A locally convex abelian group E is a commutative group and a topological space such that $(x, y) \to xy$ and x^{-1} are continuous, and in which to each neighborhood V of 0 there is a neighborhood U of zero with $\frac{1}{m} \sum_{\mu=1}^{m} U \subset V$ for every natural m, or equivalently if zero has a neighborhood basis of convex sets; here for $A, B \subset E$, $A + B := \{a + b : a \in A, b \in B\}$, $\frac{1}{m} A := \{x \in E : mx \in A\}$, $mx := \sum_{\mu=1}^{m} x$, A is *convex if* $\frac{1}{m} \sum_{\mu=1}^{m} A = A$ for each natural m, $\bar{A}$ denotes the closure of A. Since such E are uniform spaces, totally bounded is equivalent with precompact, if E is (topologically) complete, it is equivalent with relatively compact. Examples are locally convex real or complex linear spaces, especially normed spaces.

For $f : S \to E$ and $s \in S$ the *right difference* f_s is a function: $S \to E$ defined by

$$f_s(t) := f(ts) - f(t) . \tag{1}$$

If $A \subset S$, $f(A) := \{f(s) : s \in A\}$.

THEOREM A.—*If* S *is a topological semigroup,* E *a locally convex abelian group,* $f : S \to E$, *then* f *is a.p. if and only if the following three conditions hold*

(A) *For each* $s \in S$, *the right difference* f_s *is a.p. on* S;

(B) *For each* $c \in S$, $f(cS)$ *is totally bounded*;

(C) *There is a natural* m *and* $c_1, \cdots, c_m \in S$ *such that*

$$\bigcap_{d_1, \cdots, d_m \in S} \sum_{\mu=1}^{m} (-1)^{\mu+1} \overline{f(Sd_\mu)} \subset \overline{\sum_{\mu=1}^{m} (-1)^{\mu+1} f(c_\mu S)} . \tag{2}$$

For groups S or semigroups with non-empty center, for example with a unit, condition (C) is automatically fulfilled; for general S however it cannot be omitted as suitable counterexamples show.

If S = R, E = complex numbers C, then theorem A yields the theorem of Bohl and Bohr on integration of a.p. functions, since the differences of indefinite integrals of a.p. functions are easily seen to be a.p. For general E and S = R, theorem A essentially contains Bochner's extension [5] on integration of Bohr–a.p. functions. If S = group, E = C, one gets the result of Doss [8]. The special case S = abelian group, E = Banach space has also been obtained independently by R. Doss (unpublished, oral communication).

If E is a locally convex linear space, then in theorem A it is enough if (A), (B) and (C) hold only in the weak topology, provided there are c_0, d_0 with $f(c_0 S d_0)$ totally bounded, theorem A in its general form follows once it has been shown for complex valued f on S. For this one has to use

THEOREM B.—*If* S *is a topological semigroup,* E *a locally convex real or complex linear space,* $f: S \to E$, *then* f *is a.p. if and only if* f *is weakly a.p. and there are* $c_0, d_0 \in S$ *so that* $f(c_0 S d_0)$ *is totally bounded.*

f is *weakly a.p.* means f is a.p. in the weak topology of E, i.e. $\varphi \circ f$ is a.p. for each continuous linear scalar valued functional φ on E. For S = R, E = Banach space, theorem B is due to Kopec [13] and Amerio [1].

Since the total-boundedness of the range is a quite restrictive assumption, weakened versions would be of interest. One such possibility, which we formulate only for $S \subset R^n$ (see theorem 2 of [11]) is the following

THEOREM C.—*Let* S *be a convex open subsemigroup of* R^n, E *a locally convex abelian group* $f: S \to E$ *be continuous. Then* f *is a.p. if and only if all differences* f_s *are a.p. on* S *and further* f *is totally bounded on a set* $M \subset S$ *which is relatively dense in* S.

M is *relatively dense in* S if there is a $r > 0$ such that each ball with center $x \in S$ and radius r meets M.

An extension in a different direction, which has useful applications in partial differential equations, is due to Amerio [2]: If the indefinite integral f of an a.p. function $g: R \to B$ is only bounded, f is a.p. if B is a uniformly convex Banach space (for example a Hilbert space).

This can be generalized to arbitrary semigroups and f with only a.p. differences. The following theorem simultaneously shows that uniform convexity is in some sense a necessary assumption.

If $A \subset N$, N a normed real or complex linear space, A has (u) respectively $(w\omega s)$ resp. $(wn\omega s)$ means:

(u) $\sup_{b \in A} \|b\| =: M < \infty$ and to each $\varepsilon > 0$ there is a $\delta > 0$ so that $x, y \in A$, $\|x - y\| \geq \varepsilon$ imply $\|x + y\| \leq 2M \cdot (1 - \delta)$

$(w\omega s)$ If $x_0, x_1, x_2, \cdots \in A$ with $x_n \to x_0$ weakly, then $\|x_n - x_0\| \xrightarrow[n \to \infty]{} 0$

$(wn\omega s)$ If $x_0, x_1, x_2, \cdots \in A$ with $x_n \to x_0$ weakly and $\|x_n\| \to \|x_0\|$, then $\|x_n - x_0\| \xrightarrow[n \to \infty]{} 0$.

N is uniformly convex if its unit ball (or sphere) has (u), then any bounded subset of N has (u), N has even $(wns) := (wn\omega s)$ for nets instead of sequences. 'Renorming' means introducing an equivalent norm.

THEOREM D.—*Let* S *be a topological semigroup*, N *a real or complex linear space*, $f: S \to N$. *If all right differences* f_s *of* f *are a.p., if* f *satisfies* (C) *of theorem* A *with respect to the weak topology, then* (3)–(6) *below are equivalent*:

(3) f *is a.p.*

(4) N *can be renormed so that* $f(S)$ *has* (u)

(5) N *can be renormed so that* $\|f\|$ *is a.p. and* $f(S)$ [*or the closed linear hull of* $f(S)$] *has* $(wn\omega s)$

(6) $f(S)$ *is bounded and has* $(w\omega s)$.

Let us say that *the theorem of Amerio-Doss holds for* N *and* S if N is a real or complex normed linear space, S a topological semigroup, and for any $f: S \to N$ which has norm–a.p. differences f_s and which is weakly a.p., one can conclude that f is a.p.; by theorem A, "f is weakly a.p." is here equivalent with "f is bounded and (C) holds in the weak topology", so for most S boundedness of f suffices.

As a *corollary*, if N is isomorphic to a uniformly convex normed linear space, then the theorem of Amerio-Doss holds for N and arbitrary S. Examples are L^p–spaces of functions with values in a uniformly convex Banach space, $1 < p < \infty$, the Amerio-Doss theorem holds also for such L^p_{lok}–spaces.

With (5) of theorem D one gets: If N is locally uniformly convex, then $f: S \to N$ is a.p. if $\|f\|$ is a.p., all f_s are a.p. and (C) holds in the weak topology (compare with theorem VII of Amerio [1]; locally uniformly convex is a much weaker condition than uniformly convex, any separable normed linear space can be renormed locally uniformly convex).

Since the space l^1 of absolutely convergent sequences has $(w\omega s)$, the theorem of Amerio-Doss is true for l^1 and any S. For l^1–type spaces however much more holds as we will see presently.

Though the range has to be uniformly convex if boundedness implies almost periodicity, by the results of Amerio [3] there are reflexive Banach spaces B of l^p–type for which the theorem of Amerio-Doss holds with $S = R$, but which are not isomorphic to a uniformly convex Banach space (use theorem I). An extension to more general substitution spaces and arbitrary S is contained in

THEOREM E.—*Let* T *be a uniformly convex Banach space with* (7), (8), (9) *below*, S *a topological semigroup*, L *a normed real or complex linear space of type* $T|N_i$ (*see below*), $f: S \to L$. *Then the theorem of Amerio-Doss holds for* L *and* S, *if it holds for each* N_i *and* S.

Here I is an arbitrary index set, the assumptions on T and L are:

(7) T is a normed real linear space of functions $\varphi: I \to R$ under pointwise addition, scalar multiplication and equality

(8) If φ and all $\varphi_j \in T$, then $\|\varphi_j\| \leq \|\varphi\|$ for all $j \in J$

(9) If $\varphi, \psi \in T$ with $0 \leq \varphi \leq \psi$ on I, then $\|\varphi\| \leq \|\psi\|$.

J denotes the system of all finite subsets $j \subset I$, $\varphi_j(i) := \varphi(i)$ for $i \in j$, else $= 0$ in I. Examples would be $l^p(I)$, $1 \leq p < \infty$.

If T satisfies (7) and N_i, $i \in I$, are normed real or complex linear spaces, L *is of type* $T|N_i$ means L is a normed real or complex linear space of functions g on I with $g(i) \in N_i$, $i \in I$, under pointwise addition, scalar multiplication and equality, such that $g \in L$ implies $g_j \in L$ for all $j \in J$ and $|g| \in T$ with $\|g\|_L = \||g|\|_T$; $|g|(i) := \|g(i)\|_{N_i}$, g_j is defined as above. Examples are the substitution spaces $l^p(I, N_i)$, then for I = natural numbers ω, S = reals R, theorem E yields an analogue to a result of Amerio [3], which by theorem I below is in fact equivalent with it.

The case $p = 1$ is not subsumed by theorem E, but here we have

THEOREM F.—*Let* S *be a topological semigroup,* N_i *normed real or complex linear spaces for* $i \in I$, I *arbitrary index set,* $f : S \to l^1(I, N_i)$. *Then* f *is a.p. if and only if it is weakly a.p. and all its projections* $f_i : S \to N_i$ *are norm–a.p.,* $i \in I$.

For an extension to locally convex range spaces with (*wωs*) see [10, theorem 4].

If N_i = complex numbers C (or reals R) one gets as a first corollary, generalizing a result of Ricci and Rizzonelli [16] from R to arbitrary topological semigroups S:

For $f : S \to l^1(I, C)$ *weak and norm almost periodicity coincide.*

Amerio's result [3, $p = 1$] is subsumed by the following corollary:

If S *and* N_i *are as in theorem* F, *then the theorem of Amerio-Doss holds for* $l^1(I, N_i)$ *and* S *if and only if it holds for all* N_i *and* S, $i \in I$.

The case $p = \infty$ however, as Amerio has shown in [2], furnishes examples of Banach spaces where the theorem of Amerio-Doss fails to hold, even for S = R or integers.

Turning to integration of a.p. functions in R^n, let us say that *the theorem of Bohl-Amerio holds for* B *and* S if the following is true: B is a real or complex Banach space, S an open additive subsemigroup of R^n in the usual topology, $1 \leq n$ natural $< \infty$, each continuous and bounded $f : S \to B$ which has all its distribution derivatives of first order a.p. is itself a.p. on S.

For $n = 1$, S open convex subsemigroup $\subset R$, the theorem of Bohl-Amerio holds for B and S if and only if for any a.p. $g : S \to B$ with on S bounded indefinite Bochner integral $f(x) = \int_0^x g(t)\, dt$, f is a.p. on S.

For general n and S, if $\frac{\partial}{\partial x_1} f, \cdots, \frac{\partial}{\partial x_n} f$ are a.p., the difference f_s is a.p. on S for each $s \in S$ open convex $\subset R^n$; if $n = 1$, Stepanoff-almost-periodicity of f' is enough. So our theorems above apply, one has:

THEOREM G.—*If the theorem of Amerio-Doss holds for* B *and* S, S *open convex* $\subset R^n$, *then the theorem of Bohl-Amerio hods for* B *and* S. *The latter is true therefore especially for uniformly convex* B, *but also for* L *as in theorem* E *with Banach spaces* N_i *for which the Bohl-Amerio-theorem holds with respect to* S, *or for* $l^1(I, B_i)$ *with Banach spaces* B_i *which satisfy the Bohl-Amerio theorem.*

Similarly as in theorem C above, the boundedness assumption can be somewhat relaxed:

THEOREM H.—*Let* S *be a convex open subsemigroup of* R^n, M *a relatively dense subset of* S, B *a real or complex Banach space*, $f: S \to B$ *continuous with a.p. first order derivatives in* S; *if* $n = 1$, f' *need only be Stepanoff–a.p. with* $p = 1$. *Then* f *is a.p. if either* $f(M)$ *is totally bounded or* f *is bounded* (*or bounded in the mean*) *on* M *and the theorem of Bohl-Amerio holds for* B *and* S.

From this one deduces for S and M as in theorem H the equivalence of the following three statements concerning a distribution $T \in \mathfrak{D}'(S)$:

(10) T *is an a.p. distribution.*

(11) *For each* $s \in S$, $\tau_{-s}T - T$ *is a.p., and* T *is bounded on* M.

(12) *The first order derivatives of* T *are a.p. on* S, *and* T *is bounded on* M.

Theorem G and H contain as special cases results of Brauers [6], ($S = M = R^n$, $B = C$), Amerio [2], [3] ($S = M = R^1$, B uniformly convex or $l^p(\omega, B_n)$, ω = natural numbers), Vasconi [17] and Prouse [15] ($S = M = R^1$, B uniformly convex). Also a.p. solutions of the wave equation $P_x u - u_{tt} + a(x) u = f(x, t)$, P_x elliptic, can be treated if f is periodic, but not necessarily continuous in t (see [11, 11) in § 3]).

Theorem G and these applications suggest that " the theorem of Bohl-Amerio is valid " is a condition less restrictive than " the theorem of Amerio-Doss holds ". For $n = 1$ one has however

THEOREM I.—*If* B *is a real or complex Banach space, then the theorem of Amerio-Doss holds for* B *and* R *if and only if the theorem of Bohl-Amerio holds for* B *and* R.

This is a corollary of the following characterization of (unbounded) functions with a.p. differences:

THEOREM J.—*If* B *is a real or complex Banach space*, $f: R \to B$ *is bounded on a set of positive Lebesgue measure, then the following statements are equivalent*:

(13) *For each* $s \in R$, f_s *is a.p. on* R.

(14) *There are two a.p.* $g, h: R \to B$ *with* $f(x) = g(x) + \int_0^x h(t)\, dt$, $x \in R$.

(15) *There is a sequence of a.p. functions* $f_n: R \to B$ *with* $f(0) + \int_0^x f_n(t)\, dt \Rightarrow f(x)$, *uniformly in* $x \in R$.

If $B = C$, this still generalizes a result of Caracosta and Doss [7], in (13) they had to add " and f is uniformly continuous on R ". With theorem J a similar characterization of distributions with a.p. differences is possible.

Furthermore, there are extensions to $S \subset R^n$, also instead of " all differences f_s are a.p. ", under suitable assumptions it is enough if only sufficiently (finitely) many f_s are a.p. This we intend to treat elsewhere.

REFERENCES.

[1] AMERIO L., *Funzioni debolmente quasi-periodiche*, « Rend. Sem. Mat. di Padova », *30*, 288–301 (1960).

[2] AMERIO L., *Sull'integrazione delle funzioni quasi-periodiche astratte*, « Annali di Mat. pura et appl. », *53*, 371–382 (1961).

[3] AMERIO L., *Sull'integrazione delle funzioni* $l^p(X_n)$*-quasi-periodiche, con* $1 \leq p < \infty$, « Ricerche di Mat. », *12*, 3–12 (1963).

[4] AMERIO L., *Abstract almost-periodic functions and functional equations*, « Boll. Un. Mat. Ital. », (3), *20*, 287–334 (1965).

[5] BOCHNER S., *Abstrakte fastperiodische Funktionen*, « Acta Math. », *61*, 149–184 (1933).

[6] BRAUERS N., *Différentiation et intégration des fonctions presque–périodiques de plusieurs variables réelles*, « Acta Univ. Latviensis » (3), 7, 235–263 (1939).

[7] CARACOSTA G. et R. DOSS, *Sur l'intégrale d'une fonction presque-périodique*, « C. r. Acad. Sci., Paris », *246*, 3207–3208 (1958).

[8] DOSS R., *On bounded functions with almost periodic differences*, « Proc. Amer. Math. Soc. », *12*, 488–489 (1961).

[9] GÜNZLER H., *Fourierreihen vektorwertiger fastperiodischer Funktionen auf Halbgruppen*, Math. Inst. Univ. Göttingen, Sem. W. Maak, Bericht Nr. 5 (1961).

[10] GÜNZLER H., *Means over countable semigroups and almost-periodicity in* l^1, to appear.

[11] GÜNZLER H., *Integration of almost periodic functions*, to appear.

[12] KÖTHE G., *Topologische lineare Räume*, I. Berlin, Springer 1960.

[13] KOPEC J., *On linear differential equations in Banach spaces*, « Zeszyty Nauk. Univ. Mickiewicka. Mat.–Chem. », *1*, 3–7 (1957) (Polish).

[14] MAAK W., *Fastperiodische Funktionen auf Halbgruppen*, « Acta Math. », *87*, 33–58 (1952).

[15] PROUSE G., *Sulle equazioni differenziali astratte quasi–periodiche secondo Stepanoff*, « Ricerche Mat. », *11*, 254–270 (1962).

[16] RICCI M. L. e P. RIZZONELLI, *Sulle funzioni* l^1*–quasi–periodiche*, « Rend. Istit. Lombardo Sci. », *95*, 941–946 (1961).

[17] VASCONI A., *Sull'integrazione delle funzioni quasi–periodiche secondo Stepanoff, negli spazi di Clarkson*, « Rend. Istit. Lombardo Sci. », *95*, 1024–1029 (1961).