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An application of the extension theorem to a control problem

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Analisi matematica. — *An application of the extension theorem to a control problem.* Nota di GIORGIO P. SZEGÖ, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — In questo lavoro si studia il problema della stabilità assoluta di un sistema di controllo non lineare sotto opportune ipotesi sulla nonlinearità e la sua derivata prima. Mediante l'uso del teorema di estensione e di una funzione di Liapunov di nuovo tipo si ricavano condizioni sufficienti per la stabilità assoluta.

INTRODUCTION. — In Ref. [1, 2] and in particular in Ref. [3] a new set of theorems, called extension theorems, have been proved. These extension theorems apply to a particular, but highly realistic stability problem, of a compact set $M \subset E^n$, namely the case in which the stability properties "in the small" (in a sufficiently small neighbourhood) of M are known. In addition it must be assumed that such stability properties are strong, i.e., of a type for which the stability theorems of first approximation hold (either asymptotic stability or instability).

Under these circumstances the extension theorem gives conditions under which the same stability properties which hold "in the small" are true globally.

In the sequel, when not otherwise stated, capital Roman letters will denote matrices, small Roman letters vectors (notable exceptions $t =$ time, k, h, v and w which are scalars), small Greek letters scalars and script letters sets.

In what follows E^n denotes the euclidean n–space.

For completeness we shall now state the extension theorem in its strongest global form.

(1) *Extension theorem.*

Let $v = \varnothing(x)$ and $w = \psi(x)$ be real-valued functions defined on E^n. Let $M \subset E^n$ be compact. Assume that

i) $v = \varnothing(x) \in C^1$

ii) $\varnothing(x) = 0$ for $x \in M$

iii) for any sequence $\{x_n\}$, $\psi(x_n) \to 0$ implies $x_n \to M$

iv) $\psi(x) = \langle \operatorname{grad} \varnothing(x), f(x) \rangle$.

Then whatever may be the local stability properties of M for the differential equation

$$\dot{x} = f(x) \tag{2}$$

they are global.

(*) Nella seduta del 21 giugno 1967.

Notice that the major difference between the requirements of the extension theorem above and those of the Liapunov second method is that in the extension theorem no condition on the sign properties of the real-valued function $v = \varnothing(x)$ is imposed. Thus, the extension theorem is particularly well suited to the case of stability problems in which the analysis of the stability properties via the second method of Liapunov would be very difficult, if not impossible, due to the extrene difficulty of analyzing the sign properties of the Liapunov functions $v = \varnothing(x)$ used.

This situation is well illustrated by the particular problem which is the subject of this note.

2. THE CONTROL PROBLEM. — Consider the closed-loop control system represented by the equations:

$$(3) \qquad \begin{cases} \dot{x} = \mathrm{A}x - b\varphi(\sigma) \\ \sigma = 2c'x - 2\xi(t) \end{cases}$$

where the real valued function $\xi(t)$ satisfies the conditions

$$(4) \qquad \lim_{t \to +\infty} \xi(t) = 0 \quad ; \quad \xi(t), \dot{\xi}(t) \in \mathrm{L}_1[0, +\infty)$$

and the real valued function $\varphi(\sigma)$ satisfies the conditions

$$(5) \qquad 0 \leq \sigma\varphi(\sigma) \leq k^* \sigma^2 \quad ; \quad \frac{d\varphi}{d\sigma} \geq 0 \quad ; \quad \varphi(\sigma) \in \mathrm{C}^1.$$

i) for all real k with

$$(6) \qquad 0 \leq k \leq k^*$$

all linear systems

$$(7) \qquad \dot{x} = (\mathrm{A} - 2kbc')x$$

are asymptotically stable

ii) the linear part of the system (3) is completely controllable and completely observable. In particular, it is assumed that the matrix A and the vectors b, c have the form:

$$(8) \qquad \mathrm{A} = \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & 1 \\ a_1 & a_2 & \cdot & \cdots & a_n \end{vmatrix} \qquad b = \begin{vmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \\ 1 \end{vmatrix} \qquad c = \begin{vmatrix} c_1 \\ c_2 \\ \cdot \\ \cdot \\ \cdot \\ c_n \end{vmatrix}$$

Then for the system (3) we shall define the following property:

(9) *Definition.*

If the asymptotical equilibrium point $x = 0$ of the system (3) is globally attracting for all real valued functions satisfying conditions (5), then the system (3) is called *absolutely attracting*. If $\xi(t) \equiv 0$ and the rest point $x = 0$

of the system (3) is globally asymptotically stable for all real valued functions satisfying condition (5) then the system (3) is called *absolutely stable.*

We shall now prove a sufficient condition for absolute stability and attraction for the system (3). Such a condition is of the Popov type [4], i.e., it is a condition on the behaviour for all real ω of the function

$$G(j\omega) = 2c'(\mathrm{I}j\omega - \mathrm{A})^{-1} b \tag{10}$$

which is the so-called harmonic response function of the linear part of the system (3).

To derive this condition our basic instrument will then be the extension Theorem (1). The function $v = \varphi(x)$ that we shall use is derived from a function used by I. A. Iakubovich [6] and a function recently introduced by K. S. Narendra and C. P. Neuman [7]. All these functions are an improvement of the well known Lur'e function [8].

(11) THEOREM.

Let $\xi(t) \equiv 0$. *Let us denote with* $\varkappa_i$ *the real zeros of the function* $G(s)$. *Consider the real numbers* $\beta_i, \gamma_i \geq 0$ *and* $\varepsilon_i \geq 0$ *such that*

$$\varkappa_i = \frac{\gamma_i + \varepsilon_i}{\beta_i}. \tag{12}$$

If the conditions (4) (6) *and* (7) *are satisfied and in addition the inequality:*

$$\frac{1}{k^*} + \mathrm{Re}\, G(\mathrm{I}\omega)\left[1 + \beta_0 j\omega + \alpha\omega^2 + \Sigma_i \gamma\left(1 - \frac{\gamma_i}{\beta_i}\frac{1}{j\omega + \varkappa_i}\right)\right] > 0 \tag{13}$$

is satisfied for all real ω, *a real* β_0, *a real* $\alpha \geq 0$ *and real numbers* $\beta_i, \gamma_i \geq 0$ *and* $\varepsilon_i \geq 0$ *subject to the constraint* (12), *then the system* (3) *is absolutely stable for all nonlinearities of the class* (5).

Proof: Consider the real valued function:

$$v = x'\mathrm{H}x + \beta_0 \int_0^\sigma \varphi(\mu)\, d\mu + \Sigma_i \beta_i \int_0^{2x'\mathrm{D}_i c} \varphi(\mu)\, d\mu - 2\alpha\varphi x'\mathrm{A}'c + \alpha c' b \varphi^2 \tag{14}$$

where H is a symmetric matrix, β_i and α are real numbers and D_i are matrices. The total time derivative of the function (14) along the solutions of the system (3) ($\xi(t) \equiv 0$) has the form

$$\begin{aligned} \dot v = {} & x'[\mathrm{A}'\mathrm{H} + \mathrm{HA}]x - 2\varphi x'[\mathrm{H}b - \beta_0 \mathrm{A}'c + \alpha \mathrm{A}'\mathrm{A}'c - c - \gamma_i(c - \mathrm{D}_i c)] \\ & - \varphi^2\left[\frac{1}{k^*} + 2\beta c' b - 2\alpha b'\mathrm{A}'c\right] - 4\alpha\left[x'\mathrm{A}'c - \varphi b' c\right]\frac{\partial\varphi}{\partial\sigma} - \varphi\left[\sigma - \frac{1}{k^*}\varphi\right] \\ & - 2\gamma_i[(\varphi - \varphi_i)(\sigma - \sigma_i)] - 2\varphi_i[\gamma_i(\sigma - \sigma_i) - \beta_i x'\mathrm{A}'\mathrm{D}_i' c] \\ & - 2\beta_i \varphi_i \varphi b' \mathrm{D}_i' c, \end{aligned} \tag{15}$$

where we have omitted the summation signs, used the notations $\varphi_i = \varphi(2x'D_i c)$ and $\sigma_i = 2x'D_i c$ and introduced the identity

$$(16)\qquad 2\varphi x'c - \varphi\sigma + \frac{1}{k^*}\varphi^2 + 2\gamma_i(\varphi + \varphi_i)(\sigma - \sigma_i) - \frac{1}{k^*}\varphi^2 - \\ - 2\gamma_i(\varphi + \varphi_i)(\sigma - \sigma_i) = 0.$$

From condition (5) it follows that if $\gamma_i \geq 0$

$$(17)\qquad \gamma_i[(\varphi - \varphi_i)(\sigma - \sigma_i)] \geq 0.$$

The last two terms of the expression (15) will be nonpositive if the matrices D_i are chosen in such a way that for $\varepsilon_i \geq 0$ it is

$$(18)\qquad \begin{aligned} &\gamma_i(I - D_i) - \beta_i D_i A = \varepsilon_i D_i \\ &c' D_i b = 0. \end{aligned}$$

The first equation (18) has the solution

$$(19)\qquad D_i = \frac{\gamma_i}{\beta_i}(\chi_i I + A)^{-1}$$

where χ_i is defined by (12).
The second equation (18) becomes simply

$$(20)\qquad c'(\chi_i I + A)^{-1} b = 0$$

and it is satisfied if χ_i is a real zero of $G(s)$, as can be seen by comparing (20) and (10).

The Iakubovich-Kalman Lemma [9, 10] applied to the system

$$(21)\qquad \begin{cases} A'H + HA + qq' < 0 \\ Hb - (1 + \gamma_i - D_i')c - \beta_0 A'c + \alpha A'A'c = \gamma q \\ \frac{1}{k^*} + 2\beta_0 c'b - 2\alpha b'A'c = \gamma^2 \end{cases}$$

shows that a necessary and sufficient condition for the existence of real q and γ satisfying (21) is that the inequality

$$(22)\qquad \frac{1}{k^*} + 2\beta_0 c'b - \alpha b'A'c + \\ + 2\,\mathrm{Re}\,[c'(1 + \gamma_i - D_i + \beta_0 A - \alpha AA)(Ij\omega - A)^{-1} b] > 0$$

is satisfied for all real ω.

Following the techniques used in References [6, 12] the inequality (22) can be reduced to the simpler form (13).

If the condition (13) (and therefore (22)) is satisfied, then the expression (15) becomes

$$(23)\qquad \dot v \leq -[x'q + \gamma\varphi]^2 - 4\alpha[x'A'c - \varphi b'c]^2 \frac{\partial\varphi}{\partial\sigma} - \varphi\left[\sigma - \frac{1}{k^*}\varphi\right] - \\ - 2\,\Sigma_i\,\varepsilon_i\,\varphi_i\,\sigma_i - \eta x'Cx$$

where $\eta > 0$ is a sufficiently small real number and C is a positive definite matrix.

The condition (23), (6) and (5) are such that all conditions of theorem (1) are satisfied. Thus the system (3) is absolutely stable and the theorem is proved.

By the same procedure as in [11] it is possible to show that if in addition condition (4) is satisfied and $\xi(t) \not\equiv 0$, the system (3) is absolutely attracting.

3. CONCLUSIONS. — By means of the extension theorem we have derived the condition (13) for absolute stability of the system (3). Notice that it is not at all obvious that the real valued function (14) is positive definite for all real β_i for which condition (13) is satisfied, so that the application here presented is a significant example of the use of the extension theorem as compared with the use of the classical theorems on asymptotical stability.

The extension theorem has other important corollaries as, for instance, a theorem analogous to Rolle's in E^n, which will be the subject of another paper.

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