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On a uniqueness question of Levi-Civita

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NOTE PRESENTATE DA SOCI

Analisi matematica. — *On a uniqueness question of Levi-Civita* (*). Nota di DENNIS DUNNINGER (**) e MONROE H. MARTIN, presentata (***) dal Corrisp. G. FICHERA.

SUNTO. — Si considera un problema pòsto da Levi-Civita nella teoria delle onde periodiche irrotazionali in un fluido incompressibile. Esso consiste nel provare l'unicità di una funzione olomorfa nel disco unitario verificante una condizione al contorno non lineare. Vengono date diverse condizioni sotto le quali si ha la richiesta unicità.

INTRODUCTION.—In 1925 Levi-Civita [1] reduced the mathematical theory of periodic, irrotational waves of finite amplitude in an incompressible fluid of infinite depth to the following boundary problem.

Determine all functions

$$\omega(\zeta) = \theta + i\tau , \qquad \omega(0) = 0,$$

of the complex variable $\zeta = \rho e^{i\sigma}$ *holomorphic in the unit circle* $|\zeta| < 1$ *and subject to the condition*

$$\frac{d\tau}{d\sigma} = p e^{-3\tau} \sin\theta, \qquad p = \text{const.}, \tag{I}$$

on its boundary.

The constant p is non-dimensional. Actually

$$p = \frac{g\lambda}{2\pi c^2},$$

where λ is the wavelength, c is the velocity of propagation of the wave and g is the acceleration of gravity. To insure that the velocities of the fluid particles are small relative to c, one requires that $|\omega| < \delta$ for some sufficiently small positive number δ. Here θ is the angle of inclination of the flow to the horizontal and τ is related [1, p. 275] to the speed q by $\tau = \log(q/c)$.

Levi-Civita demonstrated existence and uniqueness of the solution provided p is sufficiently close to unity. He pointed out further that if one is interested only in the existence of the solution, only the values $p \leqq 1$ need be considered, since one can always regard n waves each of length λ as a single wave of length $\lambda_1 = n\lambda$ for which $p_1 = np$.

On the other hand on p. 284 of his memoir [1] he writes " Toutefois il n'est pas immediatement évident que les seules solutions de (I) pour $p > 1$ soient

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elles qui l'on peut construire de la sorte. Je crois bien qu'il ent soit ainsi, mais je n'ai pas analysé exactement la question ". The question was considered again in 1931 by Lichtenstein [2] who using the theory of non-linear integral equations demonstrated existence and uniqueness for the solution if $p = m - \varkappa$, for any positive integer m provided $\varkappa$ is sufficiently small. An excellent account of modern work on the subject and on water waves in general is given by Stoker [3].

Clearly the problem belongs to a general class of boundary problems which can be formulated for a holomorphic function $w(z) = u + iv$ of a complex variable $z = x + iy$, namely

Determine all functions $w(z) = u + iv$ *of the complex variable* $z = x + iy$, *holomorphic in a region* S *bounded by an analytic arc* C *upon which*

$$\text{(I}'\text{)} \qquad u_n = h(s) f(u, v), \qquad s = \text{arc length of C},$$

where u_n *denotes the external normal derivative of* u *on* C, *the functions* $f(u, v)$, $h(s)$, *holomorphic in their arguments, being given in advance.*

This paper is devoted to the determination of conditions under which the solution to problem (I′) is unique, with special attention paid to Levi-Civita's problem (I).

2. THE INTEGRAL IDENTITY.—Consider two functions

$$w_1(z) = u_1 + iu_3 \quad , \quad w_2(z) = u_2 + iu_4 ,$$

holomorphic in the region S. The unusual designation of their real and imaginary parts turns out to have advantages later on. Following Monge, we adopt the notation

$$p_k = \frac{\partial u_k}{\partial x} \quad , \quad q_k = \frac{\partial u_k}{\partial y} , \qquad (k = 1, 2, 3, 4),$$

and observe that the Cauchy-Riemann equations for $w_1(z)$, $w_2(z)$ take the form

$$p_1 = q_3 , q_1 = -p_3 \quad ; \quad p_2 = q_4 , q_2 = -p_4 .$$

A straightforward application of Gauss' theorem verifies the formal integral identity

$$\text{(2.1)} \qquad \int_C \tau \left(f_2 \frac{\partial u_1}{\partial n} - f_1 \frac{\partial u_2}{\partial n} \right) ds = \int_S Q \, dS,$$

where

$$\text{(2.2)} \quad Q = ap_1^2 + 2\, bp_1 p_2 + cp_2^2 + 2\, d\,(p_2 p_3 - p_1 p_4) + ap_3^2 + 2\, bp_3 p_4 + cp_4^2 ,$$

is a quadratic form in p_1, p_2, p_3, p_4, with coefficients

$$\text{(2.3)} \quad a = f_2 \tau_{u_1} , \; 2b = (f_2 \tau)_{u_2} - (f_1 \tau)_{u_1} , \; c = -f_1 \tau_{u_2} , \; 2d = -(f_1 \tau)_{u_3} - (f_2 \tau)_{u_4} ,$$

in which the function $\tau = \tau(u_1, u_2, u_3, u_4)$ is at our disposal and

$$f_1 = f_1(u_1, u_3) \quad , \quad f_2 = f_2(u_2, u_4),$$

are functions, either specified in advance arbitrarily, or related to the boundary problem (I′) by selecting

$$f_1 = f(u_1, u_3) \quad , \quad f_2 = f(u_2, u_4).$$

The identity (2.1) is the source of our uniqueness theorems. Let D denote the set of points of the four-dimensional Euclidean space E_4 upon which Q is positive definite. The functions f_1, f_2 being given, D obviously depends on the choice of τ. This function τ should, of course, not be confused with the function τ introduced by Levi-Civita and referred to in the introduction.

The following theorem illustrates the type of uniqueness theorem we can expect.

THEOREM 2.1.—*If $w_1 = u_1 + iu_3$ is a non-constant solution of the boundary problem*

$$\Delta u = 0 \text{ in } S \quad , \quad u_n = h(s) f(u, v) \text{ on } C,$$

no other solution $w_2 = u_2 + iu_4$ exists for which the integrals in (2.1) *exist and the manifold*

$$M_2: \quad u_1 = u_1(x, y) \,,\; u_2 = u_2(x, y) \,,\; u_3 = u_3(x, y) \,,\; u_4 = u_4(x, y) \quad (x, y) \in S$$

lies in the set D *of points of* E_4 *in which* Q *is positive definite.*

If a second solution w_2 exists, both sides of (2.1) vanish. Since $M_2 \in D$, this implies $p_1 = p_2 = p_3 = p_4 = 0$ and both w_1, w_2 would be constant, contrary to hypothesis.

As an example consider the identity [4].

$$(2.4) \quad \int_C \left(\frac{\frac{\partial u_1}{\partial n}}{f_1} - \frac{\frac{\partial u_2}{\partial n}}{f_2} \right) ds = \int_S \left[-\frac{f_1'}{f_1^2}(p_1^2 + p_3^2) + \frac{f_2'}{f_2^2}(p_2^2 + p_4^2) \right] dS, \quad f_1' = \frac{\partial f_1}{\partial u_1}, f_2' = \frac{\partial}{\partial}$$

which arises from (2.1) when $\tau = (f_1 f_2)^{-1}$. Q is positive definite in

$$D: \; f_1' < 0 \quad , \quad f_2' > 0$$

and the integrals in (2.1) exist provided $f_1 \neq 0$, $f_2 \neq 0$ in S + C.

In this way we obtain.

THEOREM 2.2.—*If $w_1 = u_1 + iu_3$ is a non-constant solution of the boundary problem*

$$\Delta u = 0 \text{ in } S \quad , \quad u_n = h(s) f(u, v) \text{ on } C,$$

for which $f(u_1, u_3) \neq 0$, $f'(u_1, u_3) < 0$ hold in S + C, *no other solution $w_2 = u_2 + iu_4$ exists for which $f(u_2, u_4) \neq 0$, $f'(u_2, u_4) > 0$ hold in* S + C.

The requirement that M_2 lie in D places restrictions on the uniqueness theorem and one quite naturally seeks to make these restrictions as light as possible by appropriate choice of τ.

3. THE QUADRATIC FORM Q.—From (2.3) Q changes its sign if τ changes its sign. Consequently if Q is definite, one can assume it is positive definite with no loss in generality. The matrix of Q has a curious property, for its rank can be even but not odd. Q will be positive definite, as is well known [8], if the principal minors in the leading positions are all positive. These turn out to equal

$$a \; , \; ac - b^2 \; , \; a\,(ac - b^2 - d^2) \; , \; (ac - b^2 - d^2)^2 ,$$

and consequently Q will be positive definite, provided

$$a > 0 \quad , \quad \Delta = b^2 + d^2 - ac < 0. \tag{3.1}$$

4. PARTIAL DIFFERENTIAL INEQUALITIES.—From (2.3) the two inequalities (3.1) represent partial differential inequalities for τ. Let us write the second in the form

$$x^2 + y^2 + z^2 < r^2 , \tag{4.1}$$

where, after referring to (2.3) we have put

$$\begin{aligned} a &= f_1\,\tau_{u_1} - f_2\,\tau_{u_2} + (f_1' - f_2')\,\tau \quad , & y &= f_1\,\tau_{u_3} + f_2\,\tau_{u_4} + (\dot f_1 + \dot f_2)\,\tau , \\ z &= f_1\,\tau_{u_1} + f_2\,\tau_{u_2} \quad , & r &= f_1\,\tau_{u_1} - f_2\,\tau_{u_2} . \end{aligned} \tag{4.2}$$

Here, and throughout the paper we use the abbreviations

$$f_1' = \frac{\partial f_1}{\partial u_1} \; , \; f_2' = \frac{\partial f_2}{\partial u_2} \; , \; \dot f_1 = \frac{\partial f_1}{\partial u_3} \; , \; \dot f_2 = \frac{\partial f_2}{\partial u_4} .$$

When the point x, y, z is confined to the octagon

$$|x| + |y| + |z| = r$$

it will, vertices excepted, lie in the spherical ball (4.1).

If we select the face $x + y + z = r$, we are led, from (4.2), at once to the linear partial differential equation of first order

$$f_1\,\tau_{u_1} + f_2\,\tau_{u_2} + f_1\,\tau_{u_3} + f_2\,\tau_{u_4} + (f_1' - f_1' + \dot f_1 + \dot f_2)\,\tau = 0 . \tag{4.3}$$

If $f(u, v)$ is multiplicatively separable, i.e., if

$$f(u, v) = \varphi(u)\,\psi(v) \quad , \quad f_1 = \varphi(u_1)\,\psi(u_3) \quad , \quad f_2 = \varphi(u_2)\,\psi(u_4),$$

the method of Lagrange yields a solution

$$\tau = k\,\frac{\varphi(u_2)}{\varphi(u_1)\,\psi(u_3)\,\psi(u_4)} , \qquad k = \text{const.}, \tag{4.4}$$

of (4.3). With this selection for τ from (2.3) and (3.1) the inequalities (3.1) become

$$(4.5) \qquad a = \frac{-f_1'}{f_1} f_2 \tau > 0 \quad , \quad \Delta = f_2'(f_2' - f_1') \tau^2 < 0 .$$

From Theorem 2.1 we obtain

THEOREM 4.1.—*If the boundary problem*

$$\Delta u = 0 \text{ in S} \quad , \quad u_n = h(s)\, \varphi(u)\, \psi(v) \text{ on C} ,$$

has a solution $u_1 + iu_3$ *regular analytic in* S + C, *it cannot possess a second such solution* $u_2 + iu_4$ *for which the inequalities* (4.5) *hold in* S *and the integrals in* (2.1) *exist for* τ *chosen as in* (4.4).

This theorem forms the starting point for the treatment of the problem of Levi-Civita in the next section.

5. THE PROBLEM OF LEVI-CIVITA.—This problem comes within the scope of Theorem 4.1, for by writing the boundary condition in (I) in the form

$$\theta_n = pe^{-3\tau} \sin\theta \qquad\qquad p = \text{const.},$$

the problem is identified with (I′) by placing

$$u = \theta \quad , \quad v = \tau \quad , \quad h(s) = p ,$$

by taking S to be the interior of the unit circle $\rho < 1$ and C to be its boundary $\rho = 1$. On placing

$$(5.1) \qquad f = e^{-3v} \sin u \quad , \quad f_1 = e^{-3u_3} \sin u_1 \quad , \quad f_2 = e^{-3u_4} \sin u_2 ,$$

the solution (4.4) becomes

$$(5.2) \qquad \tau = k\lambda\, e^{3(u_3+u_4)} \quad , \quad \lambda = \frac{\sin u_2}{\sin u_1} ,$$

from which it is easy to verify that

$$a = -k\lambda^2 e^{3u_3} \cos u_1 \quad , \quad b = k\lambda\, e^{3u_3} \cos u_2 \quad , \quad c = -k\, e^{3u_4} \cos u_2 \quad , \quad d = 0 ,$$

and that

$$\Delta = k^2 \lambda^2 e^{6u_3} \cos u_2 \left(e^{3u_3} \cos u_2 - e^{3u_4} \cos u_1\right) .$$

Thus the inequalities (3.1) will be satisfied by choosing $k < 0$ and requiring that

$$|u_1| < \frac{\pi}{2} \quad , \quad |u_2| < \frac{\pi}{2} \quad , \quad e^{3u_4} > e^{3u_3} \frac{\cos u_2}{\cos u_1} .$$

Furthermore the integrals in (2.1) will exist, provided λ is regular in S + C.

Introducing the directions θ_1, θ_2 and the speeds of q_1, q_2 of the two flows by placing

$$(5.3) \qquad u_1 = \theta_1 \quad , \quad u_2 = \theta_2 \quad , \quad u_3 = \log \frac{q_1}{c} \quad , \quad u_4 = \log \frac{q_2}{c} ,$$

the inequalities become

$$(5.4) \qquad |\theta_1| < \frac{\pi}{2} \quad , \quad |\theta_2| < \frac{\pi}{2} \quad , \quad q_2^3 > q_1^3 \frac{\cos u_2}{\cos u_1} .$$

If we take, q , θ to be polar coordinates in a " hodograph plane ", the inequalities (5.4) attach the shaded region in the figure below to the hodograph point (q_1 , θ_1), and the following theorem becomes apparent.

THEOREM 5.1.—*If the problem of Levi-Civita has a solution* $\omega = \omega_1(\zeta)$ *no second solution* $\omega = \omega_2(\zeta)$ *exists for which the hodograph point* (q_2 , θ_2) *always lies in the shaded region attached to the point* (q_1 , θ_1) *in fig.* 5.1 *by the inequalities* (5.4), *and the ratio* $\lambda = \sin\theta_2/\sin\theta_1$ *is regular.*

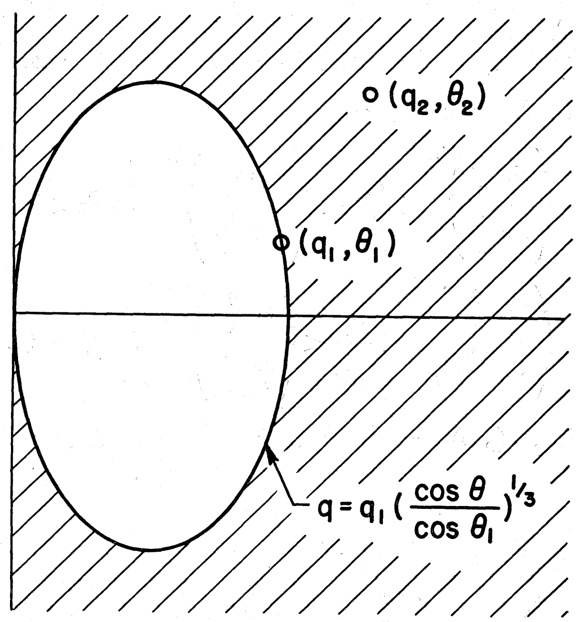

Fig. 5.1.

The solution (5.2) has the form

$$\tau = \mathrm{T}(u_1 , u_2)\, e^{3(u_3+u_4)} .$$

Any function τ of this form with f_1 , f_2 given by (5.1) renders $d = 0$ in (3.1), as is readily seen from (2.3), and from [5, p. 10] we are led to take

$$(5.5) \qquad \mathrm{T} = \frac{\cos u_2 - \cos u_1}{\sin u_1 \sin u_2} = \frac{\lambda^{-1} - \lambda}{\cos u_1 + \cos u_2} .$$

A simple calculation based on (2.3) verifies that

$$(5.6) \qquad a = \frac{1 - \cos u_1 \cos u_2}{\sin^2 u_1} r_1^2 \quad , \quad 2b = -\left(\frac{\sin u_2}{\sin u_1} r_1^2 + \frac{\sin u_1}{\sin u_2} r_2^2\right) ,$$

$$c = \frac{1 - \cos u_1 \cos u_2}{\sin^2 u_2} r_2^2 \quad , \quad d = 0 ,$$

or, if one desires to bring out the dependence on the ratio $\lambda = \sin u_2/\sin u_1$, that

$$(5.7) \qquad a = \frac{\lambda^2 + \cos^2 u_2}{1 + \cos u_1 \cos u_2} r_1^2 \quad , \quad 2b = -(\lambda r_1^2 + \lambda^{-1} r_2^2) ,$$

$$c = \frac{\lambda^{-2} + \cos^2 u_1}{1 + \cos u_1 \cos u_2} r_2^2 \quad , \quad d = 0 .$$

where, in both cases, from (5.3),

$$(5.8) \qquad r_1^2 = e^{3u_3} = \left(\frac{q_1}{c}\right)^3 \quad , \quad r_2^2 = e^{3u_4} = \left(\frac{q_2}{c}\right)^3 .$$

Using (5.6) we compute Δ from (3.1) and (5.6) as

$$\Delta = \left(r_1 \cos^2 \frac{u_2}{2} - r_2 \cos^2 \frac{u_1}{2}\right)\left(r_1 \sin^2 \frac{u_2}{2} - r_2 \sin^2 \frac{u_1}{2}\right) \cdot$$

$$\cdot \left[(r_1 + r_2)^2 - (r_1 \cos u_2 + r_2 \cos u_1)^2\right] \csc^2 u_1 \csc^2 u_2 ,$$

and consequently $\Delta < 0$ provided

$$(5.9) \qquad \left(\frac{\sin \frac{\theta_2}{2}}{\sin \frac{\theta_1}{2}}\right)^{4/3} < \frac{q_2}{q_1} < \left(\frac{\cos \frac{\theta_2}{2}}{\cos \frac{\theta_1}{2}}\right)^{4/3} \quad \text{if} \quad \theta_2 < \theta_1$$

$$\text{or} \quad \left(\frac{\cos \frac{\theta_2}{2}}{\cos \frac{\theta_1}{2}}\right)^{4/3} < \frac{q_2}{q_1} < \left(\frac{\sin \frac{\theta_2}{2}}{\sin \frac{\theta_1}{2}}\right)^{4/3} \quad \text{if} \quad \theta_2 > \theta_1 .$$

The inequalities (5.9) are portrayed by the shaded areas in fig. 5.2 below

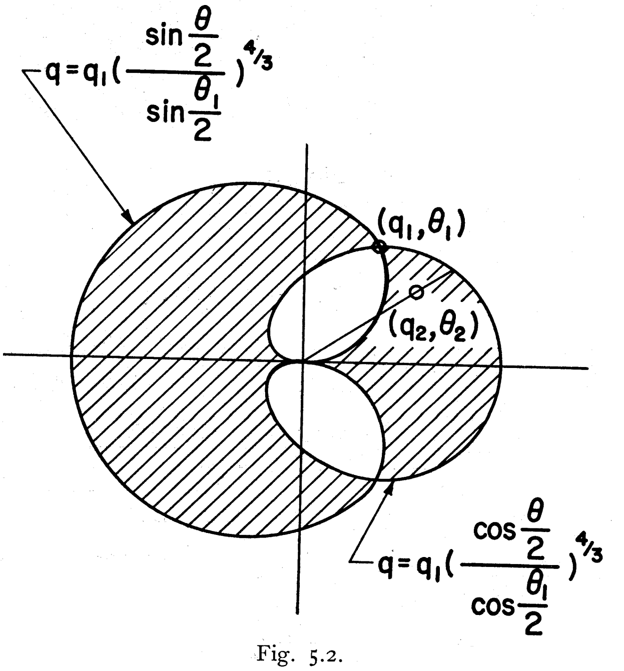

Fig. 5.2.

Given two flows

$$q_i = q_i(\rho, \sigma) \quad , \quad \theta_i = \theta_i(\rho, \sigma) , \qquad (i = 1, 2),$$

for which $|\theta_1| + |\theta_2| < \pi$ and λ^{-1}, $\lambda = \sin\theta_2/\sin\theta_1$ are regular, it is clear from (5.5) and (5.7) that the integrals in (2.1) exist. Consequently since $\Delta < 0$ in the shaded region in Fig. 5.2 we have

THEOREM 5.2.—*If the problem of Levi-Civita has a solution* $\omega_1(\zeta) = \theta_1 + i\tau_1$, *no second solution* $\omega_2(\zeta) = \theta_2 + i\tau_2$ *exists for which the hodograph point* (q_2, θ_2) *always lies in the shaded region attached to the point* (q_1, θ_1) *in* fig. (5.2) *by the inequalities* (5.9) *and for which the ratios* λ^{-1}, $\lambda = \sin\theta_2/\sin\theta_1$ *are regular in* $|\zeta| \leqq 1$ and $|\theta_1| + |\theta_2| < \pi$.

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