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Ezio Marchi

Remarks on generalized n-person games

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NOTE PRESENTATE DA SOCI

Teoria dei giochi. — *Remarks on generalized n–person games* (*). Nota di EZIO MARCHI, presentata (**) dal Socio B. SEGRE.

SUNTO. — Utilizzando la rappresentazione euclidea di un gioco generalizzato fra un qualunque numero di persone ed il teorema del punto fisso di Eilenberg Montgomery [2], si assegnano condizioni per l'esistenza di punti di equilibrio. Quelli di Debreu [1] sono un caso particolare.

1. An *e–generalized game* is given by a generalized n–person game

$$\Gamma = \{\Sigma_1, \cdots, \Sigma_n \quad ; \quad A_1, \cdots, A_n \quad ; \quad Z_1, \cdots, Z_n\}$$

where for each player $i \in N = (1, \cdots, n)$, Σ_i is the strategy set, which is assumed to be in an Euclidean space R^{m_i}, where the payoff A_i is a function defined on $\Sigma_N = \underset{j \in N}{X} \Sigma_j$, with values in the completed real line $\bar{R}$, and the multivalued function $Z_i : \Sigma_N \to \Sigma_i$ determine a non–empty compact set $Z_i(\sigma)$ for each strategy $\sigma \in \Sigma_N$ and where a multivalued function $e : N \to N$ is defined such that $e(i) \subseteq N - (i)$ for all $i \in N$. Such a game will be denoted by

$$\Gamma_e = \{\Sigma_1, \cdots, \Sigma_n \quad ; \quad A_1, \cdots, A_n \quad ; \quad Z_1, \cdots, Z_n\}.$$

For each player $i \in N$ and each strategy $\sigma \in \Sigma_N$, $f(i)$ denoted the set $N - (e(i) \cup (i))$, and

$$V_i(\sigma) = \max_{s_i \in Z_i(\sigma)} \; \min_{s_{e(i)} \in Z_{e(i)}(\sigma)} A_i(s_i, s_{e(i)}, \sigma_{f(i)})$$

and

$$V^i(\sigma) = \min_{s_{e(i)} \in Z_{e(i)}(\sigma)} \; \max_{s_i \in Z_i(\sigma)} A_i(s_i, s_{e(i)}, \sigma_{f(i)}).$$

We denote here $S_R \in Z_R = \underset{j \in R}{X} \Sigma_i$ and $Z_R = \underset{j \in R}{X} Z_j$, where R is $e(i)$ or $f(i)$.

A strategy $\bar{\sigma} \in \Sigma_N$ is an e_m–simple point of the e–generalized n–person game Γ_e if:

(i)
$$\min_{s_{e(i)} \in Z_{e(i)}(\bar{\sigma})} A_i(\bar{\sigma}_i, s_{e(i)}, \bar{\sigma}_{f(i)}) = V_i(\bar{\sigma})$$

and

$$\bar{\sigma}_i \in Z_i(\bar{\sigma}) \qquad \text{for all} \quad i \in N.$$

It is an e^m–simple point of the e–generalized n–person game Γ_e if:

(ii)
$$\max_{s_i \in Z_i(\bar{\sigma})} A_i(s_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) = V^i(\bar{\sigma})$$

and

$$\bar{\sigma}_i \in Z_i(\bar{\sigma}) \qquad \text{for all} \quad i \in N.$$

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(**) Nella seduta dell'8 aprile 1967.

The e-generalized n-person game is on the one hand an extension of the generalized n-person game due to Debreu [1], and on the other hand a generalization of the game with a simple structure function introduced in [3]. If for each $i \in N : e(i) = \varnothing$ and Z_i is independent of $\sigma_i \in \Sigma_i$, then an e_m-simple point is an equilibrium point in the sense of Debreu.

The results presented here give the general conditions under which the existence of such points is guaranteed.

As in [1], we prove the main result using as a lemma a particular case of the fixed point theorem of Eilenberg and Montgomery [2].

LEMMA: *Let W be a contractible polyhedron and* α: W $\rightarrow$ W *an upper semicontinuous multivalued function such that for every* $w \in W$ *the set* $\alpha(w)$ *is contractible. Then* α *has a fixed point.*

We recall that a polyhedron is a set in a Euclidean space R^m homeomorphic to the union of a finite number of simplexes.

A non-empty set W in a Euclidean space R^m is said to be contractible, or more precisely, deformable into a point $\bar{w} \in W$, if there exists a continuous function $\alpha : [0,1] \times W \rightarrow W$ such that $\alpha(0,w) = w$ and $\alpha(1,w) = \bar{w}$ for all $w \in W$.

A multivalued function $\alpha : W_1 \rightarrow W_2$, where W_1 and W_2 are subsets in Euclidean spaces, is said to be upper semicontinuous if the graph of the function $\alpha : G_\alpha = \{(\sigma, \tau) : \tau \in \alpha(\sigma)\}$ is closed. It is said to be lower semicontinuous if for any $\tau \in \alpha(\sigma)$ and any convergent sequence $\sigma(k) \rightarrow \sigma$ there is a convergent sequence $\tau(k) \rightarrow \tau$ such that for all $k : \tau(k) \in \alpha(\sigma(k))$. The multivalued function α is continuous if it is upper and lower semicontinuous. A fixed point of the multivalued function $\alpha : W \rightarrow W$ is a point $\bar{w}$ such that $\bar{w} \in \alpha(\bar{w})$.

2. THEOREM: *Let* Γ_e *be an e-generalized game such that for all* $i \in N$, Σ_i *is contractible polyhedron, the multivalued function* Z_i *is upper semicontinuous.*

(a) *An* e_m*-simple point of* Γ_e *exists, if: for all* $i \in N$

(a1) *the functions* V_i *in* $\sigma \in \Sigma_N$ *and*

$$\min_{s_{e(i)} \in Z_{e(i)}(\sigma)} A_i(\tau_i, s_{e(i)}, \sigma_{f(i)}) \qquad \textit{in} \quad (\sigma, \tau_i) \in G_i$$

are continuous and

(a2) *for all* $\sigma \in \Sigma_N$ *the set*

$$S_i(\sigma) = \{\tau_i \in Z_i(\sigma) : \min_{s_{e(i)} \in Z_{e(i)}(\sigma)} A_i(\tau_i, s_{e(i)}, \sigma_{f(i)}) = V_i(\sigma)\}$$

is contractible.

(b) *An* e^m*-simple point of* Γ_e *exists, if: for all* $i \in N$

(b1) *the functions* V^i *in* $\sigma \in \Sigma_N$ *and* $\max_{s_i \in Z_i(\sigma)} A_i(s_i, \tau_{e(i)}, \sigma_{f(i)})$ in $(\sigma, \tau_{e(i)}) \in G_{e(i)}$ *are continuous where* $G_{e(i)}$ *is the graph of* $Z_{e(i)}$, *and*

(b2) *for all* $\sigma \in \Sigma_N$ *the set*

$$T(\sigma) = \{\tau \in Z_N(\sigma) : \max_{s_i \in Z_i(\sigma)} A_i(s_i, \tau_{e(i)} \sigma_{e(i)}) = V^i(\sigma) \qquad \text{for all} \quad i \in N\}$$

is contractible.

(c) *An e_m-and e^m-point of Γ_e exists, if it is satisfied* (a1), (a2), (b1), (b2) *and*

(c1) *for all* $\sigma \in \Sigma$ *the set* $U(\sigma) = S(\sigma) \cap T(\sigma)$ *is contractible, where* $S(\sigma) = \{\tau \in \Sigma_N : \tau_i \in S_i(\sigma) \text{ for all } i \in N\}$.

Proof: Σ_N is a contractible polyhedron, since it is the product of a finite number of contractible polyhedra.

(a) Let $\psi = \underset{i \in N}{X} S_i : \Sigma_N \to \Sigma_N$: be a multivalued function. $\psi(\sigma)$ is contractible for all $\sigma \in \Sigma_N$, because the set $S_i(\sigma) \subseteq \Sigma_i$ is contractible for all $i \in N$ and $\sigma \in \Sigma_N$. For each $i \in N$, the set

$$S_i = \{(\sigma, \tau_i) \in \Sigma_N \times \Sigma_i : \tau_i \in S_i(\sigma)\} =$$
$$= \{(\sigma, \tau_i) \in G_i : \min_{s_{e(i)} \in Z_{e(i)}(\sigma)} A_i(\tau_i, s_{e(i)}, \sigma_{f(i)}) = V_i(\sigma)\}$$

is closed, since G_i is closed and the functions $V_i(\sigma)$ and

$$\min_{s_{e(i)} \in Z_{e(i)}(\sigma)} A_i(\tau_i, s_{e(i)}, \sigma_{f(i)})$$

are continuous. Therefore the graph of the function ψ, which is

$$G_\psi = \{(\sigma, \tau) \in \Sigma_N \times \Sigma_N : \tau \in \psi(\sigma)\} =$$
$$= \{(\sigma, \tau) \in \Sigma_N \times \Sigma_N : \tau \in S_i(\sigma) \quad \text{for all } i \in N\} =$$
$$= \{(\sigma, \tau) \in \Sigma_N \times \Sigma_N : (\sigma, \tau_i) \in S_i \quad \text{for all } i \in N\} =$$
$$= \bigcap_{i \in N} \{(\sigma, \tau) \in \Sigma_N \times \Sigma_N : (\sigma, \tau_i) \in S_i\},$$

is closed, since S_i is. The above lemma, applied to the function ψ, gives a fixed point $\bar{\sigma} \in \Sigma_N : \bar{\sigma} \in \psi(\bar{\sigma})$. Such a strategy is an e_m-simple point.

(b) Let $\psi = T : \Sigma_N \to \Sigma_N$ be a multivalued function, such that $\psi(\sigma)$ is contractible for all $\sigma \in \Sigma_N$.

$$G_{e(i)} = \{(\sigma, \tau_{e(i)}) \in \Sigma_N \times \Sigma_{e(i)} : (\sigma, \tau_j) \in G_j \quad \text{for all } j \in e(i)\},$$

which is the graph of the function $Z_{e(i)}$ is closed for all $i \in N$, since G_j is. By the same argument the graph G_N of the multivalued function Z_N is closed. Thus the graph of the function ψ,

$$G_\psi = \{(\sigma, \tau) \in \Sigma_N \times \Sigma_N : \tau \in \psi(\sigma)\} =$$
$$= \{(\sigma, \tau) \in \Sigma_N \times \Sigma_N : \tau \in Z_N(\sigma) \quad \text{and} \quad \max_{s_i \in Z_i(\sigma)} A_i(s_i, \tau_{e(i)}, \sigma_{f(i)}) = V^i(\sigma)$$
$$\text{for all } i \in N)\} =$$
$$= G_N \cap \bigcap_{i \in N} \{(\sigma, \tau) \in \Sigma_N \times \Sigma_N : (\sigma, \tau_{e(i)}) \in G_{e(i)} \text{ and } \max_{s_i \in Z_i(\sigma)} A_i(s_i, \tau_{e(i)}, \sigma_{f(i)}) = V^i(\sigma)\}$$

is closed since for all $i \in N$, the set $G_{e(i)}$ and G_N are also closed, and the functions $V^i(\sigma)$ and

$$\max_{s_i \in Z_i(\sigma)} A_i(s_i, \tau_{e(i)}, \sigma_{f(i)})$$

are continuous. The lemma applied to the function ψ, gives again a fixed point $\bar{\sigma} \in \Sigma_N$; $\bar{\sigma} \in \psi(\bar{\sigma})$. Such a strategy is an e^m–simple point.

(c) Let $\psi = U : \Sigma_N \to \Sigma_N$ be a multivalued function. For each $\sigma \in \Sigma_N$, the set $\psi(\sigma) \subseteq \Sigma_N$ is contractible. The graph of the function ψ,

$$G_\psi = \{(\sigma, \tau) \in \Sigma_N \times \Sigma_N : \tau \in \psi(\sigma)\} =$$

$$= \bigcap_{i \in N} [\{(\sigma, \tau) \in \Sigma_N \times \Sigma_N : (\sigma, \tau_i) \in G_i \text{ and } \min_{s_{e(i)} \in Z_{e(i)}(\sigma)} A_i(\tau_i, s_{e(i)}, \sigma_{f(i)}) = V_i(\sigma)\} \cap$$

$$\cap \{(\sigma, \tau) \in \Sigma_N \times \Sigma_N : (\sigma, \tau_{e(i)}) \in G_{e(i)} \text{ and } \max_{s_i \in Z_i(\sigma)} A_i(s_i, \tau_{e(i)}, \sigma_{f(i)}) = V^i(\sigma)\}]$$

is closed, since G_i, $G_{e(i)}$ are closed and the functions $V_i(\sigma)$, $V^i(\sigma)$,

$$\min_{s_{e(i)} \in Z_{e(i)}(\sigma)} A_i(\tau_i, s_{e(i)}, \sigma_{f(i)}) \quad \text{and} \quad \max_{s_i \in Z_i(\sigma)} A_i(s_i, \tau_{e(i)}, \sigma_{f(i)})$$

are continuous on G_i and $G_{e(i)}$ respectivly. Finally, the lemma applied to the function ψ, guaranteed a fixed strategy $\bar{\sigma} \in \psi(\bar{\sigma})$, which is e_m–and e^m–simple point. Q.E.D.

The continuity of the functions involved in the above results is related with the requirements on the functions A_i and Z_i.

LEMMA: *Let us suppose that for each* $i \in N$, *the graph* G_i *of the multivalued function* Z_i *is compact, the function* $A_i : \Sigma_N \to \bar{R}$ *and the multivalued function* Z_i *are continuous. Then for each* $i \in N$, *the functions*

$$B_i(\tau_i, \sigma) = \min_{s_{e(i)} \in Z_{e(i)}(\sigma)} A_i(\tau, s_{e(i)}, \sigma_{f(i)})$$

are continuous in $(\sigma, \tau_i) \in G_i$,

$$C_i(\tau_{e(i)}, \sigma) = \max_{s_i \in Z_i(\sigma)} A_i(s_i, \tau_{e(i)}, \sigma_{f(i)})$$

are continuous in $(\sigma, \tau_{e(i)}) \in G_{e(i)}$ *and* V_i *and* V^i *are continuous in* $\sigma \in \Sigma_N$.

Proof: The multivalued function $Z_{e(i)}$ has a compact graph $G_{e(i)}$, since for all $j \in e(i)$ the multivalued function Z_j has a compact graph G_j. Let $(\sigma(k), \tau_i(k)) \to (\bar{\sigma}, \bar{\tau}_i)$ be a convergent sequence in G_i. Since $Z_{e(i)}(\sigma(k))$ is compact $s_{e(i)}(k) \in Z_{e(i)}(\sigma(k))$ can be chosen such that

$$A_i(\tau_i(k), s_{e(i)}(k), \sigma_{f(i)}(k)) = B_i(\tau_i(k), \sigma(k));$$

by the compactness of the graph $G_{e(i)}$ it is possible to extract from the sequence $(\tau_i(k), s_{e(i)}(k), \sigma(k))$ a convergent subsequence $(\tau_i(k'), s_{e(i)}(k'), \sigma(k')) \to (\bar{\tau}, \bar{s}_{e(i)}, \bar{\sigma})$. By the continuity of A_i,

$$A_i(\tau_i(k'), s_{e(i)}(k'), \sigma_{f(i)}(k')) \to A_i(\bar{\tau}_i, \bar{s}_{e(i)}, \bar{\sigma}_{f(i)}) \geq B_i(\bar{\tau}_i, \bar{\sigma}).$$

Therefore, for any $\delta > 0$ there exists a m such that

$$B_i(\tau_i(k'), \sigma(k')) > B_i(\bar{\tau}_i, \bar{\sigma}) - \delta \qquad \text{for all} \quad k' > m.$$

Since any sequence $(\sigma(k), \tau_i(k)) \to (\bar{\sigma}, \bar{\tau}_i)$ in G_i has a subsequence $(\sigma(k'), \tau_i(k')) \to (\bar{\sigma}, \bar{\tau}_i)$ with the mentioned property any sequence $(\sigma(k), \tau_i(k)) \to \to (\bar{\sigma}, \bar{\tau}_i)$ in G_i has the property.

Since $Z_{e(i)}(\bar{\sigma})$ is compact

$$s_{e(i)} \in Z_{e(i)}(\bar{\sigma})$$

can be chosen such that $A_i(\bar{\tau}_i, s_{e(i)}, \bar{\sigma}_{f(i)}) = B_i(\bar{\tau}_i, \bar{\sigma})$. By the lower–continuity of the function $Z_{e(i)}$ a sequence $s_{e(i)}(k) \to \bar{s}_{e(i)}$ exists such that $s_{e(i)}(k) \in Z_{e(i)}(\sigma(k))$ for all k. By continuity of the function A_i,

$$B_i(\tau_i(k), \sigma(k)) \leq A_i(\tau_i(k), s_{e(i)}(k), \sigma_{f(i)}(k)) \to A_i(\bar{\tau}_i, \bar{s}_{e(i)}, \bar{\sigma}_{f(i)}).$$

Therefore, for any $\delta > 0$ an m exists such that

$$B_i(\tau_i(k), \sigma(k)) < B_i(\bar{\tau}_i, \bar{\sigma}) + \delta \qquad \text{for all} \quad k > m.$$

The function B_i is continuous at $(\bar{\tau}_i, \bar{\sigma}) \in G_i$.

The continuity of the functions C_i, B_i and V^i can be obtained in the same way. Q.E.D.

The above result implies:

COROLLARY: *Let Γ_e be an e–generalized game such that for all $i \in N$, Σ_i is a contractible polyhedron, the function A_i and the multivalued function Z_i are continuous.*

(a) *A e_m–simple point of Γ_e exists, if for all $i \in N$ and all $\sigma \in \Sigma_N$ the set $S_i(\sigma)$ is contractible.*

(b) *A e^m–simple point of Γ_e exists, if for all $i \in N$ and all $\sigma \in \Sigma_N$ the set $T(\sigma)$ is contractible.*

(c) *A e_m–and e^m–simple point $\bar{\sigma} \in \Sigma_N$ of Γ_e such that $V_i(\bar{\sigma}) \leq \leq A_i(\bar{\sigma}_i, \bar{\sigma}_{e(i)}, \bar{\sigma}_{f(i)}) \leq V^i(\bar{\sigma})$ for all $i \in N$, exists if for all $i \in N$ and all $\sigma \in \Sigma_N$ the set $U(\sigma)$ is contractible.*

A very particular case arises when for each $i \in N$ and each $\sigma \in \Sigma_N$: $Z_i(\sigma) = \Sigma_i$ is convex, the function $B_i(\tau_i, \sigma) = B_i(\tau_i, \sigma_{f(i)})$ is quasi–concave in $\tau_i \in \Sigma_i$ for fixed $\sigma_{f(i)} \in \Sigma_{f(i)}$ and the function $C_i(\tau_{e(i)}, \sigma) = = C_i(\tau_{e(i)}, \sigma_{f(i)})$ is quasi–convex in $\tau_{e(i)} \in \Sigma_{e(i)}$ for fixed $\sigma_{f(i)} \in \Sigma_{f(i)}$. Applying these conditions in the previous corollary a generalization of the main results given in [3] is obtained.

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