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The Quasi-Commuting Inverses for a Square Matrix

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Algebra lineare. — *The Quasi-Commuting Inverses for a Square Matrix.* Nota di IVAN ERDELYI, presentata (*) dal Socio B. SEGRE.

SUNTO. — Il presente lavoro concerne il sistema più generale di equazioni a termini matriciali, anche singolari, per il quale il sottospazio degli autovettori è invariante; tale sistema implica un concetto generale di quasi-commutatività ed è strettamente legato ai diversi tipi di funzioni inverse matriciali. Qui si studia l'esistenzà e la forma generale delle soluzioni e si considera, come applicazione, una classe di isometrie parziali più generale della classe di isometrie parziali normali.

INTRODUCTION.

We are concerned in this paper with the matrix system:

$$\text{(I)} \quad \text{(i) } AXA=A, \quad \text{(ii) } XAX=X, \quad \text{(iii) } A^kX=XA^k, \quad \text{(iv) } AX^k=X^kA,$$

for a given A and some positive integers k.

System (I) is closely related to various inverse-type matrix functions. Bjerhammar [1] defined, by the sole equation (i) the set of the generalized inverses, and by both equations (i) and (ii), the reciprocal inverses of the given matrix A. Further developments of the generalized and reciprocal inverses are due to Frame [2], Cline [3], Korganoff and Pavel-Parvu [4], and Englefield [5]. The Moore-Penrose inverse, in Penrose's formulation [6], is the reciprocal inverse $X = A^+$, for which both idempotents AA^+ and A^+A are Hermitian. The particular case of a possible commuting reciprocal inverse, i.e. $AX = XA$, had been studied in [5] and [7].

The requirement of commutativity may be relaxed for extending the existence of the reciprocal inverse for any square matrix over the complex field.

We shall, therefore, substitute commutativity by some quasi-commuting relations for the prospect of preserving the invariance of the modal matrix and of inverting the nonsingular part of the spectral matrix. Unfortunately, such an extension entails sacrificing the uniqueness of the inverse. The class of the quasi-commuting inverses contains, in particular, the unique Scroggs-Odell inverse [8] which, however, is not defined by equations expressible in a semigroup language.

Here there are some *preliminary properties* of system (I):

1. (I) is invariant under similarity, hence geometry of linear transformations can be used to advantage.

2. (I) implies that the largest subspace on which A is invariant is also invariant under X. Likewise is the orthogonal complement and hence direct sums.

3. The unique solution of (I) for the nonsingular part is the ordinary inverse.

(*) Nella seduta del 13 maggio 1967.

THE EXISTENCE OF SOLUTIONS.

It is well-known that every linear transformation is the direct sum of a nonsingular and a nilpotent transformation.

Let

$$A = TBT^{-1} \tag{1}$$

$$B = \left[\begin{array}{c:c} M & \\ \hdashline & N \end{array}\right], \tag{2}$$

with M nonsingular of rank r, and N a pth order nilpotent matrix of index q, i.e. $N^q = o$, $N^{q-1} \neq o$, be a decomposition of the given matrix A.

A possible solution X, for system (I), may be represented in the form:

$$X = TYT^{-1} \tag{3}$$

corresponding to the similarity transformation (1). Equations (I), then become equivalent to the following system:

(II) (i′) $BYB = B$, (ii′) $YBY = Y$, (iii′) $B^kY = YB^k$, (iv′) $BY^k = Y^kB$.

In particular, when B is nonsingular ($p = o$), the unique solution $Y = B^{-1}$ to system (II) will correspond. If B is the null matrix o ($r = o$, $N = o$), then $Y = o$ is the solution.

In the general case, let

$$Y = \left[\begin{array}{c:c} V & P \\ \hdashline Q & Z \end{array}\right] \tag{4}$$

be the partitioned form of a possible solution for system (II), conformable to the partition of the given B, (2). Then, by introducing the partitioned forms (2) and (4) for B and Y, respectively, into the equations (i′) and (iii′) of the set (II), we are led to the following relations:

$$\begin{gathered} MVM = M \quad , \quad MPN = o \quad , \quad NQM = o, \\ NZN = N, \end{gathered} \tag{5}$$

$$\begin{gathered} M^kP = PN^k \quad , \quad N^kQ = QM^k \quad , \quad M^kV = VM^k, \\ N^kZ = ZN^k. \end{gathered} \tag{6}$$

where o designates the null matrix of appropriate size. Since M is nonsingular, we obtain:

$$\begin{gathered} V = M^{-1}; \\ PN = o \quad , \quad NQ = o, \\ P = M^{-k}(PN)N^{k-1}, \\ Q = N^{k-1}(NQ)M^{-k} \end{gathered}$$

and hence

$$P = 0 \quad , \quad Q = 0.$$

Thus the solutions Y, when they exist, assume the partitioned block-diagonal form:

$$Y = \left[\begin{array}{c:c} M^{-1} & \\ \hdashline & Z \end{array}\right]. \tag{7}$$

Next, the remaining equations (ii′) and (iv′) of the set (II), together with (5) and (6), establish the following subsystem of equations:

(III) (i″) $NZN = N$, (ii″) $ZNZ = Z$, (iii″) $N^k Z = ZN^k$, (iv″) $NZ^k = Z^k N$.

The couples of equations (i″), (iii″) and (ii″), (iv″) lead us to the following equations, respectively:

$$N^h = N^{h+1} Z = ZN^{h+1}, \tag{8}$$

$$Z^h = Z^{h+1} N = NZ^{h+1}, \qquad \text{for all} \quad h \geq k. \tag{9}$$

Furthermore, with the help of (i″) and (8) we get successively:

$$N^k = N^{k-1}(NZN) = (N^k Z) N = ZN^{k+1} = \cdots = Z^2 N^{k+2} = \cdots,$$

thus

$$N^k = Z^s N^{k+s}, \qquad s = 1, 2, \cdots \tag{10}$$

and similarly, (ii″) and (9) lead us to the following:

$$Z^k = N^s Z^{k+s}, \qquad s = 1, 2, \cdots \tag{11}$$

We are now in a position to prove

THEOREM 1.—*The matrix system* (I) *possesses at least one solution if and only if*

$$k \geq q, \tag{12}$$

where q is the index of the nilpotent term N.

PROOF.—Assume that N is nilpotent of index q. If $k < q$ then system (II) and subsequently (I) fail to have solutions. In this circumstance system (III) is incompatible. In fact, if $k < q$, then $N^k \neq 0$. Then by (10) we have a contradiction when $s = q - k$, thereby making condition (12) necessary.

As for sufficiency, by admitting condition (12), we shall show that the matrix

$$Y = B^+ = \left[\begin{array}{c:c} M^{-1} & \\ \hdashline & N^+ \end{array}\right] \tag{14}$$

satisfies system (II). In fact, by equations (12), the third and the fourth equations of (III) are void and hence we are back to the problem of solutions to the first two Penrose equations. Thus $Y = B^+$ is a solution of system (II), and X, (3) verifies system (I).

If k is unrestricted, system (I) always admits solutions.

ON THE SOLUTIONS OF SYSTEM (I).

Representations for the reciprocal inverses were obtained by Cline [3] through a matrix factorization, due to Scroggs and Odell [8]. This concerns the representation of the given matrix A, of rank $t \geq r$, and likewise its reciprocal inverses X as a product of three matrices:

$$\begin{aligned} A &= CED \\ X &= D^{-1} EC^{-1}, \end{aligned} \tag{15}$$

where

$$E = \left[\begin{array}{c:c} I_t & \\ \hdashline & O \end{array}\right], \text{ and } I_t \text{ is the identity matrix of order } t.$$

Since E is idempotent, we have

$$A^k = C\,[E\,(DC)\,E]^{k-1}\,D,$$

$$X^k = D^{-1}\,[E\,(DC)^{-1}\,E]^{k-1}\,C^{-1}, \text{ for any integer } k > 1.$$

If we represent the product DC and its inverse $(DC)^{-1}$ in the partitioned forms:

$$DC = \left[\begin{array}{c:c} P & \cdots \\ \hdashline \cdots & \cdots \end{array}\right], \qquad (DC)^{-1} = \left[\begin{array}{c:c} Q & \cdots \\ \hdashline \cdots & \cdots \end{array}\right],$$

where P and Q are t by t matrices, the *kth* powers of A and X become expressible as

$$A^k = C\left[\begin{array}{c:c} P^{k-1} & \\ \hdashline & O \end{array}\right]D, \quad X^k = D^{-1}\left[\begin{array}{c:c} Q^{k-1} & \\ \hdashline & O \end{array}\right]C^{-1}, \qquad k > 1$$

and then, the remaining equations (iii) and (iv) of system (I) might be used for delimiting the solutions of (I) among the reciprocal inverses. Thus after substituting the foregoing factorizations of A, X, A^k and X^k into equations (iii) and (iv) of system (I), we obtain

$$DC\left[\begin{array}{c:c} P^{k-1} & \\ \hdashline & O \end{array}\right] = \left[\begin{array}{c:c} P^{k-1} & \\ \hdashline & O \end{array}\right]DC$$

$$DC\left[\begin{array}{c:c} Q^{k-1} & \\ \hdashline & O \end{array}\right] = \left[\begin{array}{c:c} Q^{k-1} & \\ \hdashline & O \end{array}\right]DC.$$

These are the delimiting conditions for all couples of matrices C and D, which perform the factorization (15).

In at least one particular case, it can easily be seen that all solutions of system (I) enjoy a noticeable spectral property. We shall, therefore, confine the given A to the class of matrices which are equivalent to the direct sum of a nonsingular and a nilpotent matrix of either minimal or maximal index.

For these matrices, system (I) will provide all matrices which preserve the complete set of principal vectors, invert the nonzero, and conserve the zero eigenvalues with the algebraic multiplicities properly counted. Such solutions of (I) were used for reducing the matrix equation $Ax = \lambda Bx$, in the general case, to the ordinary eigenvector problem [7].

In view of the representation (1), (2) of the given matrix A, the nilpotent term N, when of maximal index p, is always reducible, by similarity, to a p*th* order Jordan matrix J_0, having 1's immediately above (or below) the main diagonal and 0's elsewhere. It is convenient now to consider N the zero matrix, when of minimal index $q = 1$; and in the reduced Jordan form J_0, when it is of maximal index $q = p$.

Let $\mathcal{A}^-$ be the set of all matrices A^- satisfying system (I), and B represent any of the possible reductions (1) of A in the direct sum of a nonsingular matrix M and a nilpotent matrix N of minimal or maximal index as considered above. We then have

THEOREM 2.—*All solutions* $A^- \in \mathcal{A}^-$ *of system* (I) *are equivalent to* B^+, *under similarity.*

PROOF.—As we have already seen, the partitioned form of any solution Y for system (II) is given by (7) consistent with the conditions (III).

If N is the zero matrix, $Z = 0$ and hence

$$Y = \left[\begin{array}{c:c} M^{-1} & \\ \hdashline & 0 \end{array}\right] = B^+ \tag{16}$$

is the unique solution of (II).

Assume now that N is nilpotent of maximal index p. First we shall prove that the corresponding Z is also nilpotent of maximal index p. This is a straightforward consequence of system (III). In fact, relation (11), for $k = s = p$, gives:

$$Z^p = 0.$$

Next, we need to show that the smallest power m which causes Z to vanish is p.

Let us suppose that

$$Z^m = 0\,, \qquad \text{for } m < p. \tag{17}$$

Now we may interchange the roles of matrices N and Z in system (III). Given Z and if (17) is true, then by Theorem 1, it follows that there exists one or more solutions N which correspond, in particular, to $k = m < p$. Thus, for all solutions N, we have $N^m = 0$, and this contradicts the definition of N.

Furthermore, for a nilpotent matrix of maximal index p like Z, there always exists a similarity transformation:

$$Z = RJ_0^* R^{-1} = RN^+ R^{-1} \tag{18}$$

where the asterisk denotes the conjugate transpose.

The replacement of (18) in (7), gives

$$Y = \left[\begin{array}{c:c} M^{-1} & \\ \hdashline & RN^+R^{-1} \end{array}\right].$$

Then, with the help of the matrix

$$S = \left[\begin{array}{c|c} I_r & \\ \hline & R \end{array}\right],$$

Y achieves the form

$$Y = S\left[\begin{array}{c|c} M^{-1} & \\ \hline & N^+ \end{array}\right]S^{-1} = SB^+S^{-1}. \tag{19}$$

Finally, we return to the original system (I). In both cases (16) and (19) the solutions X, by the transformation (3), are equivalent to B^+, under similarity. Notice that neither T nor R, which performs the similarity reductions (1) and (18), respectively, is unique.

The solutions of system (I) possess a spectral property which might be regarded as a generalization of a corresponding property of the ordinary inverse for matrices. This may be stated as follows:

If λ_i are the eigenvalues of the given matrix A, then all solutions $A^- \in \mathfrak{A}^-$ have as eigenvalues:

$$\lambda_i^+ = \begin{cases} \lambda_i^{-1}, & \text{if } \lambda_i \neq 0; \\ 0, & \text{if } \lambda_i = 0, \end{cases}$$

with the same algebraic multiplicities. Moreover, the eigenvalues λ_i and λ_i^+ pairwise have the same elementary divisors, and the same eigenvectors.

QUASI-COMMUTATIVITY.

Two square matrices A and X are called, by McCoy (e.g. [9, pp. 250–254]) quasi-commutative if they satisfy the following relations:

$$(AX - XA)A = A(AX - XA) \tag{20}$$

$$(AX - XA)X = X(AX - XA). \tag{21}$$

In order to further elaborate on the matrix system (I), we shall consider a slight generalization of the previous quasi-commutative relations, that is

$$(AX - XA)A^k = A^k(AX - XA) \tag{22}$$

$$(AX - XA)X^k = X^k(AX - XA), \tag{23}$$

for some positive integers k. We shall call any couple of matrices A, X, which verify relations (22) and (23), quasi-commutative of index k. It is obvious that the quasi-commutativity of index 1 coincides with McCoy's original concept.

It is a matter of elementary algebra to show that a reciprocal inverse X of A is a solution of system (I), if and only if A and X are quasi-commuting matrices of index k.

APPLICATION TO PARTIAL ISOMETRIES.

Among the solutions $A^- \in \mathcal{A}^-$, there may be one equal to the conjugate transpose of A,

$$A^- = A^*. \tag{24}$$

Since $A^- A$ and AA^-, by condition (24), are Hermitian, this solution, when it exists, coincides with the Moore-Penrose inverse A^+ and subsequently it is unique. Furthermore, from

$$A^- = A^+ = A^*,$$

it follows that the matrix A, which produces the solution defined by (24), is a partial isometry.

Even more than this is true. Since the spectrum of a partial isometry covers the closed unit disc (e.g. [4, 10, 11]), the nonzero eigenvalues of the partial isometry defined by (24) have modulus 1. This is in common with the normal partial isometries (e.g. [12]).

The partial isometry A (24) and its conjugate transpose are quasi-commutative of index k. The class $\mathcal{Q}$ of these partial isometries may be regarded as a generalization of finite normal partial isometries. In particular, if a partial isometry A, together with its conjugate transpose A^*, verifies the McCoy quasi-commutative relations (20) and (21), it is normal.

The unitary equivalence holds for the class $\mathcal{Q}$ of the considered partial isometries. It takes merely a simple computation to show that if $A \in \mathcal{Q}$, then every $B = UAU^* \in \mathcal{Q}$, with U unitary. This may be proved by direct substitution of B and B^*; either in the matrix system (I) or, in the quasi-commutative relations (22) and (23).

In the particular case, when a partial isometry $A \in \mathcal{Q}$ is unitarily similar to the direct sum of a nonsingular and a nilpotent matrix, then the nonsingular term is unitary.

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