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Boundary Layer Flows of viscoelastic materials

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Chimica (Principi di Ingegneria Chimica). — *Boundary Layer Flows of viscoelastic materials*. Nota di GIANNI ASTARITA e GIUSEPPE MARRUCCI, presentata (*) dal Corrisp. G. MALQUORI.

RIASSUNTO. — Sulla base dei concetti sviluppati in una recente analisi dei flussi stazionarii di liquidi viscoelastici, viene esaminato il caso di flussi di strato limite. Tale analisi mostra che lo strato limite si estende in modo tale da far sì che il campo di flusso esterno o idrodinamico sia « lento ».

L'analisi viene successivamente ristretta al caso del flusso lungo una lastra piana ad angolo di incidenza nullo. Viene dimostrato che lo strato limite si sviluppa a monte del punto di ristagno, e che la zona interessata a tale comportamento anomalo ha dimensioni dell'ordine di grandezza del prodotto della velocità del campo esterno per il tempo naturale del liquido considerato. Tali dimensioni sono assai maggiori di quelle della zona di non validità delle analisi di strato limite per liquidi Newtoniani. La distribuzione di velocità nella zona del punto di ristagno viene valutata qualitativamente.

INTRODUCTION.

Boundary layer flows of viscoelastic materials have been considered by several Authors [1–5]. Although these analyses are of considerable interest, and show both the difficulty of the problem to be studied and the possible ways of fruitful attack, the approach chosen has been in general a traditional one. The customary boundary layer approximations have been made, and the resulting equations of motion have been integrated after insertion of more or less sophisticated constitutive equations of the viscoelastic type. Such a procedure, although obviously correct in analyzing materials whose rheological behavior is indeed governed by the assumed constitutive equation, avoids attacking two fundamental problems. These are:

(1) Are the usual boundary layer approximations valid for viscoelastic liquids, at least as much as for Newtonian liquids, and, if not, what type of approximations need to be made instead?

(2) Considering that constitutive equations usually assumed in analyzing flow phenomena have only asymptotic validity for real fluids, is a boundary layer flow one for which such constitutive equations may indeed be used with some degree of confidence?

A partial answer to these problems has recently been presented by Metzner and Astarita [1]; in this note, a further step in this direction is hopefully contributed.

The analysis given below is based on the results recently published by Astarita [6]. Analyzing in general terms steady flows of viscoelastic materials,

(*) Nella seduta del 12 novembre 1966.

Astarita defines two dimensionless groups, N_1 and N_2, which are relevant in the analysis of such flows. If I_X is the invariant of a tensor X_{ij} defined by:

$$I_X = 2\, X_{ij}\, X^{ij} \tag{1}$$

and if e_{ij} and w_{ij} are the rate-of-strain and the vorticity tensor, respectively, the groups N_1 and N_2 are defined as:

$$N_1 = \sqrt{I_e}\,\theta \tag{2}$$

$$N_2 = \sqrt{I_e - I_w}\,\theta \tag{3}$$

where θ is the "natural time" of the liquid [7–10].

The two most important conclusions reached by Astarita [6] are:

A) When the group N_1 is smaller than unity, the flow is "slow", in the sense that asymptotic expansions of the "*n*–order fluid" type [11] are valid.

B) Fluids which are characterized by an asymptotically exponential rate of stress relaxation may flow steadily at any value of N_1, but will develop infinite stresses at values of N_2 approaching some critical upper limit of the order of unity.

Concepts derived from recent analyses of unsteady flows [2, 6, 9, 10, 12–14] will also be used. In particular, if τ^* is the time elapsed from startup of flow, a Deborah number τ^*/θ can be defined; the following general conclusion is now generally accepted:

C) When the Deborah number approaches unity, the rheological response of viscoelastic materials is solidlike.

Internal and external flow fields.

In considering a boundary layer flow, the flow field may conceptually be divided into an "external" and an "internal" field. The external field is defined as that region of space where the hydrodynamic (or inviscid) equations of motion correctly portray the actual flow field; the internal field is the remainder of the entire flow field. The term "boundary layer" is equivalent to internal field.

If the flow field is uniform at infinity, the entire external field is irrotational, because the hydrodynamic equations of motion are circulation-preserving. Thus, in the entire external field:

$$I_w = 0 \tag{4}$$

and, considering equations 2 and 3:

$$N_1 = N_2\,. \tag{5}$$

On the basis of conclusion B), equation 5 implies that N_1 is bound to be less than unity in the external field. Thus, considering A), the flow in the external field is slow. The boundary layer region extends far enough so that the entire external field is slow.

This conclusion was in part envisaged is a discussion of flow around spherical gas bubbles [15], as well as in the analysis of Metzner and Astarita [1] of the flow around small diameter cylinders, such as used in hot-wire velocity probes. In the more general form obtained here, this conclusion implies anomalous kinematics of motion in every stagnation region.

FLAT PLATE. PERMISSIBLE APPROXIMATIONS.

Now restrict the attention to flow past a flat plate at zero incidence. Let x be the distance from the leading edge, y the distance from the plate, u and v the velocity components in the x and y directions, and $y < \delta(x)$ the boundary layer region.

The customary boundary layer approximation is the assumption that:

$$\delta(0) = 0. \tag{6}$$

In the following, it will be shown that equation 6 may lead to appreciable errors in the case of liquids characterized by non-negligible values of θ.

The flow pattern considered is a two-dimensional one, so that only four components of the rate-of-strain tensor are to be considered. Of these, by definition of the boundary layer thickness δ, $\partial u/\partial y$ is:

$$\frac{\partial u}{\partial y} = O\left(\frac{U}{\delta}\right) \tag{7}$$

where U is the free stream velocity. On the basis of hypothesis 6 one obtains:

$$\frac{\partial u}{\partial x} = O\left(\frac{U}{x}\right) \tag{8}$$

and, by requirement of continuity,

$$\frac{\partial v}{\partial y} = O\left(\frac{U}{x}\right). \tag{9}$$

Integration of equation 9 yields:

$$v = O\left(\frac{U\delta}{x}\right) \tag{10}$$

and thus:

$$\frac{\partial v}{\partial x} = O\left(\frac{U\delta}{x^2}\right). \tag{11}$$

Substitution of Equations 7–11 into 2 and 3 yields:

$$N_1 = U\theta \frac{x^2 + \delta^2}{x^2\,\delta} \tag{12}$$

$$N_2 = U\,\theta/x. \tag{13}$$

Equation [13] is, in view of conclusion B), impossible to hold true at $x < U\theta$. Thus, the only hypothesis made, namely, equation 6, must be rejected, or at least an analysis based on the same can be accepted only at $x \gg U\theta$.

There may be some difficulty in accepting results obtained from an analysis of steady flows, such as A) and B), as applied to a boundary layer flow, which is unsteady in a Lagrangian sense. Although in fact the physical reasoning leading to A) and B) suggests that at least qualitative validity is to be expected also for unsteady flows, it is worthwhile considering the problem from a different viewpoint.

If equation 6 is accepted, a material element is essentially undeformed up to the leading edge. Thus, the boundary layer flow can be regarded as a suddenly accelerated flow with a value of τ^* given by U/x. In conclusion, the Deborah number is:

$$N_{De} = U\,\theta/x = N_2\,. \tag{14}$$

In the region $x < U\theta$, solidlike behavior is expected, in view of conclusion C). Thus, stresses would be governed by the total deformation of a material element, with the configuration at $x < O$ as the ground state. This would lead to abnormally high stresses, so that again Equation 6 is seen to be unrealistic in the region $x < U\theta$.

The flow in the leading edge region should be analyzed in terms of a realistic velocity distribution, such as discussed in the next section, which does not assume the validity of equation 6.

Leading edge kinematics.

Assume that the boundary layer extends in the upstream region for a distance of the order of $U\theta$, so that the velocity distributions $u\,(y)$ are of the form sketched in fig. 1. By a reasoning analogous to the one leading to equations (7)–(11), one obtains, for the region downstream of the leading edge:

$$x > 0, \qquad \left\{ \begin{aligned} \frac{\partial u}{\partial y} &= O\left(\frac{U}{\delta}\right) \\ \frac{\partial u}{\partial x} &= O\left(\frac{U}{x + U\theta}\right) \\ \frac{\partial v}{\partial y} &= O\left(\frac{U}{x + U\theta}\right) \\ \frac{\partial v}{\partial x} &= O\left(\frac{U\delta}{[x + U\theta]^2}\right) \end{aligned} \right. \tag{15}$$

so that the values of N_1 and N_2 are:

$$x > 0, \qquad \left\{ \begin{aligned} N_1 &= U\theta\,\frac{(x + U\theta)^2 + \delta^2}{(x + U\theta)^2\,\delta} \\ N_2 &= \frac{U\theta}{x + U\theta} \end{aligned} \right. \tag{16}$$

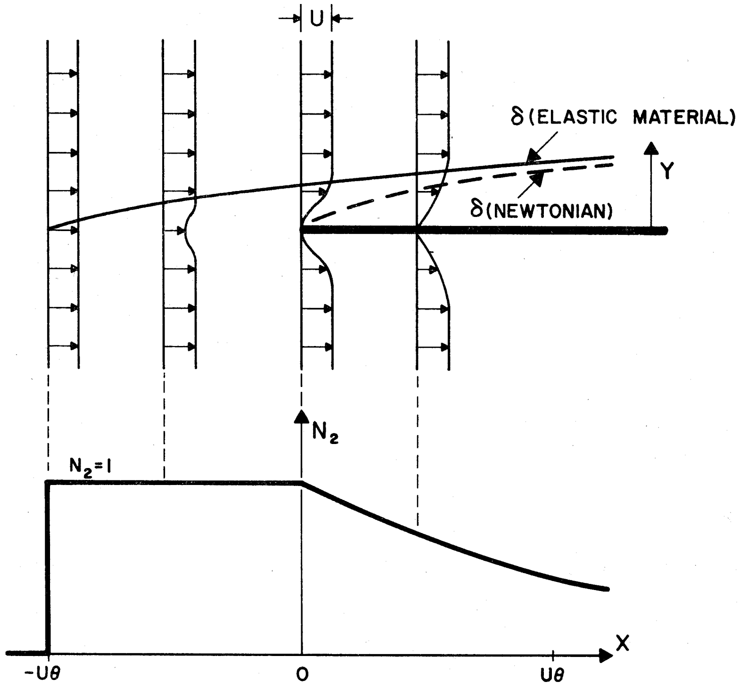

Fig. 1.

Let us now turn our attention to the region upstream of the leading edge. If the velocity distribution defect develops linearly, the components of the velocity gradient tensor are:

$$
(17) \qquad x < 0, \qquad \left\{ \begin{aligned} \frac{\partial u}{\partial x} &= O\left(\frac{1}{\theta}\right) \\ \frac{\partial v}{\partial y} &= O\left(\frac{1}{\theta}\right) \\ \frac{\partial u}{\partial y} &= O\left(\frac{U\theta + x}{\delta\theta}\right) \\ \frac{\partial v}{\partial x} &= O\left(\frac{\delta}{\theta\,[U\theta + x]}\right) \end{aligned} \right.
$$

Thus, the groups N_2 and N_{De} are easily calculated:

$$
(18) \qquad \left\{ \begin{aligned} N_2 &= 1 \\ N_{De} &= \frac{U\theta + x}{U\theta \cdot \dfrac{U\theta + x}{U\theta}} = 1 \end{aligned} \right.
$$

and are seen again to coincide.

Equation (18) shows that the assumed velocity profile is indeed realistic, because it implies that the boundary layer develops upstream of the leading edge, but as close to the latter as the elastic character of the liquid allows.

It remains to be discussed whether the leading edge region where anomalous behavior is being observed is less or more important than in the case of Newtonian liquids. In fact, even for the latter the boundary layer analysis is not valid at the leading edge; but the region concerned is so small that this early criticism of classical hydrodynamicists to the boundary layer theory has been dismissed as largely immaterial.

In the case of Newtonian liquids, the boundary layer approximation breaks down in the region where the Reynolds number based on x is not large: say, in first approximation, is lower than unity. Thus, if ν is the kinematic viscosity, the critical distance x_{cN} is:

$$x_{cN} = O(\nu/U). \tag{19}$$

On the basis of equations (13) and (14), the critical distance in the case of viscoelastic liquids, x_{cVE}, is:

$$x_{cVE} = O(U\theta). \tag{20}$$

Thus, the considerations developed in this note may have relevance provided that:

$$U\theta \gg \nu/U \tag{21}$$

or, equivalently,

$$\theta \gg \nu/U^2. \tag{22}$$

Even if ν is taken as 10^{-1} cm²/sec, i.e., if a rather viscous liquid is considered, condition 22 is fulfilled, at a free stream velocity of 100 cm/sec, when the natural time exceeds 10^{-5} sec. Considering that θ values of $10^{-3} \div 10^{-2}$ sec. are not unfrequent in viscoelastic liquids, the conclusion is drawn that the leading edge region is much larger in viscoelastic liquids than in Newtonian liquids, so that the development in this paper is of interest even if the boundary layer approximations would be satisfactory for Newtonian liquids. This point can also be considered directly from equations (19) and (20), which show that, if $\theta = 10^{-2}$ sec, $\nu = 10^{-1}$ cm²/sec, and $U = 100$ cm/sec, the value of x_{cN} is only 10^{-3} cm, while the value of x_{cVE} is of the order of 1 cm.

DOWNSTREAM REGION.

Equations (16) degenerate, at $x \gg U\theta$, into equations (12) and (13). This shows that an analysis based on equation (6) is asymptotically valid when $x \gg U\theta$. Therefore, traditional approaches to boundary layer analyses of viscoelastic liquids are not without merit, because they portray the behavior of such liquids in the far downstream region. Considering that, at $x \gg U\theta$, the boundary layer thickness δ is presumably much smaller than the distance from the leading edge x, one has from equation (12):

$$x \gg U\theta \quad , \quad N_1 \cong U\theta/\delta. \tag{23}$$

Equation 23 shows that a region exists where the leading edge effects have been damped out, yet the value of N_1 is still large, so that a viscoelastic analysis of the traditional type is both justified and useful.

Yet, when dealing with fluids characterized by natural times exceeding 10^{-3} sec, the most conspicuous effects of elasticity are to be expected in the leading edge region. An analysis of such effects is beyond the scope of the present note: it is hoped that the qualitative indications given here, concerning the kinematics of motion in the leading edge region, may constitute the framework on which such analyses may be based.

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