Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

James H. Bramble, Lawrence E. Payne

Bounds for the first derivatives of Green's function

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **42** (1967), n.5, p. 604–610.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1967_8_42_5_604_0>

NOTE PRESENTATE DA SOCI

Analisi matematica. — *Bounds for the first derivatives of Green's function.* Nota di JAMES H. BRAMBLE (*) e LAWRENCE E. PAYNE (**), presentata (***) dal Corrisp. G. FICHERA.

RIASSUNTO. — Vengono dimostrate formule di maggiorazione, della forma $|\partial G(x,y)/\partial x_i| \leq C r_{xy}^{1-n}$ per le derivate prime usando un metodo che si basa sul principio di massimo.

Nel caso in cui la frontiera appartiene alla classe $C^{1,1}$ il metodo ottiene facilmente un valore numerico per la costante C.

I. INTRODUCTION.—Let $G(x,y)$ be the Green function for Laplace's operator defined on a finite n dimensional region R with Liapunov boundary ∂R. The purpose of this note is to give a complete proof of the inequality

$$\left|\frac{\partial G(x,y)}{\partial x_i}\right| \leq C r_{xy}^{1-n}, \qquad i = 1, \cdots, n, \tag{1.1}$$

where C is a constant depending only on R, and r_{xy} is the distance from the point $x = (x_1, \cdots, x_n)$ to the point $y = (y_1, \cdots, y_n)$.

This question is discussed elsewhere in the literature. For example, Eidus [3] presents (1.1) and states without proof the inequality (2.1) which is crucial in his derivation of (1.1). Giraud [5], however, gives a proof of (2.1) so that if the results of these two authors were combined (1.1) could be considered as having been proved. The method presented here is simpler and in certain cases allows us to easily determine the constant C in (1.1). More recently Fichera [4] has treated the case of second derivatives in two dimensions using a method of conformal mapping even for non-simply connected domains. His method can be applied to obtain an inequality analogous to (1.1) for partial derivatives of any order

By the methods used in obtaining (1.1) we can obtain

$$\left|\frac{\partial^2 G(x,y)}{\partial n_x \partial n_y}\right| \leq C r_{xy}^{-n} \tag{1.2}$$

where $\partial/\partial n_x$ and $\partial/\partial n_y$ are outward directed normal derivatives at x and y, respectively, and, of course, x and y lie on ∂R. This, we show, leads to a bound for the Dirichlet integral of a harmonic function whose boundary values are Hölder continuous with exponent greater than 1/2. That this is a sufficient

(*) Supported in part by NSF GP–3666. Some of this reasearch was completed while this author was an NSF Senior Postdoctoral Fellow, visiting the University of Rome, Rome Italy.

(**) Supported in part by NSF GP–5882.

(***) Nella seduta dell'8 aprile 1967.

condition for the existence of the Dirichlet integral was shown by Miranda [8]. For further interesting work on this subject see De Vito [2].

Throughout we shall use C as a generic constant, not necessarily the same in any two places.

II. BOUNDARY ESTIMATES.—We shall assume that the boundary ∂R satisfies the following conditions of Liapunov.

(1) At every point of ∂R there exists a uniquely defined tangent plane.

(2) There exist two constants $C > 0$ and $\lambda, 0 < \lambda \leq 1$, such that for any two points P_1 and P_2 on the surface, $\theta < Cr^{\lambda}$, where θ is the angle between the normals through P_1 and P_2 and r is the distance between P_1 and P_2.

(3) There exists a constant $d > 0$ such that if Σ is a sphere with radius d and center at $P \in \partial R$, every line parallel to the normal at P intersects ∂R at most once inside Σ (1).

In this section we shall show that

$$\left| \frac{\partial G(x, y)}{\partial n_x} \right| \leq C r_{xy}^{1-n} \tag{2.1}$$

for $x \in \partial R$, $y \in R$. In the case $n = 2$ (2.1) is easily obtained using conformal mapping and the maximum principle. We shall consider now $n \geq 3$ and at first prove (2.1) for $\lambda = 1$ since this case can be treated in a somewhat simpler manner.

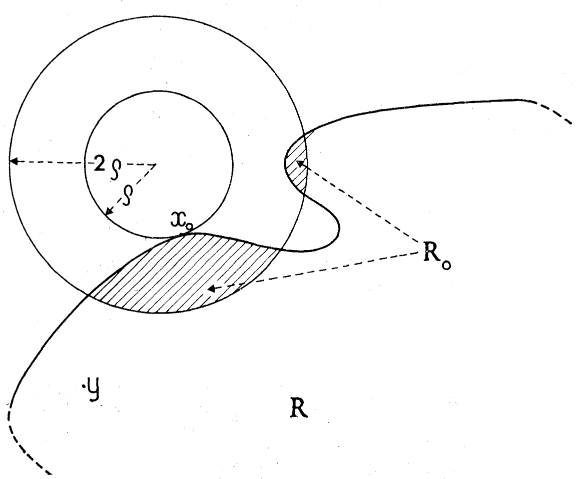

Fig. 1.

(a) $\lambda = 1$. Consider an arbitrary but fixed $y \in R$ and arbitrary $x_0 \in \partial R$. We choose ε such that $0 < \varepsilon < 1/2$ and such that it is possible to construct the sphere S of radius $\rho = \varepsilon r_{x_0 y}$ outside R, tangent to ∂R at x_0. Because ∂R is a Liapunov boundary and $\lambda = 1$ this can always be done. Since R is

(1) Because of the finiteness of R the conditions of Liapunov are not independent. We state them here in the usual manner (c. f. [6]) for convenience.

finite ε may be chosen independent of x_0 and y. For the moment we take the center of this sphere to be the origin. Let r be the distance from the origin to a point x. Let $R_0 = R \cap [x \mid r < 2\rho]$, $C_0 = \partial R \cap [x \mid r < 2\rho]$ and $C_1 = R \cap [x \mid r = 2\rho]$ (see fig. 1).

Now consider the function of x in R_0 defined by

$$w(x) = \max_{z \in C_1} |G(z, y)| \frac{\rho^{-n} - r^{-n}}{\rho^{-n} - (2\rho)^{-n}}. \tag{2.2}$$

This function has the following properties:

1. $w(x_0) = 0$
2. $w(x) \geq 0$, $x \in R_0$
3. $w(x) = \max_{z \in C_1} |G(z, y)|$, $x \in C_1$
4. $\Delta w \leq 0$ in R_0.

(Note that w is just a local barrier at x_0, cf. Kellogg [7]).

Thus since $G(x, y) = 0$ for $x \in C_0$, it follows from the maximum principle that

$$G(x, y) \leq w(x) \quad , \quad x \in R_0. \tag{2.3}$$

But since $G(x_0, y) = w(x_0) = 0$ we conclude that

$$\left| \frac{\partial G(x_0, y)}{\partial n_x} \right| \leq \left| \frac{\partial w(x_0)}{\partial n} \right|. \tag{2.4}$$

Now we can calculate $\frac{\partial w(x_0)}{\partial n}$ as

$$\frac{\partial w(x_0)}{\partial n} = -\frac{\partial w(x_0)}{\partial r} = -\frac{n}{\rho}\left(\frac{2^n}{2^n - 1}\right) \max_{z \in C_1} |G(z, y)|. \tag{2.5}$$

It is well known that

$$0 \leq G(z, y) \leq \frac{1}{(n-2)\,\omega_n} r_{zy}^{2-n} \tag{2.6}$$

and hence by the construction

$$\max_{z \in C_1} |G(z, y)| \leq \frac{1}{(n-2)\,\omega_n} (1 - 2\varepsilon)^{2-n} r_{x_0 y}^{2-n}. \tag{2.7}$$

Combining (2.4), (2.5) and (2.7) we obtain

$$\left| \frac{\partial G(x, y)}{\partial n_x} \right| \leq \frac{n}{\varepsilon(n-2)\,\omega_n} (1 - 2\varepsilon)^{2-n} \left(\frac{2^n}{2^n - 1}\right) r_{xy}^{1-n} \tag{2.8}$$

for any $x \in \partial R$ and $y \in R$.

The same method in two dimensions would yield

$$\left| \frac{\partial G(x, y)}{\partial n_x} \right| \leq C r_{xy}^{-1} |\ln r_{xy}| \tag{2.9}$$

which is not optimal with respect to r_{xy}.

Note that we could have introduced another parameter $\delta > 1$ into the construction and restricted ε so that $0 < \varepsilon < 1/\delta_0 = \min\ [1/\delta\ , d/2$ (diameter of R)] and defined R_0 as $R \cap [x \mid r < \delta_0\, \rho]$, etc. We could obtain better constants by optimizing first with respect to ε and then with respect δ_0. An interesting case is that in which $S \cap R = \varnothing$ even if $\rho >$ (diameter of R) as would be true for example for convex domains. The optimum value of the constant of (2.8) then turns out to be $\frac{1}{\omega_n}\left(\frac{n-1}{n-2}\right)^{n-1}(n+1)^{n/1}\,\frac{n+1}{n}$. For $n = 3$ we obtain $4^{4/3}/3\,\pi$.

(b) $0 < \lambda < 1$. For simplicity and clarity we shall consider $n = 3$. The case of general n is completely analogous.

We first observe that ∂R will be a Liapunov surface for any positive $\lambda' \leq \lambda$ so that we can take, without loss, $\lambda = \frac{1}{2m-1}$, where m is a positive integer. Also d may be chosen small.

Consider an arbitrary point $P \in \partial R$ and take it to be the origin. Let the positive z–axis be along the outward normal and the (x, y) plane be tangent to ∂R at $(0, 0, 0)$. Inside $x^2 + y^2 + z^2 \leq d$ we may represent ∂R as $z = \varphi(x, y)$. As is shown in Smirnov [10] p. 571, if $C d^{\lambda} \leq 1$ then

$$|\varphi(x, y)| \leq c\,(x^2 + y^2)^{(1+\lambda)/2}. \tag{2.10}$$

We consider for a positive number $z_0 \leq d/2 < 1$ the "sphere"

$$(z - z_0)^2 + x^{1+\lambda} + y^{1+\lambda} = z_0^2. \tag{2.11}$$

The set $S : (z - z_0)^2 + x^{1+\lambda} + y^{1+\lambda} \leq z_0^2$ lies inside the Liapunov sphere. Now we have for $(x, y, z) \in S$, $z < z_0$

$$\begin{aligned} |\varphi(x, y)| &\leq C\,(x^{1+\lambda} + y^{1+\lambda}) \leq C\,[z_0^2 - (z - z_0)^2] \\ &= C\,[2\,z_0\,z - z^2] \leq 2\,C\,z_0\,z. \end{aligned} \tag{2.12}$$

Thus if $z_0 < 1/2\,C$ we have

$$|\varphi(x, y)| \leq z \tag{2.13}$$

which means that $S \cap R$ is empty.

We now are in a position to construct a barrier at $(0, 0, 0)$ which exhibits the proper behaviour. For this purpose we consider the function

$$f(x, y, z) = (z - z_0)^2 + x^{1+\lambda} + y^{1+\lambda} - z_0^2. \tag{2.14}$$

Now $f \in C^{1,\lambda}$ in $\bar{R}$ but does not have second derivatives everywhere in R. But $\int_{\Sigma} \Delta f\, dV$ exists and is positive for every subdomain $\Omega \subset R$ so that f is a subharmonic function in R. Hence $f \leq h$, where h is the harmonic function taking the values f on ∂R. By a theorem in Günter [6], Satz 1 p. 212, since ∂R is a Liapunov boundary and $f \in C^{1,\lambda}(\bar{R})$, it follows that h has continuous first derivatives in $\bar{R}$.

We now consider Q to be an arbitrary point of R and let r_Q be the distance from Q to the origin. Let R_0 be the intersection of R with $(z-z_0)^2+x^{1+\lambda}+y^{1+\lambda}\leq(z_0+\varepsilon r_Q)^2$, where $\varepsilon>0$ is to be chosen. Let C_1 be the intersection of R with $(z-z_0)^2+x^{1+\lambda}+y^{1+\lambda}=(z_0+\varepsilon r_Q)^2$. It is not difficult to see that (diameter of R_0) $\leq C\,\varepsilon r_Q$ so that ε can be chosen so that R_0 lies in a sphere about the origin of radius δr_Q with $\delta<1$.

Now let

$$w(x,y,z)=\frac{h(x,y,z)\sup_{T\in C_1}|G(T,Q)|}{(z_0+\varepsilon r_Q)^2-z_0^2} \tag{2.15}$$

in $\overline{R}_0$. The function w satisfies

$$\Delta w=0 \quad \text{in } R_0$$
$$w(0,0,0)=0$$
$$w\geq 0 \quad \text{in } R_0$$

and on C_1

$$w\geq\frac{f\sup_{T\in C_1}|G(T,Q)|}{(z_0+\varepsilon r_Q)^2-z_0^2}\geq\sup_{T\in C_1}|G(T,Q)|.$$

Thus it follows from the maximum principle that

$$G(P,Q)\leq w(P) \tag{2.16}$$

for $P\in R_0$. Since $G(0,Q)=w(0)=0$ we have

$$\left|\frac{\partial G(0,Q)}{\partial n_p}\right|\leq\left|\frac{\partial w(0)}{\partial n}\right|=\frac{\sup_{T\in C_1}|G(T,Q)|}{(z_0+\varepsilon r_Q)^2-z_0^2}\left|\frac{\partial h}{\partial n}\right|. \tag{2.17}$$

But $\sup_{T\in C_1}|G(T,Q)|\leq C/r_Q$, $(z_0+\varepsilon r_Q)^2-z_0^2\geq C r_Q$ and $\left|\frac{\partial h(0)}{\partial n}\right|\leq C$, so that finally,

$$\left|\frac{\partial G(0,Q)}{\partial n_p}\right|\leq C/r_Q^2 \tag{2.18}$$

where the constant in (2.18) depends only on R.

By exactly the same types of arguments as in (a) and (b) we can establish

$$\left|\frac{\partial^2 G(x,y)}{\partial n_x\,\partial n_y}\right|\leq C r_{xy}^{-n} \tag{2.19}$$

where for example, in case (a) we use (2.8) in place of (2.6).

III. Interior estimates.—In this section we make use of the following observation. Let the origin O be an aribitrary point of R and h be a function, harmonic in R except at the origin. Then

$$\Delta(r^{n-2}h)-2(n-2)\,x_i\,r^{-2}(r^{n-2}h)_{,i}=0 \tag{3.1}$$

in R — O, where here the notation, i means partial differentiation with respect to x_i, and summation from 1 to n over a repeated index is understood. The symbol Δ means, of course, Laplace's operator. Now from the maximum principle for second order elliptic equations it follows that

$$(3.2) \qquad |r^{n-2} h| \leq \max [\sup_{x \in \partial R} |r^{n-2} h(x)|, \limsup_{x \to 0} |r^{n-2} h(x)|].$$

Since the origin was taken to be an arbitrary point, we can assume, without loss, that the point y is the origin. Let us consider

$$(3.3) \qquad h(x) = x_i G_{,i}(x, 0),$$

and verify by calculation that $\Delta h = 0$ if $x \neq 0$. Thus using the definition of $G(x, y)$, and the fact that $G(x, 0)$ is of class C^1 in $R \cup \partial R$ except at the origin (cf. Günter [5]),

$$(3.4) \qquad |r^{n-2} x_i G_{,i}(x, 0)| \leq \max \left[\sup_{x \in \partial R} \left| r^{n-2} x_i n_i \frac{\partial G(x, 0)}{\partial n} \right|, \frac{1}{\omega_n} \right],$$

where n_i is the component of the outward unit normal in the direction x_i. Noting that $|x_i n_i / r| \leq 1$ and using (2.8) (or 2.18) we have

$$(3.5) \qquad |r^{n-2} x_i G_{,i}(x, 0)| \leq C,$$

where C is a constant depending only on R. In fact C can be calculated from (2.8) (or (2.18)) and (3.4). In exactly the same manner we take

$$(3.6) \qquad h(x) = x_i G_{,j}(x, 0) - x_j G_{,i}(x, 0)$$

for arbitrary but fixed i and j. As before we observe that $\Delta h = 0$ in $R - O$ and conclude finally that

$$(3.7) \qquad |r^{n-2}(x_i G_{,j}(x, 0) - x_j G_{,i}(x, 0)| \leq C.$$

We now note that

$$(3.8) \quad r^{n-1} G_{,j}(x, 0) = \frac{x_j}{r} r^{n-2} x_i G_{,i}(x, 0) + \frac{x_i}{r} r^{n-2}(x_i G_{,j}(x, 0) - x_j G_{,i}(x, 0)).$$

It follows at once from (3.5), (3.7) and (3.8) that

$$(3.9) \qquad |r^{n-1} G_{,j}(x, 0)| \leq C$$

where C is another constant. Since the origin was chosen arbitrarily we have the result

$$(3.10) \qquad \left| \frac{\partial G(x, y)}{\partial x_i} \right| \leq C r_{xy}^{1-n}.$$

IV. THE DIRICHLET INTEGRAL.—Let u be a harmonic function in R which is Hölder continuous with exponent $\alpha > 1/2$ in $R \cup \partial R$. The following expression for the Dirichlet integral, $D(u, u) = \int_R u_{,i} u_{,i} \, dx$, is easily derived

$$(4.1) \qquad D(u, u) = \int_{\partial R(x)} \int_{\partial R(y)} [u(x) - u(y)]^2 \frac{\partial^2 G(x, y)}{\partial n_x \partial n_y} dS_x \, dS_y$$

(cf. Slobodetski and Babich [9]). The inequality (2.19) thus leads to

$$D(u, u) \leq c \int_{\partial R(x)} \int_{\partial R(y)} [u(x) - u(y)]^2 \, r_{xy}^{-n} \, dS_x \, dS_y . \tag{4.2}$$

Such an inequality can be used to provide error bounds for the approximation of the Dirichlet integral of one harmonic function in terms of another. Since the inequality is quadratic the Rayleigh-Ritz technique can be applied to systematically improve the approximations (cf. Bramble and Payne [1]).

BIBLIOGRAPHY.

[1] BRAMBLE J. H. and PAYNE L. E., *Bounds for solutions of second order elliptic partial differential equations*, «Contrib. to Diff. Eq.», *1*, 95–127 (1963).

[2] DE VITO L., *Sulle funzioni ad integrale di Dirichlet finito*, «Annali della Scuola Norm. Sup. di Pisa», ser. III, *12*, fasc. I–II, 55–127 (1958).

[3] EIDUS D. M., *Estimations of the derivatives of Green's function (Ocenki proizwodnyh funkcii Grina)*, «Doklady Akad. Nauk. SSSR», *106*, No. 1, 207–209 (1956).

[4] FICHERA G., *Estimation for the second derivatives of Green's function of a two dimensional domain*, manuscript (1966).

[5] GIRAUD G., *Généralisation des problèmes sur les operations du type elliptique*, «Bull. Sc. Mat.», *56*, 338–340 (1932).

[6] GÜNTER N. M., *Die Potentialtheorie*, Teubner, Leipzig (1957).

[7] KELLOGG O. D., *Foundations of Potential Theory*, Springer, Berlin (1929), also Dover, N. Y. (1953).

[8] MIRANDA C., *Sulla sommabilità delle derivate di una funzione armonica hölderiana*, «Rend. Acc. Sc. Fis. e Matem. di Napoli», ser. IV, *18*, 96–98 (1948).

[9] SLOBODETSKIJ L. N. and BABICH V. M., *On bounds for the Dirichlet integral*, «Doklady Akad. Nauk. S.S.S.R.», *106*, No. 9 (1956).

[10] SMIRNOV V. I., *A course of higher mathematics*, IV, Pergamon Press (1964).