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Triple transitivity in finite Möbius planes

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Matematica. — *Triple transitivity in finite Möbius planes.* Nota di JUDITA COFMAN, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Sia $\mathfrak{M}$ un piano finito di Möbius di ordine n. In base di risultati di Dembowski [5] e Hering [9], se $\mathfrak{M}$ ammette un gruppo di automorfismi 3–transitivo sui punti di $\mathfrak{M}$, allora $\mathfrak{M}$ è miqueliano. In questa Nota si dimostra che, se $\mathfrak{M}$ ammette un gruppo di automorfismi che trasformi un insieme $\mathfrak{V}$ di $k > n + 1$ punti in sè e che sia 3–transitivo sui punti di $\mathfrak{V}$, allora $\mathfrak{V}$ contiene tutti i punti del piano $\mathfrak{M}$ sicché $\mathfrak{M}$ è miqueliano.

A Möbius plane is an incidence structure consisting of points and circles and an incidence relation satisfying the following axioms (see for instance Benz [3]):

(I) *Any three distint points are incident with exactly one circle.*

(II) *If c is a circle,* A *a point on c and* B *a point not on c then there exists exactly one circle d containing* A *and* B *such that* $c \cap d = \{A\}$.

(III) *There exist at least four non-concyclic points. Any circle is incident with at least one point.*

An *automorphism* of a Möbius plane is a permutation of the points of the plane mapping circles onto circles.

Let P be any point of a Möbius plane $\mathfrak{M}$. Consider the following incidence structure $\mathfrak{M}_P$:

the points of $\mathfrak{M}_P$ *are the points of* $\mathfrak{M}$ *distinct from* P;
the lines of $\mathfrak{M}_P$ *are the circles of* $\mathfrak{M}$ *through* P;
incidence in $\mathfrak{M}_P$ *is equivalent to incidence in* $\mathfrak{M}$.

It is easy to see that $\mathfrak{M}_P$ is an affine plane; it is called *the affine subplane of* $\mathfrak{M}$ *at* P. The order of $\mathfrak{M}_P$ does not depend on the point P; it is called *the order of* $\mathfrak{M}$.

A Möbius plane $\mathfrak{M}$ is said to be *finite* if the number of points in $\mathfrak{M}$ is finite.

A Möbius plane is called *miquelian* if in the plane the THEOREM OF MIQUEL (see e.g. [3]) is satisfied.

Let $\mathfrak{M}$ be a finite Möbius plane of order n satisfying the following condition:

(**A**) $\mathfrak{M}$ *contains a set* $\mathfrak{V}$ *of k points and admits an automorphism group* Δ *such that* Δ *maps* $\mathfrak{V}$ *onto itself and induces a triply transitive permutation group on the points of* $\mathfrak{V}$.

It is known that finite miquelian Möbius planes $\mathfrak{M}$ of order n satisfy condition (**A**) for $k = n^2 + 1$ and $k = n + 1$ (see [3]); in the first case $\mathfrak{V}$ consists

(*) Nella seduta del 13 maggio 1967.

of all points of $\mathfrak{M}$ in the latter the points of $\mathfrak{V}$ are the points of a circle in $\mathfrak{M}$.

Moreover Dembowski [5] and Hering [9] proved that finite Möbius planes of order n satisfying condition (**A**) for $k = n^2 + 1$ are miquelian.

The aim of the present note is to show that there are no finite Möbius planes of order n satisfying condition (**A**) for $n + 1 < k < n^2 + 1$. Thus the following generalization of the above result of Dembowski and Hering is obtained:

If $\mathfrak{M}$ is a finite möbius plane of order n admitting an automorphism group Δ, which maps a set $\mathfrak{V}$ of $k > n + 1$ distinct points of $\mathfrak{M}$ onto itself and induces a triply transitive permutation group on the elements of $\mathfrak{V}$ then $\mathfrak{V}$ consists of all points of $\mathfrak{M}$ and $\mathfrak{M}$ is miquelian.

DEFINITIONS AND PRELIMINARY RESULTS.

Let P be an arbitrary point of a Möbius plane $\mathfrak{M}$ and let $\alpha \neq 1$ be an automorphism of $\mathfrak{M}$ fixing P. Then α induces a collineation α_P in the affine plane $\mathfrak{M}_P$. If $\mathfrak{M}_P$ is a perspectivity with an affine axis c in $\mathfrak{M}_P$ then α is called an *inversion of $\mathfrak{M}$ with axis c*. If α_P is a perspectivity with improper axis in $\mathfrak{M}_P$ and an affine (improper) centre then α is called a *dilatation* (*translation*) *of* $\mathfrak{M}$.

For our proofs the following results will be needed:

Result 1 (Dembowski [5] (5.3) and Zusatz 5): Let $\mathfrak{M}$ be a Möbius plane and let c be a circle of $\mathfrak{M}$. Then there exists at most one inversion in $\mathfrak{M}$ with axis c. Any inversion of $\mathfrak{M}$ is involutorial.

Result 2 (Dembowski [6] Satz 2.3): Let $\mathfrak{M}$ be a finite Möbius plane of order n and let α be an involution (i.e. an automorphism of order 2) of $\mathfrak{M}$ which is not an inversion. Then

if n is even α is a translation, and

if n is odd α is either a dilatation or a fixed point free automorphism of $\mathfrak{M}$.

Results 3 and 4 can be immediately deduced from Results 1–2:

Result 3: Let $\mathfrak{M}_P$ be the affine subplane of a Möbius plane $\mathfrak{M}$ at an arbitrary point $P \in \mathfrak{M}$ and let c be any affine line of $\mathfrak{M}_P$. Then there exists at most one perspectivity in $\mathfrak{M}_P$ with axis c. Any perspectivity of $\mathfrak{M}_P$ with affine axis is involutorial.

Result 4: Let $\mathfrak{M}_P$ be the affine subplane of a finite Möbius plane $\mathfrak{M}$ at an arbitrary point $P \in \mathfrak{M}$. Then all involutions of $\mathfrak{M}_P$ are perspectivities.

Result 5 (Dembowski [5] Satz 3): The affine subplane $\mathfrak{M}_P$ of a finite Möbius plane $\mathfrak{M}$ of even order at an arbitrary point P is desarguesian.

Result 6 (Dembowski [5] Satz 5): A finite Möbius plane $\mathfrak{M}$ of even order admitting an automorphism group, which is triply transitive on the points of $\mathfrak{M}$, is miquelian.

Result 7 (Hering [9]): A finite Möbius plane $\mathfrak{M}$ of odd order admitting an automorphism group which is doubly transitive on the points of $\mathfrak{M}$, is miquelian.

Result 8 (Cofman [4] Theorem 1): Let $\mathfrak{A}$ be a finite affine plane of order n and let $\mathfrak{S}$ be a set of $l > n + 1$ affine points in $\mathfrak{A}$. If $\mathfrak{A}$ admits a collineation group Γ which maps $\mathfrak{S}$ onto itself and is doubly transitive on the points of $\mathfrak{S}$ and if the involutions of Γ are perspectivities, then $\mathfrak{A}$ is a translation plane and $\mathfrak{S}$ consists of all affine points of $\mathfrak{A}$.

Main results.

We start our investigations by proving several lemmas about finite affine planes.

Let $\mathfrak{A}$ be a finite affine plane of order n satisfying the condition:

(**B**) *$\mathfrak{A}$ admits a collineation group Γ which maps a set $\mathfrak{S}$ of l affine points of $\mathfrak{A}$ onto itself and induces a doubly transitive permutation group on the elements of $\mathfrak{S}$.* The following can be shown:

LEMMA 1.—*If $\mathfrak{A}$ is an affine plane of order n satisfying condition* (**B**) *for $l > n$ then Γ is transitive on the improper points of $\mathfrak{A}$.*

Proof.—An arbitrary improper point of $\mathfrak{A}$ is incident with exactly n affine lines of $\mathfrak{A}$. Since $\mathfrak{S}$ contains more than n points this implies that through any improper point of $\mathfrak{A}$ there exists at least one affine line carrying more than one point of $\mathfrak{S}$. From the doubly transitivity of Γ on the points of $\mathfrak{S}$ then it follows that Γ is transitive on the improper points of $\mathfrak{A}$.

LEMMA 2.—*If $\mathfrak{A}$ is an affine plane of order n satisfying condition* (**B**) *for $l > n$ and if Γ contains a non-identical homology with improper axis then Γ contains non-identical elations with improper axis.*

Proof.—Let α be a non-identical homology of $\mathfrak{A}$ with centre A and improper axis l_∞. Let B_0 be a point of $\mathfrak{S}$ distinct from A; denote by B'_0 the image of B_0 under α. Clearly $B'_0 \neq B_0$. Since $\mathfrak{S}$ contains more than n points the points of $\mathfrak{S}$ are not collinear. Let C_0 be a point of S not on $B_0 B'_0$ and let φ be a collineation of Γ fixing B'_0 and mapping B_0 onto C_0. Then $\varphi^{-1} \alpha\varphi$ is a homology with axis l_∞ and centre $A\varphi \neq A$. According to André [1] Satz 3 the group generated by α and $\varphi^{-1} \alpha\varphi$ contains an elation with axis l_∞ mapping A onto $A\varphi$; q.e.d.

LEMMA 3.—*Let $\mathfrak{M}_P$ be the affine subplane of a finite Möbius plane $\mathfrak{M}$ of odd order n at a point $P \in \mathfrak{M}$. Then $\mathfrak{M}_P$ cannot satisfy condition* (**B**) *for $l = n + 1$.*

Proof.—Assume that $\mathfrak{M}_P$ satisfies condition (**B**) for $l = n + 1$. At first we shall show that

(i) *the collineation group Γ of $\mathfrak{M}_P$ contains non-identical homologies with improper axis.*

Γ is of even order thus it contains a non-identical involution α_P. According to Result 2 the involution α_P is a homology of $\mathfrak{M}_P$.

If the axis of α_P is the improper line of $\mathfrak{M}_P$ then (i) is proved.

Therefore it remains to investigate the case when the axis of α_P is an affine line a of $\mathfrak{M}_P$. Then the centre of α_P is an improper point A_∞ of $\mathfrak{M}_P$. By Lemma 1 the group Γ is transitive on the improper points of $\mathfrak{M}_P$; this implies that Γ contains an involution β_P whose centre B_∞ is the intersection of a with the improper line of $\mathfrak{M}_P$.

Suppose that the axis b of β_P is not incident with A_∞. Then $\beta_P^{-1}\,\alpha_P\,\beta_P$ is an involutory homology with axis a and centre $A_\infty\,\beta_P \neq A_\infty$. Thus Γ contains two distinct perspectivities of $\mathfrak{M}_P$ with the same axis, a contradiction to Result 3.

Hence $b \ni A_\infty$. The product $\alpha_P\,\beta_P$ is, according to Ostrom [10] Lemma 6, an involutory homology with centre $a \cap b$ and axis $A_\infty B_\infty$.

This proves (i).

From (i), applying Lemma 2, it follows that $\mathfrak{M}_P$ contains non-identical translations. According to Gleason [7], Lemma 1.6. this implies together with the transitivity of Γ on the improper points of $\mathfrak{M}_P$ that Γ contains the translation group of $\mathfrak{M}_P$. Thus Γ is transitive on the affine points of $\mathfrak{M}_P$ which contradicts the fact that Γ has an orbit $\mathcal{S}$ of $n+1$ affine points. This establishes the proof of Lemma 3.

LEMMA 4: *Let $\mathfrak{M}_P$ be the affine subplane of a finite Möbius plane $\mathfrak{M}$ of even order n at a point P of $\mathfrak{M}$. Then $\mathfrak{M}_P$ cannot satisfy condition* (**B**) *for $l = n+1$.*

Proof.—Assume that $\mathfrak{M}_P$ satisfies condition (**B**) for $l = n+1$.

The group Γ is of even order hence, by Result 2 and Lemma 1, any improper point of $\mathfrak{M}_P$ is the centre of at least one involutory elation of Γ.

Let A_0 be one of the $n+1$ points of $\mathcal{S}$. Among the $n+1$ lines joining A_0 to the improper points of $\mathfrak{M}_P$ clearly there exists at least one line b carrying no point of $\mathcal{S}$ distinct from A_0. Denote by B_∞ the intersection of b with the improper line of $\mathfrak{M}_P$. Obviously the axis of any elation with centre B_∞ must contain A_0. Hence, according to Result 3, it follows:

(i) *Any improper point of $\mathfrak{M}_P$ is the centre of exactly one non-identical involution of Γ.*

Since Γ is doubly transitive on the points of $\mathcal{S}$, for any two points of $\mathcal{S}$ there exists an involution of Γ mapping one onto the other. This implies, in view of (i), that no three distinct points of $\mathcal{S}$ are collinear.

However according to Result 5, the plane $\mathfrak{M}_P$ is desarguesian and therefore it cannot admit a collineation group which is doubly transitive on a set of $n+1$ points no three of which are collinear (see Hartley [8]).

This contradiction proves Lemma 4.

We are now able to prove our

THEOREM.—*Let $\mathfrak{M}$ be a finite Möbius plane of order n and let $\mathcal{V}$ be a set of $k > n+1$ points in $\mathfrak{M}$. If $\mathfrak{M}$ admits an automorphism group Δ which maps $\mathcal{V}$ onto itself and is triply transitive on the points of $\mathcal{V}$ then $\mathcal{V}$ consists of all points of $\mathfrak{M}$ and $\mathfrak{M}$ is miquelian.*

Proof.—Let P_0 be an arbitrary point of $\mathfrak{M}$. The stabilizer of P_0 in Δ induces an automorphism group Δ_{P_0} in the affine subplane $\mathfrak{M}_{P_0}$ of $\mathfrak{M}$ at P_0 with the properties:

(1) *Δ_{P_0} maps the set $\mathcal{V}_{P_0} = \mathcal{V} \setminus \{P_0\}$ onto itself and is doubly transitive on the points of $\mathcal{V}_{P_0}$;*

(2) *the involutions of Δ_{P_0} are perspectivities* (according to Result 4).

We shall distinguish two cases:

(a) $$k = n + 2 ;$$

(b) $$k > n + 2 .$$

In *Case* (a) the plane $\mathfrak{M}_{P_0}$ would satisfy Condition (**B**) for $l = n + 1$. However this is impossible according to Lemmas 3-4. Thus Case (a) cannot occur.

In *Case* (b) the set $\mathfrak{V}_{P_0}$ consists of more than $n + 1$ points. Δ_{P_0} has the properties (1)-(2) hence the affine plane $\mathfrak{M}_{P_0}$ satisfies the assumptions of Result 8. Thus, by Result 8, the set $\mathfrak{V}_{P_0}$ consists of all affine points of $\mathfrak{M}_{P_0}$. The point P_0 is not fixed by Δ therefore Δ is triply transitive on the points of $\mathfrak{M}$. The application of Results 6-7 completes the proof of the Theorem.

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