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A Helly-type Theorem for Polygons

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Geometria. — *A Helly-type Theorem for Polygons.* Nota di DOUGLAS DERRY, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Dopo di avere richiamato la definizione ed alcune proprietà inerenti all'*ordine* di un poligono chiuso di uno spazio proiettivo reale L_n, si dimostra che, se $A_1, A_2, \cdots, A_m$ sono m punti di L_n in posizione generica, con $m \geq n+4$, tali che $n+4$ qualsivogliano (purché distinti) di essi siano vertici di un poligono d'ordine n di L_n, allora anche gli m punti $A_1, A_2, \cdots, A_m$ risultano vertici di un poligono d'ordine n.

1. INTRODUCTION.

A unique norm curve [1] exists which contains $n+3$ points in general position in real projective n-space L_n. If these $n+3$ points are the vertices of a polygon inscribed in the norm curve then this polygon has order n. Consequently any $n+3$ points in general position in L_n are the vertices of a polygon of order n in L_n. This result forms the background for the following theorem which answers a query posed by Prof. B. Grünbaum. If $A_1, A_2, \cdots, A_m$, $m \geqq n+4$, are points of L_n in general position with the property that any $n+4$ of these points are the vertices of a polygon of order n in L_n, then there is a polygon of order n in L_n with the vertices $A_1, A_2, \cdots, A_m$. This result, analogous to Helly's theorem for convex bodies, is the subject of the present note. This is a best possible result for, if $n+4$ were replaced by $n+3$, the condition would place no restriction on the points as was seen above. However the result would not necessarily be true as there are sets of $n+4$ points in L_n which are not the vertices of any polygon of order n in L_n [3].

Section 2 contains definitions and easily established results used in the proof. In the following section it is shown that if a polygon of order n exists in L_n with the vertices $A_1, A_2, \cdots, A_m$, $m \geqq n+3$, in general position in L_n, then this polygon is unique. This fact can be established by induction from the case $m = n+3$ previously considered by the author [3]. However, partly to make the present note self-contained, a complete and essentially different proof of this polygon property is given. In the final section the previous material is assembled to establish the theorem.

2. PRELIMINARIES.

2.1. The symbol $[Q_1, Q_2, \cdots]$, where $Q_1, Q_2, \cdots$ are either points or sets of points in L_n, denotes the linear subspace of L_n spanned by $Q_1, Q_2, \cdots$.

2.2. The symbol $\pi : A_1 A_2 \cdots A_m$ denotes a *closed* polygon in L_n with the vertices $A_1, A_2, \cdots, A_m$ in general position. The subscripts are computed modulo m, e.g. A_1 and A_{1+m} represent the same vertex.

(*) Nella seduta del 12 novembre 1966.

2.3. If, for a polygon $\pi : A_1 A_2 \cdots A_m$ in L_n, L_{n-1} is a hyperplane in L_n such that $A_i \notin L_{n-1}$, $1 \leqq i \leqq m$, then the maximum number of points of $L_{n-1} \cap \pi$ for all such L_{n-1} is defined to be the *order* of π.

The number of points of $L_{n-1} \cap \pi$ is either even for all L_{n-1} or odd for all L_{n-1}. Thus π is defined to be *even* or *odd* according as a hyperplane L_{n-1}, $A_i \notin L_{n-1}$, $1 \leqq i \leqq m$, exists for which $L_{n-1} \cap \pi$ is even or odd.

2.4. If $A_{i-1} A_i A_{i+1}$ is an arc of a polygon $\pi : A_1 A_2 \cdots A_m$ in L_n, $n > 1$, $m > 3$, let $A_{i+1} A_{i-1}$ be the line segment which closes $A_{i-1} A_i A_{i+1}$ so that it becomes an even triangle. The *contraction* of π with respect to the vertex A_i is defined to be the arc $A_{i+1} A_{i+2} \cdots A_{i-1+m}$ of π closed by the segment $A_{i+1} A_{i-1}$. A contraction is even or odd according as π itself is even or odd. The order of a contraction of π is at most that of π.

2.5. If $A_{i-1} A_i A_{i+1}$ is an arc of a polygon $\pi : A_1 A_2 \cdots A_m$ in L_n, $n > 1$, $m \geqq 3$, let $\overline{A_{i+1} A_{i-1}}$ be the line segment which closes $A_{i-1} A_i A_{i+1}$ so that it becomes an even triangle. A line L_1 for which $A_i \in L_1 \subseteqq [A_{i-1}, A_i, A_{i+1}]$, $L_1 \cap \overline{A_{i+1} A_{i-1}} = \Phi$, is defined to be tangent of π with the point of contact A_i. It is convenient to represent such a tangent by the symbol $L(A_i)$.

2.6. If A_i is any vertex of a polygon $\pi : A_1 A_2 \cdots A_m$ in L_n, $n > 1$, the projection from A_i of the arc $A_{i+1} A_{i+2} \cdots A_{i-1+m}$ of π closed by that of the set of all the tangents $L(A_i)$ defines a polygon in the projected space L_{n-1} which we shall call the projection of π from A_i.

2.7. Two polygons which have the same set of vertices $A_1, A_2, \cdots, A_m$ in general position in L_n, $n > 1$, coincide if they both have an arc $A_u A_1 A_v$ in common and their projections from A_1 coincide.

2.8. A polygon $\pi : A_1 A_2 \cdots A_m$ in L_n for which $m > n$ has at least order n. Such a polygon with the minimum order n will be denoted by the symbol π_n.

2.9. If $m > n + 1$ it follows with the use of 2.4 that the contraction of a polygon $\pi_n : A_1 A_2 \cdots A_m$ with respect to any one of its vertices is also a polygon of order n in L_n.

2.10. If, for a polygon $\pi_n : A_1 A_2 \cdots A_m$ in L_n, $n > 1$, $A'_i A'_{i+1}$ denotes the projection of the side $A_i A_{i+1}$ of π_n, $i \neq j$, $i + 1 \neq j$, from the vertex A_j and $A'_{j-1} A'_{j+1}$ that of the tangent set $L(A_j)$ together with the projections of the points A_{j-1}, A_{j+1} then the projection of π_n can be written as $A'_1 A'_2 A'_3 \cdots A'_{j-1} A'_{j+1} \cdots A'_m$ in accordance with 2.6. This projection has order $n - 1$ in the projected space and so can be written as π_{n-1} [2].

2.11. *If* S *is set of* n *distinct vertices* $A_{p_1}, A_{p_2}, \cdots, A_{p_n}$ *of a polygon* $\pi_n : A_1 A_2 \cdots A_m$ *in* L_n, $n > 1$, *then sequences* $X_1, X_2, \cdots, X_n$ *of points of* π_n *exist for which* X_i *is a point of the side* $A_{p_i} A_{p_i+1}$ *and* $X_i \to A_{p_i}$, $1 \leqq i \leqq n$, *with the property that* $A_j \notin [X_1, X_2, \cdots, X_n]$, $1 \leqq j \leqq m$, *provided only that* X_i *be sufficiently close to* A_{p_i}, $1 \leqq i \leqq n$.

Proof. The subscripts can be adjusted so that $1 \leqq p_1 < p_2 < \cdots < p_n < m$. Let X_i be an interior point of the side $A_{p_i} A_{p_i+1}$ of π_n which approaches A_{p_i}, $1 \leqq i \leqq n$. As the vertices of π_n are in general position $A_{p_1}, A_{p_2}, \cdots, A_{p_n}$ span a hyperplane which contains these vertices but no other vertex of π_n. As $X_i \rightarrow A_{p_i}$, $1 \leqq i \leqq n$, it follows that $[X_1, X_2, \cdots, X_n]$ is a hyperplane L_{n-1} which also contains no vertex of π_n which is not within S provided only that X_i is sufficiently close to A_{p_i}, $1 \leqq i \leqq n$. Suppose at least one vertex of S is within L_{n-1}. In this case let A_{p_j} be the vertex of S with the maximum subscript in L_{n-1}. As $X_j \in L_{n-1}$, $[A_{p_j}, A_{p_j+1}] = [A_{p_j}, X_j] \subseteqq L_{n-1}$. Hence $A_{p_j+1} \in L_{n-1}$. Because of the adjustment of the subscripts $p_j < p_j + 1 \leqq m$. As A_{p_j} was the point of S with the maximum subscript in L_{n-1} $A_{p_j+1} \notin S$. This contradicts the fact that L_{n-1} contains no vertex of π_n which is not within S. Hence the supposition that L_{n-1} contain a vertex of S is false. Therefore $A_j \notin L_{n-1}$, $1 \leqq j \leqq m$, provided only X_i is sufficiently close to A_{p_i}, $1 \leqq i \leqq n$. The proof is now complete.

3. A UNIQUENESS PROPERTY FOR POLYGONS π_n.

3.1. *If* A_i, A_j *are distinct vertices of a polygon* $\pi_n : A_1 A_2 \cdots A_m$ in L_n, $n > 1$, $m \geqq n+3$, *then, for a line segment* $\overline{A_i A_j}$ *which is not a side of* π_n, *vertices* $A_{p_1}, A_{p_2}, \cdots, A_{p_n}$ *exist which span a hyperplane* L_{n-1} *for which* $L_{n-1} \cap \overline{A_i A_j} \neq \Phi$, $A_i \notin L_{n-1}$, $A_j \notin L_{n-1}$.

Proof. π_n is the union of the two of its arcs with the endpoints A_i, A_j. Let α be one of these arcs which contains at least three vertices. The subscripts can be adjusted so that A_i, A_j become A_1, A_k, respectively, and α has the form $A_1 A_2 \cdots A_k$ in which case $\overline{A_i A_j}$ becomes $\overline{A_1 A_k}$ and $k \geqq 3$. Let β be the arc $A_k A_{k+1} \cdots A_{1+m}$ complementary to α. As $m \geqq n+3$ there are at least $n+1$ vertices of π_n different from A_1, A_k. Let Q be a set of n distinct vertices of π_n for which $A_1 \notin Q$, $A_k \notin Q$ and for which $Q \cap \alpha$ contains an even or odd number of vertices according as the polygon obtained by closing α with the segment $\overline{A_1 A_k}$ is odd or even. Suppose first that β is a single side of π_n. Hence it consists of $A_1 A_k = A_1 A_m$. In this case as a $\overline{A_1 A_k}$ is, by the hypothesis, not a side of π_n it must be the complement of the interior of $A_1 A_k$ in the projective line $[A_1, A_k]$. The order of π_n is even or odd according as n is even or odd. That of the polygon obtained by closing α with the segment $\overline{A_1 A_k}$ is therefore odd or even according as n is even or odd. Hence, if Q is any set of n distinct vertices of π_n different from A_1, A_k, Q satisfies the above conditions. If β contains at least one vertex different from A_1, A_k then n vertices can be chosen so that $Q \cap \alpha$ is even or odd. Hence Q can be chosen to satisfy the above conditions in both cases.

Let $A_{p_1}, A_{p_2}, \cdots, A_{p_n}$ be the vertices of the set Q and L_{n-1} the hyperplane spanned by them. By 2.11 sequences of points X_i of π_n exist for which $X_i \in A_{p_i} A_{p_i+1}$, $X_i \rightarrow A_{p_i}$, $1 \leqq i \leqq n$, and $A_j \notin [X_1, X_2, \cdots, X_n]$, $1 \leqq j \leqq m$, provided only that X_i is sufficiently close to A_{p_i}, $1 \leqq i \leqq n$. Now, as

$A_{p_i} \neq A_1$, $A_{p_i} \neq A_k$ the interior of $A_{p_i} A_{p_i+1}$ is in α or β according as A_{p_i} itself is in α or β. Consequently, as $X_i \in A_{p_i} A_{p+1}$, X_i is in α or β according as A_{p_i} is in α or β. As π_n has order n it follows that $[X_1, X_2, \cdots, X_n] \cap \pi_n$ is the set $X_1, X_2, \cdots, X_n$. Moreover $[X_1, X_2, \cdots, X_n] \cap \alpha$ and $Q \cap \alpha$ contain the same number of points. But this number is odd or even according as the polygon obtained by closing α with $\overline{A_1 A_k}$ is even or odd. Therefore the hyperplane $[X_1, X_2, \cdots, X_n]$ must intersect this polygon in a point not on α i.e. in a point of $\overline{A_1 A_k}$. Now if P be the intersection $L_{n-1} \cap [A_1, A_k]$, P is a single point as $A_1 \notin L_{n-1}$, $A_k \notin L_{n-1}$. As $[X_1, X_2, \cdots, X_n] \to L_{n-1}$ it follows then that $P \in \overline{A_1 A_k}$ and the result is established.

3.2. *If, for* $m \geqq n + 3$, π_n *is a polygon with the vertices* $A_1, A_2, \cdots, A_m$ *in* L_n, *then* π_n *is the only polygon of order n with these vertices.*

Proof. The points $A_1, A_2, \cdots, A_m$ are in general position as they are the vertices of a polygon π_n. If $n = 1$ π_1 is the projective line and so is unique. If, for $n > 1$, π'_n, $\pi'_n \neq \pi_n$, is a polygon of order n with the vertices $A_1, A_2, \cdots, A_m$ then a side $\overline{A_h A_k}$ of π'_n exists which is not a side of π_n. By 3.1, applied to π_n, a hyperplane $L_{n-1} = [A_{p_1}, A_{p_2}, \cdots, A_{p_n}]$ exists which intersects $\overline{A_h A_k}$ and for which $A_h \notin L_{n-1}$, $A_k \notin L_{n-1}$. By 2.11, applied to π'_n, sequences of points X_i of π'_n exist for which $X_i \to A_{p_i}$, $1 \leqq i \leqq n$, and for which $A_j \notin [X_1, X_2, \cdots, X_n]$, $1 \leqq j \leqq m$, provided only that X_i is sufficiently close to A_{p_i}, $1 \leqq i \leqq n$. Consequently, as $[X_1, X_2, \cdots, X_n] \to [A_{p_1}, A_{p_2}, \cdots, A_{p_n}]$, $[X_1, X_2, \cdots, X_n]$ is a hyperplane which intersects $\overline{A_h A_k}$ while A_h, $A_k \notin [X_1, X_2, \cdots, X_n]$ provided only that X_i is sufficiently close to A_{p_i}, $1 \leqq i \leqq n$. $X_i \notin [A_h, A_k]$, $1 \leqq i \leqq n$, provided only that X_i is sufficiently close to A_{p_i} for otherwise, as $X_i \to A_{p_i}$, this would imply that $A_{p_i} \in [A_h, A_k]$. A_{p_i}, A_h, A_k are distinct as $A_{p_i} \in L_{n-1}$ but A_h, $A_k \notin L_{n-1}$. Hence $A_1, A_2, \cdots, A_m$ would not be in general position contrary to the hypothesis. Therefore $[X_1, X_2, \cdots, X_n]$ intersects π'_n in $n + 1$ distinct points, namely $X_1, X_2, \cdots, X_n$ and an interior point of $\overline{A_h A_k}$ in contradiction to the order of π'_n. Hence no such polygon π'_n exists and the result is established.

4. THE THEOREM.

If every $n + 4$ *points of a set of points* $A_1, A_2, \cdots, A_m$ *in general position in* L_n, $m \geqq n + 4$, *are the vertices of a polygon of order n in* L_n, *then the points* $A_1, A_2, \cdots, A_m$ *themselves are the vertices of a polygon of order n in* L_n.

Proof. We assume the result to be false and obtain a contradiction. It is true in L_1 as, in this case, the projective line is the required polygon. Consequently if n is defined to be the least number for which the theorem is false in L_n, then $n > 1$. For $m = n + 4$ the result is included in the hypothesis. If m is defined to be the least number for which a set of points $A_1, A_2, \cdots, A_m$ exists in L_n for which the theorem is false then $m > n + 4$.

For m so defined let $A_1, A_2, \cdots, A_m$ be a set of points in L_n which satisfies the hypothesis but for which no polygon of order n exists with these points as vertices. Any subset $A_1, A_2, \cdots, A_{i-1}, A_{i+1}, \cdots, A_m$ of this set satisfies the hypothesis as $m - 1 \geqq n + 4$. Hence, by the definition of m, a polygon $\pi_n(i)$ exists of order n in L_n with the vertices $A_1, A_2, \cdots, A_{i-1}, A_{i+1}, \cdots, A_m$, $1 \leqq i \leqq m$. If $A'_2, A'_3, \cdots, A'_m$ be the projections of $A_2, A_3, \cdots, A_m$, respectively, from A_1 in the projected space L_{n-1}, then these points also satisfy the hypothesis of the theorem in L_{n-1}. The $m - 1$ projected points are in general position in L_{n-1} otherwise the original points would not be in general position in $L_n \cdot m - 1 \geqq (n-1) + 4$ follows from the assumption that $m > n + 4$. Finally we check that any set Q of $(n-1) + 4 = n + 3$ points of $A'_2, A'_3, \cdots, A'_m$ are the vertices of a polygon of order $n - 1$ in L_{n-1}. There is, by the hypothesis, a polygon of order n the vertices of which are the $n + 3$ points of $A_1, A_2, \cdots, A_m$ which are projected into Q and the point A_1. The projection of such a polygon from its vertex A_1 is, by 2.10, a polygon of order $n - 1$ with the set of vertices Q. As the theorem is valid in L_{n-1} because of the definition of n we may now apply it to the set $A'_2, A'_3, \cdots, A'_m$ and so obtain the result that a polygon π_{n-1} of order $n - 1$ exists with the vertices $A'_2, A'_3, \cdots, A'_m$. From now onward we assume that the subscripts have been adjusted so that π_{n-1} can be written as $A'_2 A'_3 \cdots A'_m$.

Let $\pi_n(i, j)$ be the contraction of $\pi_n(i)$ with respect to the vertex $A_j, A_i \neq A_i$. $\pi_n(i, j)$ has order n by 2.9 as $\pi_n(i)$ has $m - 1$ vertices and $m - 1 > n + 1$ follows from the assumption that $m > n + 4$. Let $\pi_{n-1}(i)$ be the contraction of π_{n-1} with respect to A'_i, $i \neq 1$, and $\pi_{n-1}(i, j)$ that of $\pi_{n-1}(i)$ with respect to A'_j, $j \neq 1$, $A'_i \neq A'_j$. Again by 2.9 these contractions have order $n - 1$ as, in the first case, $m - 1 > (n-1) + 1$ while in the second $m - 2 > (n-1) + 1$ because $m > n + 4$. By 3.2 the polygons $\pi_n(i)$, $\pi_n(i, j)$ are the only polygons of order n with their respective vertices as they both have at least $m - 2$ vertices and $m - 2 \geqq n + 3$. The same is true for the polygons $\pi_{n-1}(i)$, $\pi_{n-1}(i, j)$ of order $n - 1$ as both have at least $m - 3$ vertices and $m - 3 \geqq (n-1) + 3$. The projection of $\pi_n(i)$ from A_1 is, by 2.10, a polygon of order $n - 1$ with the same vertices as $\pi_{n-1}(i)$; that of $\pi_n(i, j)$ is a polygon of order $n - 1$ with the same vertices as $\pi_{n-1}(i, j)$. Hence the projections of $\pi_n(i)$, $\pi_n(i, j)$ from A_1 are $\pi_{n-1}(i)$, $\pi_{n-1}(i, j)$, respectively.

With the use of these results we show that a polygon $\pi_n(p)$ exists for which $2 < p < m$ and of which a segment $A_{p-1} A_{p+1}$ is a side. It is sufficient to construct such a polygon for which $2 < p < 7$ because, from the assumptions $n > 1$, $m > n + 4$, it follows that $m > 6$. By the previous paragraph the projection of $\pi_n(3,6)$ from A_1 is the contraction $\pi_{n-1}(3.6)$: $A'_2 A'_4 A'_5 A'_7 \cdots A'_m$ of π_{n-1}. If the projection of the tangents $L(A_1)$ of $\pi_n(3.6)$ is not the interior of $A'_5 A'_7$ let $p = 6$ otherwise let $p = 3$. Clearly these values of p satisfy the inequalities $2 < p < 7$. Again by the previous paragraph the projection of $\pi_n(6)$ from A_1 is the contraction $\pi_{n-1}(6)$: $A'_2 A'_3 A'_4 A'_5 A'_7 \cdots A'_m$ of π_{n-1}. It follows from 2.10 that if $A_5 A_7$ is not a side of $\pi_n(6)$ that $A_5 A_1 A_7$ is an arc of $\pi_n(6)$ which must then have the form $A_2 A_3 A_4 A_5 A_1 A_7 \cdots A_m$. Consequently $\pi_n(6,3)$ becomes $A_2 A_4 A_5 A_1 A_7 \cdots A_m$. But $\pi_n(6.3)$ and $\pi_n(3.6)$

coincide by 3.2 as the both have order n and the same $m-2$ vertices because $m-2 \geqq n+3$. By 2.10 the projection of the tangents $L(A_1)$ of π (3.6) is the side $A'_5 A'_7$ of π_{n-1} (3.6) contrary to the choice of p. Hence $A_5 A_7$ is a side of π_n (6). In the second case a similar argument shows that $A_2 A_4$ is a side of π_n (3). Thus the required polygon $\pi_n(p)$ has been constructed.

We may now construct a polygon π with vertices $A_1, A_2, \cdots, A_m$ which will subsequently be shown to have order n. As $2 < p < m$, A_1 is different from each of the points A_{p-1}, A_p, A_{p+1}. As $A_1, A_2, \cdots, A_m$ are in general position $A_1 \notin [A_{p-1}, A_p]$, $A_1 \notin [A_p, A_{p+1}]$. Hence we may define $A_{p-1} A_p$, $A_p A_{p+1}$ as the line segments the projections of which from A_1 are sides $A'_{p-1} A'_p$, $A'_p A'_{p+1}$ of π_{n-1}. If $A_{p+1} \cdots A_1 \cdots A_{p-1}$ is the arc of $\pi_n(p)$ complementary to its side $A_{p-1} A_{p+1}$ let π be the polygon obtained by closing this arc by the segments $A_{p-1} A_p$, $A_p A_{p+1}$. The projection of π from A_1 is the arc $A'_{p+1} \cdots A'_m A'_2 \cdots A'_{p-1}$ of $\pi_{n-1}(p)$ closed by the segments $A'_{p-1} A'_p$, $A'_p A'_{p+1}$, i.e. π_{n-1} itself. $\pi_{n-1}(p)$ is, by its definition, the contraction of π_{n-1} with respect to the vertex A_p. Therefore the side $A'_{p-1} A'_{p+1}$ of $\pi_{n-1}(p)$ and $A'_{p-1} A'_p$, $A'_p A'_{p+1}$ form an even triangle. But this triangle is the projection of the side $A_{p-1} A_{p+1}$ of $\pi_n(p)$ together with those of the sides $A_{p-1} A_p$, $A_p A_{p+1}$ of π. As the projection of an odd triangle is an odd triangle it follows that the three segments $A_{p-1} A_{p+1}$, $A_{p-1} A_p$, $A_p A_{p+1}$ form an even triangle. In other words the contraction of π with respect to its vertex A_p is $\pi_n(p)$.

Let $\pi(k)$ denote the contraction of π with respect to the vertex A_k. If $A_u A_1 A_v$ be the arc of π which contains A_1 and its neighboring vertices we show that if A_k is different from each A_p, A_u, A_1, A_v that $\pi(k) = \pi_n(k)$. To prove this it is sufficient, by 2.7, to show that the projections of $\pi(k)$ and $\pi_n(k)$ from A_1 coincide and that $A_u A_1 A_v$ is an arc of both $\pi(k)$ and $\pi_n(k)$. To show that the projections from A_1 of $\pi(k)$ and $\pi_n(k)$ coincide we consider the arc $A_r A_k A_s$ of π which contains A_k and its neighboring vertices. $A_r \neq A_1$, $A_s \neq A_1$ for otherwise A_k would be one of A_u or A_v. Therefore the projection of $A_r A_k A_s$ from A_1 is the arc $A'_r A'_k A'_s$ of the projection π_{n-1} of π. Moreover if $A_r A_s$ is the segment which closes $A_r A_k A_s$ so that it becomes an even triangle the projection $A'_r A'_s$ of this segment will close $A'_r A'_k A'_s$ so that it also becomes an even triangle. Hence the projection of the contraction $\pi(k)$ is the contraction $\pi_{n-1}(k)$ which was proved above to be the projection of $\pi_n(k)$ from A_1. It remains now to check that $A_u A_1 A_v$ is an arc of $\pi(k)$ and of $\pi_n(k)$. $A_u A_1 A_v$ is an arc of $\pi(k)$ because of the definition of a contraction and because A_k is different from each of A_u, A_1, A_v. Because $2 < p < m$ and $A_{p-1} A_p A_{p+1}$ is an arc of π, it follows that $A_u \neq A_p$, $A_v \neq A_p$. Hence $A_u A_1 A_v$ is an arc of $\pi(p)$. We proved in the previous paragraph that $\pi(p) = \pi_n(p)$. Hence $A_u A_1 A_v$ is an arc of $\pi_n(p)$ and also of $\pi_n(p, k)$ as A_k is different from each of A_u, A_1, A_v. But $\pi_n(p, k) = \pi_n(k, p)$ as both polygons have order n and the same $m-2$ vertices. This means that $A_u A_1 A_v$ is an arc of the contraction $\pi_n(k, p)$ of $\pi_n(k)$. It follows, then, from the definition of a contraction that either $A_u A_1 A_v$ is an arc of $\pi_n(k)$ or that $\pi_n(k)$ contains an arc of the type $A_u A_p A_1 A_v$ or $A_u A_1 A_p A_v$. In the latter two cases the projection $\pi_{n-1}(k)$ of $\pi_n(k)$ would contain an arc $A'_u A'_p A'_v$. This would imply that no segment $A'_u A'_v$ could be a side of $\pi_{n-1}(k)$ as this polygon contains more than three sides for $m - 2 \geqq n + 3 > 3$. This is impossible as $A'_u A'_v$ is a side of $\pi_{n-1}(k)$. To see this we note, by 2.10, it is a side of the projection π_{n-1} of π as the latter contains the arc $A_u A_1 A_v$. Consequently $A'_u A'_v$ is a side of the contraction $\pi_{n-1}(k)$ of π_{n-1} for $A_k \neq A_u$, $A_k \neq A_v$. Therefore $A_u A_1 A_v$ is an arc of $\pi_n(k)$ and the proof that $\pi(k) = \pi_n(k)$ is now complete.

To show that π has order n let M_{n-1} be any hyperplane of L_n for which $A_i \notin M_{n-1}$, $1 \leqq i \leqq m$. If M_{n-1} intersects exactly one of the sides $A_{p-1} A_p$, $A_p A_{p+1}$ of π it will also intersect the segment $A_{p-1} A_{p+1}$ which closes the arc $A_{p-1} A_p A_{p+1}$ so that it becomes an even triangle. Therefore in the case in which M_{n-1} intersects neither side $A_{p-1} A_p$, $A_p A_{p+1}$ or exactly one of them M_{n-1} intersects π and its contraction $\pi_n(p)$ in exactly the same number of points. As $\pi_n(p)$ has order n this means that M_{n-1} intersects π in at most n points. If M_{n-1} intersects the arc $A_{p-1} A_p A_{p+1}$ in two points then, as every other point of $M_{n-1} \cap \pi$ is included in $M_{n-1} \cap \pi_n(p)$, M_{n-1} intersects π in at most $n + 2$ points. As π has at least $n + 5$ sides, there exists, in this case, at least one side of π of which A_1 is not an endpoint and which has no point in common with M_{n-1}. Let A_k be an endpoint of such a side which is not a vertex of the arc $A_u A_1 A_v$ of π. M_{n-1} intersects π and its contraction $\pi(k)$ in the same number of points. We have proved above that $\pi(k) = \pi_n(k)$ provided that $A_k \neq A_1$, $A_k \neq A_u$, $A_k \neq A_v$. Consequently M_{n-1} intersects $\pi_n(k)$ and so also π in at most n points. Hence π has order n. This contradicts the assumption that $A_1, A_2, \cdots, A_m$ were not the vertices of any polygon of order n and so completes the proof of the theorem.

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