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The Lanczos Identity and its Generalizations

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Matematica. — *The Lanczos Identity and its Generalizations.* Nota di DAVID LOVELOCK, presentata [(*)] dal Socio B. SEGRE.

SUNTO. — C. Lanczos ha stabilito in [1] una notevole relazione quadratica per le componenti del tensore di curvatura di una varietà riemanniana a quattro dimensioni; ma la sua dimostrazione non è estendibile alle varietà di dimensione qualsiasi. Nella presente Nota una via per ottenere una siffatta estensione viene indicata coll'uso opportuno del tensore di curvatura conforme di H. Weyl; ciò fornisce, accanto ad una diversa dimostrazione e formulazione dell'identità di Lanczos, una successione di nuove identità dipendenti in modo essenziale dalla dimensione dello spazio, la complessità delle quali cresce colla dimensione. Come applicazione, si ricava una delle condizioni di G. Y. Rainich [5] per il tensore del momento di energia elettromagnetica.

1. INTRODUCTION.—Our considerations are based on an n-dimensional Riemannian space with line element [(1)]

$$ds^2 = g_{ij}\, dx^i\, dx^j .$$

The curvature tensor is introduced by means of the commutation relations

$$X^j{}_{;kl} - X^j{}_{;lk} = R_i{}^j{}_{kl}\, X^i ,$$

where a semi-colon denotes partial covariant differentiation and X^i is any contravariant vector field. We shall define the Ricci tensor and the curvature scalar by

$$R_{ij} = R_i{}^h{}_{jh} ,$$

and

$$R = g^{ij} R_{ij} ,$$

respectively. The Weyl conformal curvature tensor is defined by

$$C_j{}^l{}_{im} = R_j{}^l{}_{im} + \frac{1}{n-2}\left(R^l{}_i\, g_{jm} + R_{jm}\, \delta^l_i - R_{ij}\, \delta^l_m - R^l_m\, g_{ij}\right) + \tag{1.1}$$
$$+ \frac{R}{(n-1)(n-2)}\left(g_{ij}\, \delta^l_m - g_{jm}\, \delta^l_i\right),$$

and enjoys all the symmetry properties of the curvature tensor together with

$$C_j{}^l{}_{il} = 0 . \tag{1.2}$$

We shall always raise and lower indices by means of the contravariant and covariant metric tensors, respectively.

(*) Nella seduta dell'11 febbraio 1967.

(1) The summation convention is used throughout this paper.

Lanczos [2] has shown that if $n = 4$ then

$$(1.3) \qquad 2\,R_{jh}\,R^{ih} + 2\,R^{hk}\,R_{h\ kj}^{\ i} - RR^{i}_{j} - R_{hjlk}\,R^{hilk} = \\ = \frac{1}{4}\,(4\,R_{lh}\,R^{lh} - R^2 - R_{lmkh}\,R^{lmkh})\,\delta^{i}_{j},$$

—a remarkable identity, which henceforth we shall term *the Lanczos identity*. However, the proof suggested by Lanczos depends crucially on the dimension of the space being four and does not admit of any obvious generalizations for arbitrary n, either in the proof or in the form of the identity.

Nevertheless progress can be made since, by using the Weyl conformal curvature tensor (1.1), the identity (1.3) may be expressed in the particularly simple form

$$(1.4) \qquad C^{ihkl}\,C_{jhkl} = \frac{1}{4}\,(C^{rhkl}\,C_{rhkl})\,\delta^{i}_{j}.$$

It is the generalizations of this form of the identity which we shall investigate for arbitrary n. In the next section identities between certain tensors involving an integral parameter m are established. In the final section it is shown that by imposing restrictions on the dimensionality of the space generalizations of (1.4) are obtainable. The evenness and oddness of dimensionality appears to play an unexpectedly important role.

In order that our results be as general as possible, we shall consider a tensor K^{ij}_{kl} which is anti-symmetric in (i, j) and (k, l) respectively, i.e.

$$(1.5) \qquad K^{ij}_{kl} = -K^{ij}_{lk} = -K^{ji}_{kl},$$

and which, furthermore, enjoys the following property

$$(1.6) \qquad K^{ij}_{jl} = 0.$$

An example of a tensor which satisfies (1.5) and (1.6) is the Weyl conformal curvature tensor C^{ij}_{kl}. We shall frequently use the properties (1.5) and (1.6) without explicit reference to them.

It is easily seen that the number of independent components of K^{ij}_{kl} is

$$\left[\frac{n(n-1)}{2}\right]^2 - n^2 = \frac{n^2}{4}\,(n-3)\,(n+1).$$

We thus have

LEMMA 1: *If* $n = 3$ *then* $K^{ij}_{kl} = 0$.

In particular of course, this contains the well-known result due to Weyl [3] according to which the Weyl conformal curvature tensor vanishes for $n = 3$.

(2) In the notation of Lanczos, the vanishing of S^{i}_{j} (defined in [1], p. 847, equation (4.10)) is equivalent to (1.3) above.

(3) See, for example [3], p. 306.

2. IDENTITIES INVOLVING K^{ij}_{kl}.—In this section we shall derive certain properties associated with the three tensors $T^{i}_{.j}[m]$, $S^{i}_{.j}[m]$ and $P^{i}_{.j}[m]$ defined for m an integer, $m \geq 1$, by

$$(2.1)\qquad T^{i}_{.j}[m] \equiv \delta^{i_1 i_2 \cdots i_{2m-1} i}_{j_1 j_2 \cdots j_{2m-1} j_{2m}} K^{j_1 j_2}_{\quad i_1 i_2} K^{j_3 j_4}_{\quad i_3 i_4} \cdots K^{j_{2m-1} j_{2m}}_{\qquad i_{2m-1} j},$$

$$(2.2)\qquad S^{i}_{.j}[m] \equiv \begin{cases} \delta^{i_1 i_2 \cdots i_{2m-1}}_{j_1 j_2 \cdots j_{2m-1}} K^{j_1 j_2}_{\quad i_1 i_2} \cdots K^{j_{2m-3} i}_{\qquad i_{2m-3} i_{2m-2}} K^{j_{2m-2} j_{2m-1}}_{\qquad i_{2m-1} j} & \text{for } m \geq 2, \\ 0 \quad \text{for } m = 1, \end{cases}$$

and

$$(2.3)\qquad P^{i}_{.j}[m] \equiv \delta^{i_1 i_2 \cdots i_{2m} i}_{j_1 j_2 \cdots j_{2m} j} K^{j_1 j_2}_{\quad i_1 i_2} \cdots K^{j_{2m-1} j_{2m}}_{\qquad i_{2m-1} i_{2m}},$$

respectively, where $\delta^{i_1 \cdots i_N}_{j_1 \cdots j_N}$ is the generalized Kronecker delta ([2], pp. 242 and 278) which may be written in the form

$$(2.4)\qquad \delta^{i_1 \cdots i_N}_{j_1 \cdots j_N} = \det. \begin{vmatrix} \delta^{i_1}_{j_1} & \delta^{i_1}_{j_2} & \cdots & \delta^{i_1}_{j_N} \\ \vdots & & & \\ \delta^{i_N}_{j_1} & \delta^{i_N}_{j_2} & \cdots & \delta^{i_N}_{j_N} \end{vmatrix}.$$

In the sequel it is usually easier to follow the arguments if the right hand side of (2.4) is used explicitly whenever the generalized Kronecker delta occurs. In connection with (2.1), (2.2) and (2.3) we remark that the quantity in square brackets m indicates the degree of linear homogeneity of K^{ij}_{kl} while the determinantal parts of $T^{i}_{.j}[m]$, $S^{i}_{.j}[m]$ and $P^{i}_{.j}[m]$ are respectively $2m \times 2m$, $(2m-1)\times(2m-1)$ and $(2m+1)\times(2m+1)$.

We now establish a relation between (2.1) and (2.2) which we state in the form of

THEOREM 1: *For* $m \geq 1$

$$(2.5)\qquad T^{i}_{.j}[m] = -4(m-1) K^{ki}_{\ hj} T^{h}_{.k}[m-1] + 2(m-1) S^{i}_{.j}[m].$$

Proof: By expanding $T^{i}_{.j}[m]$ about the last row of its determinant, and using (1.6), we find that

$$T^{i}_{.j}[m] = 2\,\delta^{i_1 \cdots i_{2m-1}}_{j_1 \cdots j_{2m-1}} K^{j_1 j_2}_{\quad i_1 i_2} \cdots K^{j_{2m-1} i}_{\qquad i_{2m-1} j} +$$

$$+ 2 \sum_{\mu=1}^{m-1} \delta^{i_1 \cdots\cdots\cdots i_{2m-1}}_{j_1 \cdots j_{2\mu-1} j_{2\mu+1} \cdots j_{2m}} K^{j_1 j_2}_{\quad i_1 i_2} \cdots K^{j_{2\mu-1} i}_{\qquad i_{2\mu-1} i_{2\mu}} \cdots K^{j_{2m-1} j_{2m}}_{\qquad i_{2m-1} j}.$$

By expanding the first term on the right hand side about the last column of its determinant we have

$$(2.6)\quad T^{i}_{\cdot j}[m] = 4\sum_{\mu=1}^{m-1} \delta^{i_1 i_2 \cdots i_{2\mu-2} i_{2\mu} \cdots i_{2m-2} i_{2m-1}}_{j_1 j_2 \cdots\cdots\cdots\cdots\cdots j_{2m-2}} K^{j_1 j_2}_{i_1 i_2} \cdots K^{j_{2\mu-1} j_{2\mu}}_{i_{2\mu-1} i_{2\mu}} \cdots \cdots K^{j_{2m-3} j_{2m-2}}_{i_{2m-3} i_{2m-2}} K^{i_{2\mu-1} i}_{i_{2m-1} j} +$$

$$+ 2\sum_{\mu=1}^{m-1} \delta^{i_1 \cdots i_{2m-1}}_{j_1 \cdots j_{2\mu-1} j_{2\mu+1} \cdots j_{2m}} K^{j_1 j_2}_{i_1 i_2} \cdots K^{j_{2\mu-1} i}_{i_{2\mu-1} i_{2\mu}} \cdots K^{j_{2m-1} j_{2m}}_{i_{2m-1} j} .$$

In each of these sums we change the dummy indices according to the following scheme:

$$i_{2\mu-1} \longleftrightarrow i_{2m-3} ,$$
$$i_{2\mu} \longleftrightarrow i_{2m-2} ,$$
$$j_{2\mu-1} \longleftrightarrow j_{2m-3} ,$$
$$j_{2\mu} \longleftrightarrow j_{2m-2} .$$

Hence (2.6) may be written in the form

$$(2.7)\quad T^{i}_{\cdot j}[m] = 4\sum_{\mu=1}^{m-1} \delta^{i_1 i_2 \cdots i_{2\mu-2} i_{2m-2} i_{2\mu+1} \cdots i_{2m-4} i_{2\mu-1} i_{2\mu} i_{2m-1}}_{j_1 j_2 \cdots j_{2\mu-2} j_{2m-3} j_{2m-2} j_{2\mu+1} \cdots j_{2m-4} j_{2\mu-1} j_{2\mu}} K^{j_1 j_2}_{i_2 i_2} \cdots \cdots K^{j_{2\mu-1} j_{2\mu}}_{i_{2\mu-1} i_{2\mu}} \cdots K^{j_{2m-3} j_{2m-2}}_{i_{2m-3} i_{2m-2}} K^{i_{2m-3} i}_{i_{2m-1} j} +$$

$$+ 2\sum_{\mu=1}^{m-1} \delta^{i_1 i_2 \cdots i_{2\mu-2} i_{2m-3} i_{2m-2} i_{2\mu+1} \cdots i_{2m-4} i_{2\mu-1} i_{2\mu} i_{2m-1}}_{j_1 j_2 \cdots j_{2\mu-2} j_{2m-3} j_{2\mu+1} \cdots j_{2m-4} j_{2\mu-1} j_{2\mu} j_{2m-1} j_{2m}} K^{j_1 j_2}_{i_1 i_2} \cdots \cdots K^{j_{2\mu-1} j_{2\mu}}_{i_{2\mu-1} i_{2\mu}} \cdots K^{j_{2m-3} i}_{i_{2m-3} i_{2m-2}} K^{j_{2m-1} j_{2m}}_{i_{2m-1} j} .$$

By suitable interchange of rows with rows and columns with columns in the determinants of the right hand side of (2.7), it is easily seen that

$$(2.8)\quad T^{i}_{\cdot j}[m] = 4\sum_{\mu=1}^{m-1} \delta^{i_1 i_2 \cdots i_{2m-4} i_{2m-2} i_{2m-1}}_{j_1 j_2 \cdots j_{2m-4} j_{2m-3} j_{2m-2}} K^{j_1 j_2}_{i_1 i_2} \cdots K^{j_{2m-3} j_{2m-2}}_{k i_{2m-2}} K^{ki}_{i_{2m-1} j} +$$

$$+ 2\sum_{\mu=1}^{m-1} \delta^{i_1 i_2 \cdots i_{2m-2} i_{2m-1}}_{j_1 j_2 \cdots j_{2m-4} j_{2m-3} j_{2m-1} j_{2m}} K^{j_1 j_2}_{i_1 i_2} \cdots K^{j_{2m-3} i}_{i_{2m-3} i_{2m-2}} K^{j_{2m-1} j_{2m}}_{i_{2m-1} j} .$$

In view of (2.1) and (2.2), (2.8) is (2.5) for $m \geq 2$. However, for $m = 1$ (2.5) also holds since in this case both sides vanish identically. This completes the proof of Theorem 1.

It is easily seen that Theorem 1 enables us to calculate $T^{i}_{\cdot j}[m]$ in terms of $S^{i}_{\cdot j}[m]$, $S^{i}_{\cdot j}[m-1]$, $\cdots$, $S^{i}_{\cdot j}[2]$, by an iterative procedure since

$$S^{i}_{\cdot j}[1] = 0,$$

and

$$(2.9)\qquad T^{i}_{\cdot j}[1] = 0.$$

In the sequel we shall consider the special cases of $m = 2$ and $m = 3$ in some detail in order to illustrate the general theory. It is not difficult to show that

$$\mathrm{S}^{i}_{\cdot j}[2] = 2\,\mathrm{K}^{hi}_{\ \ kl}\,\mathrm{K}^{kl}_{\ \ hj}, \tag{2.10}$$

and

$$\mathrm{S}^{i}_{\cdot j}[3] = 4\,\mathrm{K}^{mi}_{\ \ \ hk}\,\mathrm{K}^{hk}_{\ \ lr}\,\mathrm{K}^{lr}_{\ \ mj} + 16\,\mathrm{K}^{li}_{\ \ km}\,\mathrm{K}^{hk}_{\ \ lr}\,\mathrm{K}^{rm}_{\ \ \ hj}\,. \tag{2.11}$$

From (2.5), (2.9), (2.10) and (2.11) we therefore find

$$\mathrm{T}^{i}_{\cdot j}[2] = 4\,\mathrm{K}^{hi}_{\ \ kl}\,\mathrm{K}^{kl}_{\ \ hj}\,, \tag{2.12}$$

and

$$\mathrm{T}^{i}_{\cdot j}[3] = 32\,\mathrm{K}^{ik}_{\ \ hj}\,\mathrm{K}^{rh}_{\ \ st}\,\mathrm{K}^{st}_{\ \ rk} + 16\,\mathrm{K}^{mi}_{\ \ \ hk}\,\mathrm{K}^{hk}_{\ \ lr}\,\mathrm{K}^{lr}_{\ \ mj} + 64\,\mathrm{K}^{li}_{\ \ km}\,\mathrm{K}^{hk}_{\ \ lr}\,\mathrm{K}^{rm}_{\ \ \ hj}\,. \tag{2.13}$$

We shall now prove a result which relates (2.3) to (2.1) and (2.2), and which will subsequently form the cornerstone of our general theory.

THEOREM 2: *For* $m \geq 1$

$$\mathrm{P}^{i}_{\cdot j}[m] = 2\,(m - 1)\,\delta^{i}_{j}\,\mathrm{S}^{h}_{\cdot h}[m] - 2m\,\mathrm{T}^{i}_{\cdot j}[m]\,. \tag{2.14}$$

Proof: The proof of this Theorem is very similar to that of Theorem 1 and we shall merely display the salient features. $\mathrm{P}^{i}_{\cdot j}[m]$ is initially expanded about the last column of its determinant and subsequently that determinant which is associated with δ^{i}_{j} is expanded about its last row. This gives rise to the relation

$$\mathrm{P}^{i}_{\cdot j}[m] = 2\,\delta^{i}_{j} \sum_{\mu=1}^{m-1} \delta^{i_1 i_2 \cdots\cdots\cdots\cdots\cdots i_{2m-1}}_{j_1 \cdots j_{2\mu-1} j_{2\mu+1} \cdots j_{2m-1} j_{2m}}\, \mathrm{K}^{j_1 j_2}_{\ \ \ \ i_1 i_2} \cdots \mathrm{K}^{j_{2m-1} j_{2m}}_{\ \ \ \ \ \ \ \ \ \ i_{2m-1} j_{2\mu}} +$$

$$- 2 \sum_{\mu=1}^{m} \delta^{i_1 \cdots i_{2\mu-1} i_{2\mu+1} \cdots i_{2m-1} i_{2m} i}_{j_1 j_2 \cdots\cdots\cdots\cdots\cdots j_{2m}}\, \mathrm{K}^{j_1 j_2}_{\ \ \ \ i_1 i_2} \cdots \mathrm{K}^{j_{2\mu-1} j_{2\mu}}_{\ \ \ \ \ \ \ \ \ i_{2\mu-1} j} \cdots \mathrm{K}^{j_{2m-1} j_{2m}}_{\ \ \ \ \ \ \ \ \ \ i_{2m-1} i_{2m}}\,,$$

which, with a similar change of dummy indices as in Theorem 1, leads precisely to (2.14).

3. IDENTITIES ARISING FROM DIMENSIONALITY RESTRICTIONS.—We shall here discuss the consequences which follow from restricting the dimensionality n by inequalities involving m, the linear homogeneity of $\mathrm{T}^{i}_{\cdot j}[m]$, $\mathrm{S}^{i}_{\cdot j}[m]$ and $\mathrm{P}^{i}_{\cdot j}[m]$ in $\mathrm{K}^{hk}_{\ \ rs}$. The results of this section depend on the fact that *if* $n \leq \mathrm{N} - 1$ *then*

$$\delta^{i_1 \cdots i_{\mathrm{N}}}_{j_1 \cdots j_{\mathrm{N}}} = 0\,.$$

We apply this result to (2.1), (2.2) and (2.3) and obtain
LEMMA 2:

$$\textit{If}\quad n \leq 2m \qquad \textit{then}\quad P^{i}_{.j}[m] = 0.$$

$$\textit{If}\quad n \leq 2m - 1 \quad \textit{then}\quad T^{i}_{.j}[m] = 0.$$

$$\textit{If}\quad n \leq 2m - 2 \quad \textit{then}\quad S^{i}_{.j}[m] = 0.$$

We combine these results with Theorems 1 and 2 to prove our main result.

THEOREM 3: *If* $n = 2m$ *then*

$$T^{i}_{.j}[m] = \frac{m-1}{m}\,\delta^{i}_{j} S^{h}_{.h}[m]. \tag{3.1}$$

If $n = 2m - 1$ *then*

$$S^{i}_{.j}[m] = 2\,K^{ki}_{\ \ hj} T^{h}_{.k}[m-1]. \tag{3.2}$$

Proof: If $n = 2m$ then (3.1) follows from Lemma 2 and Theorem 2.
If $n = 2m - 1$ then (3.2) follows from Lemma 2 and Theorem 1.

Of course (3.1) and (3.2) are also valid for $n < 2m$ and $n < 2m - 1$ respectively, but in each case the left hand sides of the corresponding identities vanish identically.

COROLLARY: *If* $n \leq 2m$ *then* $K^{ki}_{\ \ hj} T^{h}_{.k}[m] = 0$.

If $n \leq 2m - 1$ *then* $S^{h}_{.h}[m] = 0$.

In order to illustrate Theorem 3 we shall consider the special cases of $m = 2$ and $m = 3$.

Case 1: $m = 2$.

In this case (3.1) will be valid for $n = 4$ and in view of (2.10) and (2.12) will reduce to

$$K^{hi}_{\ \ kl} K^{kl}_{\ \ hj} = \frac{1}{4}\,\delta^{i}_{j}\,(K^{hr}_{\ \ kl} K^{kl}_{\ \ hr}). \tag{3.3}$$

(3.2) provides nothing new when $m = 2$, since in this case $n = 3$ and, by Lemma 1,

$$K^{ij}_{\ \ kl} = 0.$$

Case 2: $m = 3$.

For $n = 6$ we find from (3.1), (2.11) and (2.13) that

$$4\,K^{ik}_{\ \ hj} K^{rh}_{\ \ st} K^{st}_{\ \ rk} + 2\,K^{mi}_{\ \ hk} K^{hk}_{\ \ lr} K^{lr}_{\ \ mj} + 8\,K^{li}_{\ \ km} K^{hk}_{\ \ lr} K^{rm}_{\ \ hj} = \tag{3.4}$$

$$= \frac{1}{3}\,\delta^{i}_{j}\,(K^{ms}_{\ \ hk} K^{hk}_{\ \ lr} K^{lr}_{\ \ ms} + 4\,K^{ls}_{\ \ km} K^{hk}_{\ \ lr} K^{rm}_{\ \ hs}).$$

For $n = 5$, (3.2) reduces to

$$K^{mi}{}_{hk} K^{hk}{}_{lr} K^{lr}{}_{mj} + 4 K^{li}{}_{km} K^{hk}{}_{lr} K^{rm}{}_{hj} = 2 K^{ki}{}_{hj} K^{rh}{}_{st} K^{st}{}_{rk} . \tag{3.5}$$

Naturally (3.5) is also valid for $n = 4$. However, by virtue of (3.3) it simplifies to

$$K^{mi}{}_{hk} K^{hk}{}_{lr} K^{lr}{}_{mj} + 4 K^{li}{}_{km} K^{hk}{}_{lr} K^{rm}{}_{hj} = 0 . \tag{3.6}$$

In fact (3.6) follows directly from Lemma 2 since $n = 2m - 2$ for $n = 4$ and $m = 3$.

We stress that (3.1) and (3.2), and therefore (3.3)-(3.6), are (dimensionaly dependent) identities for any tensor satisfying (1.5) and (1.6). We shall briefly discuss two applications of (3.3) which, with very little further calculation, gives rise to well-known results for $n = 4$.

Firstly, we replace $K^{ij}{}_{rs}$ by the Weyl conformal curvature tensor $C^{ij}{}_{rs}$. Under these circumstances (3.3) reduces to (1.4)—the Lanczos identity (4). *We may therefore regard* (3.1) *and* (3.2) *as generalizations of the Lanczos identity.*

Secondly, we consider the tensor $H^{ij}{}_{hk}$ defined by

$$H^{ij}{}_{hk} = -12 f^{ij} f_{hk} + 6 (\delta^i_h T^j_k + \delta^j_k T^i_h - \delta^j_h T^i_k - \delta^i_k T^j_h) + (f_{rs} f^{rs}) (\delta^i_h \delta^j_k - \delta^j_h \delta^i_k) ,$$

where

$$f_{ij} = -f_{ji}$$

and

$$T^i_j = f^{ri} f_{rj} - \frac{1}{4} (f_{rs} f^{rs}) \delta^i_j .$$

If f_{ij} is the electromagnetic field tensor then T^i_j is the energy momentum tensor for the electromagnetic field in General Relativity ([4], p. 227). When $n = 4$, $H^{ij}{}_{hk}$ has the properties

$$H^{ij}{}_{hk} = -H^{ji}{}_{hk} = -H^{ij}{}_{kh} ,$$

and

$$H^{ij}{}_{jk} = 0 .$$

Thus we may replace $K^{ij}{}_{hk}$ by $H^{ij}{}_{hk}$ in (3.3). To do this we require

$$H^{ij}{}_{hk} H^{hk}{}_{il} = -(24)^2 T^j_i T^i_l + \delta^j_l [30 (f_{rs} f^{rs})^2 + 72 (T^r_s T^s_r)] . \tag{3.7}$$

(4) A very short proof of this result is obtained directly from the expansion of $P^i{}_{.j}[2] = 0$ with $K^{ij}{}_{rs}$ replaced by $C^{ij}{}_{rs}$.

When (3.7) is substituted in (3.3) we find

$$T_i^j T_l^i = \frac{1}{4} \delta_l^j (T_s^r T_r^s) . \tag{3.8}$$

(3.8) *is a well-known result which arises when electromagnetic fields are introduced in the General Theory of Relativity.* It is of fundamental importance to the so-called " already unified " field theory since it is (3.8) (together with the field equations) which gives rise to the Rainich conditions [5]. However, we have obtained (3.8) without using either the concept of duality rotations ([4], p. 237, et seq.) or eigenvalue properties of certain 4×4 matrices ([6], p. 417, et seq.).

It is remarkable that both the Lanczos identity (1.4) *and the identity* (3.8) *involving the square of the electromagnetic energy momentum tensor* T_j^i *are derivable from* (3.3).

In a subsequent paper [7] we hope to discuss certain applications of the above results with particular reference to variational principles of the type employed in General Relativity (5).

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