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Structure Theory in s-d-Rings. Nota I

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NOTE PRESENTATE DA SOCI

Matematica. — *Structure Theory in s–d–Rings.* Nota I di ESAYAS GEORGE KUNDERT, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Operazioni di semi–derivazione e di semi–integrazione vennero recentemente introdotte ed investigate in [1] entro certo anelli. In questa Nota I si prosegue lo studio delle relative strutture, in vista anche di ulteriori approfondimenti ed applicazioni cui sarà dedicata una successiva Nota II.

1. d–RINGS (RINGS WITH SEMI-DERIVATION).—Definition: A commutative ring $\mathfrak{A}$ with 1 shall be called a d–ring, if there exists a mapping $d : \mathfrak{A} \to \mathfrak{A}$ such that:

1. d is onto,
2. $d(a + b) = da + db$,
3. $d(ab) = a \cdot db + b \cdot da - da \cdot db$,
4. $d(1) = 0$,
5. $d^{(m)} a = 0$ for some $m < \infty$, $a \in \mathfrak{A}$.

The mapping d we shall call semi-derivation. (See [1]). If $d^{(m-1)} a \neq 0$ but $d^{(m)} a = 0$ then $m - 1$ is called the "degree of a". Let $R = \{ x \mid x \in \mathfrak{A}, dx = 0 \}$. R is *a* subring of $\mathfrak{A}$. To indicate that an element of $\mathfrak{A}$ belongs to R, we denote it by a Greek letter and call it a "constant".

We frequently consider $\mathfrak{A}$ as an R–Algebra (in the natural way). Terms like submodule, subalgebra are always meant with respect to this particular Algebra.

By homomorphism we always mean ring-homomorphism, by R–homomorphism we always mean module-homomorphism. (d is clearly a R–homomorphism.)

2. s–d–RINGS (d–RINGS WITH A DEFINITE INTEGRATION).—Let $\mathfrak{A}$ be a d–ring. Definition: We shall call $\mathfrak{A}$ a s–d–ring, if there exists a ring-homomorphism $\sigma : \mathfrak{A} \to R$ with the property $\sigma(\alpha) = \alpha$ for all $\alpha \in R$. In every s–d–ring we can define the following mapping $s : \mathfrak{A} \to \mathfrak{A}$, with $s(a) = a' - \sigma(a')$ and a' an element of $\mathfrak{A}$ such that $da' = a$. This mapping is well defined, because from $da' = da'' \Rightarrow d(a' - a'') = 0 \Rightarrow a' - a'' \in R \Rightarrow \sigma(a' - a'') = \sigma(a') - \sigma(a'') = a' - a'' \Rightarrow a' - \sigma(a') = a'' - \sigma(a'')$. This mapping s we call a "definite integration in $\mathfrak{A}$". It has the following properties:

a. $ds(a) = 0$,
b. $sd(a) = a - \sigma(a)$,
c. $\sigma s(a) = 0$,
d. s is *a* R–homomorphism,
e. Formula (F): $s(ab) = a \cdot s(b) - s[da \cdot s(b)] + s[da \cdot b]$.

(*) Nella seduta del 12 novembre 1966.

The proofs are straightforward. (For formula (F) use the product-formula 1.3 replacing b by $s(b)$, then apply s on both sides and use a. through d. above).

We define now:

$$x_0 = 1, \; x_i = s(x_{i-1}); \qquad i = 1, 2, 3, \cdots$$

We have

$$dx_i = x_{i-1} \text{ for } i \geq 1 \quad \text{and} \quad deg(x_i) = i \text{ for } i \geq 0.$$

PROPOSITION: $\{x_0, x_1, \cdots, x_i, \cdots\}$ *forms a free module basis for* $\mathfrak{A}$.
Proof: $a \in \mathfrak{A}$, deg $(a) = m$. Let $\alpha_m = d^{(m)} a \in R$:

$$d^{(m)}(a - \alpha_m x_m) = \alpha_m - \alpha_m = 0 \Rightarrow d^{(m-1)}(a - \alpha_m x_m) = \alpha_{m-1} \in R,$$

$$d^{(m-1)}(a - \alpha_m x_m - \alpha_{m-1} x_{m-1}) = 0 \cdots \text{ etc.},$$

$$d(a - \alpha_m x_m - \cdots - \alpha_1 x_1) = 0 \Rightarrow a - \alpha_m x_m - \cdots - \alpha_1 x_1 = \alpha_0 \in R,$$

or $a = \alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_m x_m$. So $\{x_i\}$ is a basis. If $\sum_{i=0}^{m} \alpha_i x_i = 0$ apply $d^{(m)}$ on both sides $\Rightarrow \alpha_m = 0 \Rightarrow \sum_{i=0}^{m-1} \alpha_i x_i = 0$. Repeating the above procedure, we get $\alpha_i = 0$ for $i = 0, 1, 2, \cdots$ which proves that our basis is free.

We shall call α_i the "i–th component of a".

COROLLARY: $\sigma(x_i) = 0$ for $i \geq 1$ and $\sigma(a) = \alpha_0$.

Proof: $\sigma(x_i) = \sigma s(x_{i-1}) = 0$ and $\sigma(a) = \sigma \sum_{i=0}^{m} \alpha_i x_i = \sum_{i=0}^{m} \alpha_i \sigma(x_i) = \alpha_0$ by the properties 2 c. and 2 d.

Using formula (F) we can recursively calculate the representations for the products $x_i \cdot x_j$. For example: Taking: $a = x_1$, $b = x_0 = 1$ we have by (F): $s(x_1 \cdot 1) = x_1 \cdot s(1) - s(dx_1 \cdot s(1)) + s(dx_1 \cdot 1)$ or $x_2 = x_1 \cdot x_1 - x_2 + x_1 \Rightarrow$ $\Rightarrow x_1 \cdot x_1 = 2x_2 - x_1$. Taking: $a = x_1$, $b = x_1$ we find: $x_1 \cdot x_2 = 3x_3 - 2x_2$, etc.

EXAMPLES of s–d–rings:

a. Let A be any commutative ring with 1. There always exists a s–d–ring for which A plays the rôle of the ring of constants, namely $\mathfrak{A}(A) = \mathfrak{C}_{1\infty}(A)$ as defined in ([1]). We shall call this ring "the s–d–ring absolutely associated with A".

b. Let A be as above and $\mathfrak{m}$ an ideal in A subject to the conditions stated in ([1]). There always exists a s–d–ring $\mathfrak{A}_{\mathfrak{m}}(A)$ cannonicaly associated with the pair $(A, \mathfrak{m})$, namely $\mathfrak{A}_{\mathfrak{m}}(A) = \mathfrak{C}_{11}(A, \mathfrak{m})$ as defined in ([1]). In this case the rôle of the ring of constants is played by the $\mathfrak{m}$–completion of A. We shall call the rings $\mathfrak{A}_{\mathfrak{m}}(A)$ "the s–d–rings relatively associated with A". Our main interest is actually in these rings and their applications, but in this paper we shall not directly deal with them. I mention only the following facts: Even for the ring of integers the associated s–d–rings are not unique factorization domains, nor are they noetherian with respect to module– and ideal-structure.

We now introduce objects which are more intimately related to the s–d–structure and allow a nontrivial, but relatively simple structure theory.

3. INTULES AND INTEALS.—Let $\mathfrak{A}$ always be a s–d–ring.

Definition 1: A subset B is called an intule, if it is:
1. a submodule of $\mathfrak{A}$,
2. $b \in B \Rightarrow s(b) \in B$.

Definition 2: An intule is called an inteal if it is also an ideal in the ring $\mathfrak{A}$.

PROPOSITION 1: *Let* A, B, C *be intules (inteals) then*
a. $A \cap B$, $A + B$ *are also intules (inteals)*,
b. $A \supseteq B \Rightarrow A \cap (B + C) = B + (A \cap C)$.

a. is obvious and b. is already true for submodules (ideals).

Definition 3: Let E be a subset of $\mathfrak{A}$.

$$\langle E \rangle = \left\{ \sum_{\text{finite}} \gamma_i \, s^{(n_i)}(e_i) \;\middle|\; \begin{matrix} \gamma_i \in R \\ e_i \in E \end{matrix} \right\} = \text{" intule generated by E "},$$

$(E) =$ " ideal generated by E ",
$[E] =$ " module generated by E ".

PROPOSITION 2: a. $\langle E \rangle = [\langle E \rangle] = \langle [E] \rangle$,
b. $\langle \langle E \rangle \rangle = \langle E \rangle$,
c. $\langle E \rangle$ *is the smallest intule containing* E.

The proofs are straightforward and are omitted.

PROPOSITION 3: a. B *intule* $\Rightarrow$ (B) *is inteal*,
b. B *ideal* $\Rightarrow \langle B \rangle$ *is inteal.*

Proof: (a) We only have to show that $s(ab) \in (B)$ if $a \in \mathfrak{A}$ and $b \in B$. By formula (F): $s(ab) = a \cdot s(b) - s[da(s(b) - b)]$, but $s(b) \in B \Rightarrow a \cdot s(b) \in (B)$. Using now induction on the degree of a, it follows from: $\deg(da) < \deg(a)$ and $s(b) - b \in B$ and $s(0) = 0$ that $s(ab) \in (B)$.

(b) It is clearly sufficient to prove that $a \cdot s^{(n)}(b) \in \langle B \rangle$ for all $a \in \mathfrak{A}$ $b \in B$; $n = 0, 1, 2, \cdots$.

For $n = 0$ this is true since B is an ideal. Assume true for $n - 1$.

By formula (F) substituting $s^{(n-1)}(b)$ instead of b we have: $a \cdot s^{(n)}(b) = s[a \cdot s^{(n-1)}(b)] + s[da \cdot (s^{(n)}(b) - s^{(n-1)}(b)]$ but by hypothesis $a \cdot s^{(n-1)}(b) \in \langle B \rangle \Rightarrow s[a \cdot s^{(n-1)}(b)] \in \langle B \rangle$. Using now induction on the degree of a, it follows as under a. that $a \cdot s^{(n)}(b) \in \langle B \rangle$.

COROLLARY 1: *Let* E *be a subset of* $\mathfrak{A}$; *then*:
a. $\langle (\langle E \rangle) \rangle = (\langle E \rangle)$,
b. $(\langle (E) \rangle) = \langle (E) \rangle$.

COROLLARY 2: *Let* E *be a subset of* $\mathfrak{A}$. *Then there exists a smallest inteal* E^* *containing* E, *namely* $E^* = (\langle E \rangle) = \langle (E) \rangle$.

Proof: $E \subseteq (E) \Rightarrow \langle E \rangle \subseteq \langle (E) \rangle \Rightarrow (\langle E \rangle) \subseteq (\langle (E) \rangle) = \langle (E) \rangle$ and $E \subseteq \langle E \rangle \Rightarrow$ $\Rightarrow (E) \subseteq (\langle E \rangle) \Rightarrow \langle (E) \rangle \subseteq \langle (\langle E \rangle) \rangle = (\langle E \rangle) \Rightarrow (\langle E \rangle) = \langle (E) \rangle$. Since every inteal containing E contains E^* and E^* is an inteal $\Rightarrow E^*$ is the smallest inteal containing E.

Definition 4: E^* is called "the inteal generated by E".

We have of course

PROPOSITION 4: a. $E^{**} = E^*$,

b. $E^* = \langle E \rangle^* = (E)^*$,

c. $A = A^*$ *if and only if* A *is an inteal.*

PROPOSITION 5: $E \subseteq R \Rightarrow (E) = \langle E \rangle = E^*$,

Proof: $a \in (E) \Rightarrow a = \sum_{i=1}^{n} a_i e_i$, $a_i \in \mathfrak{A}$, $e_i \in E$. Let $a_i = \sum_{j=0}^{n_i} \alpha_{ij} x_j$, $\alpha_{ij} \in R \Rightarrow$ $\Rightarrow a = \sum_{i,j} \alpha_{ij} x_j e_i = \sum_{i,j} b_{ij} s^{(m_j)}(e_i)$, $b_{ij} \in R$. $\Rightarrow a \in \langle E \rangle$ and vice versa.

Furthermore: $E^* = \langle (E) \rangle = \langle \langle E \rangle \rangle = \langle E \rangle$.

PROPOSITION 6: a. A *intule* $\Rightarrow s(A)$ *is also intule,*

b. A *inteal* $\Rightarrow s(A)$ *is also inteal.*

Proof: a. follows immediately and to prove b. one has to show that $c \cdot s(a) \in s(A)$ for $c \in \mathfrak{A}$, $a \in A$, this is done with help of formula (F).

If A is a submodule then dA is also a submodule, but if A is an intule the dA is in general not an intule.

Definition 5: A is called a differentiable intule (inteal) if *both* A and dA are intules (resp. A is inteal and dA is intule).

Definition 6: A is called an analytic intule (inteal) if $d^{(n)}A$ is an intule for all $n = 0, 1, 2, \cdots$ (resp. A inteal and $d^{(n)}A$ intule for $n \geq 1$).

Before stating some properties of differentiable intules, we introduce certain ideals in the ring of constants, which are associated with each submodule of $\mathfrak{A}$. Let A be a submodule of $\mathfrak{A}$.

Definition 7:

$A_i(A) = \{ \alpha_i \mid \text{where } \alpha_i \text{ is the } i - th \text{ component of some element } a \in A \}$.

For the purpose of reference, we collect some trivial properties into the following

PROPOSITION 7: a. $A_i(A)$ *is an ideal in* R,

b. $A_0(A) = \sigma(A)$,

c. $A \subseteq B \Rightarrow A_i \subseteq B_i$,

d. $A_i(dA) = A_{i+1}(A)$,

e. $A + A_0 = sdA \oplus A_0$,

f. *If* A *is an intule* $\Rightarrow A_i \subseteq A_{i+1}$.

PROPOSITION 8: a. A *intule* $\Rightarrow s(\mathrm{A})$ *differentiable intule*,
b. A *intule* $\Rightarrow \mathrm{A} \subseteq d\mathrm{A}$,
c. A *differentiable intule* $\Rightarrow \mathrm{A}_0 \subset d\mathrm{A}$,
d. A *differentiable inteal* $\Rightarrow d\mathrm{A}$ *inteal.*
e. A *analytic inteal* $\Rightarrow d\mathrm{A}$ *analytic inteal*,
f. A *differentiable intule* $\Longleftrightarrow \mathrm{A} + \mathrm{A}_0 = \mathrm{A} + \langle \mathrm{A}_0 \rangle$ $\Longleftrightarrow \mathrm{A} + \mathrm{A}_0$ *is intule.*

Proof:

a. and b. are obvious.

c. $\alpha_0 \in \mathrm{A}_0 \Rightarrow \exists\, a = \sum_{i=0}^{n} \alpha_i x_i \in \mathrm{A} \subseteq d\mathrm{A} \Rightarrow sda = \sum_{i=1}^{n} \alpha_i x_i \in d\mathrm{A} \Rightarrow \alpha_0 \in d\mathrm{A}$.

d. We have to show that $c \cdot a \in d\mathrm{A}$ for $c \in \mathfrak{A}$, $a \in d\mathrm{A}$. Using the product-formula we have: $d(c \cdot s(a)) = c \cdot a + dc \cdot s(a) - dc \cdot a$. Now $a \in d\mathrm{A} \Rightarrow \exists\, b \in \mathrm{A} \ni a = db \Rightarrow b - \sigma(b) = s(a)$. Therefore: $c \cdot a = d(c \cdot b) + dc \cdot (a - s(a) - \sigma(b))$, but $c \cdot b \in \mathrm{A} \Rightarrow d(c \cdot b) \in d\mathrm{A}$ and $a - s(a) - \sigma(b) \in d\mathrm{A}$ since $s(a) \in d\mathrm{A}$ (intule) and $\sigma(b) = \beta_0 \in d\mathrm{A}$ by c.) above. Using now induction on the degree of $c \Rightarrow c \cdot a \in d\mathrm{A}$.

e. follows at once from d.

f. We prove: A diff. intule $\underset{(1)}{\Rightarrow} \mathrm{A} + \mathrm{A}_0 = \mathrm{A} + \langle \mathrm{A}_0 \rangle \underset{(2)}{\Rightarrow} \mathrm{A} + \mathrm{A}_0$ intule $\underset{(3)}{\Rightarrow}$ A diff. intule

(1) It is sufficient to prove: $c \in \langle \mathrm{A}_0 \rangle \Rightarrow c \in \mathrm{A} + \mathrm{A}_0$. Since $c = \Sigma s^{(n_i)}(\gamma_i)$, $\gamma_i \in \mathrm{A}_0 \subseteq \mathrm{R}$, it will be enough to show: $\gamma \in \mathrm{A}_0 \Rightarrow s(\gamma) \in \mathrm{A} + \mathrm{A}_0$ because by repetition it follows then $s^{(n)}(\gamma) \in \mathrm{A} + \mathrm{A}_0$. Now $\gamma \in \mathrm{A}_0 \subset d\mathrm{A}$ by c.) above since A is differentiable $\Rightarrow \gamma = db$, $b \in \mathrm{A}$, but $s(\gamma) = sdb = b - \sigma(b)$; $\sigma(b) \in \mathrm{A}_0 \Rightarrow s(\gamma) \in \mathrm{A} + \mathrm{A}_0$;

(2) is trivial;

(3) $a \in d\mathrm{A} \Rightarrow a = db$, $b \in \mathrm{A}$. Therefore $s(a) = b - \sigma(b) \Rightarrow s(a) \in \mathrm{A} + \mathrm{A}_0 \Rightarrow s^{(2)}(a) \in \mathrm{A} + \mathrm{A}_0$ (intule!) $\Rightarrow s^{(2)}(a) = c + \gamma_0$; $c \in \mathrm{A}$, $\gamma_0 \in \mathrm{A}_0$. Applying d on both sides $\Rightarrow s(a) = dc \in d\mathrm{A} \Rightarrow \mathrm{A}$ is diff.

Definition 8: Let $\mathfrak{C}$ be a set of inteals in $\mathfrak{A}$. An inteal M is called maximal in $\mathfrak{C}$ if and only if from $\mathrm{B} \supset \mathrm{M} \Rightarrow \mathrm{B} \notin \mathfrak{C}$, where B is an inteal in A.

Let $\mathfrak{F}$ denote the set of all inteals in $\mathfrak{A}$ excluding $\mathfrak{A}$ itself. An inteal which is maximal in $\mathfrak{F}$ is shortly called a maximal inteal in $\mathfrak{A}$.

PROPOSITION 9: *Let A be a fixed inteal in* $\mathfrak{F}$. *Define*:

$$\mathfrak{G}(\mathrm{A}) = \{\mathrm{C} \mid \mathrm{C} \supseteq \mathrm{A}\,;\, \mathrm{C} \in \mathfrak{F}\}$$

$\Rightarrow \exists$ *maximal inteals in* $\mathfrak{G}(\mathrm{A})$.

Proof: $\mathfrak{G}(\mathrm{A})$ is partially ordered with respect to inclusion. $\mathfrak{G}(\mathrm{A})$ is also inductive, because if $\mathrm{C}_1 \subseteq \mathrm{C}_2 \subseteq \cdots$ is a well ordered subset of $\mathfrak{G}(\mathrm{A})$, then $\mathrm{C} = \cup \mathrm{C}_i$ is an upper bound in $\mathfrak{G}(\mathrm{A})$ for this subset, because clearly $\mathrm{C}_i \subseteq \mathrm{C}$ and $\mathrm{A} \subseteq \mathrm{C}$, but since C is an ideal and $1 \notin \mathrm{C} \Rightarrow \mathrm{C} \neq \mathfrak{A}$. Furthermore C is an intule, since from $c \in \mathrm{C} \Rightarrow c \in \mathrm{C}_i$ for some $i \Rightarrow s(c) \in \mathrm{C}_i \subseteq \mathrm{C}$, therefore $\mathrm{C} \in \mathfrak{G}(\mathrm{A})$. Now use Zorn's lemma, etc.

COROLLARY: *There exist maximal inteals in* $\mathfrak{A}$.

Proof: Take for $\mathrm{A} = 0^*$.

Definition 9: An intule (respectively inteal) A is called finitely generated if there exists a finite set E such that $A = \langle E \rangle$ (resp. $A = E^*$).

A *s–d*–ring $\mathfrak{A}$ is called intule– (resp. inteal–) noetherian if every intule (resp. inteal) is finitely generated.

PROPOSITION 9:

$\mathfrak{A}$ intule– (resp. inteal–) noetherian $\Longleftrightarrow$ ascending chain condition holds for intules (resp. inteals) $\Longleftrightarrow$ maximum condition holds for intules (resp. inteals).

Proof: The proof of this proposition is completely analogous to the proof of the same proposition for submodules and is therefore omitted. (See for example ([2]).

THEOREM I.—*If R is ring-noetherian $\Rightarrow \mathfrak{A}$ is intule-noetherian $\Rightarrow \mathfrak{A}$ is inteal-noetherian.*

Proof: The second part of the theorem is immediate: Let A be an inteal $\Rightarrow A = (\langle A \rangle)$, but $\langle A \rangle = \langle E \rangle$, where E is a finite set $\Rightarrow A = (\langle E \rangle) = E^*$.

To prove the first part of the theorem, we show that the ascending chain condition holds in $\mathfrak{A}$. This can be achieved parallel to the proof of the ascending chain condition in Polynomialrings over noetherian rings. We bring the proof to the point where complete analogy is restored. (For comparison see for example [3]).

Let A be an intule. Define:

$\mathfrak{a}_n = \{ \alpha_n \mid \alpha_n \in R \,,\, \alpha_n = n\text{—}th$ component of some $a \in A$ and $\deg(a) \leq n \}$.

LEMMA 1: *$\mathfrak{a}_n$ is an ideal in* R.

Proof: $\alpha_n \,,\, \beta_n \in \mathfrak{a}_n \Rightarrow \exists\, a = \sum_{i=0}^{n} \alpha_i x_i \in A \,,\; b = \sum_{i=0}^{n} \beta_i x_i \in A$ $a - b \in A$ has $\alpha_n - \beta_n$ as $n\text{—}th$ component and $\deg(a - b) \leq n \Rightarrow \alpha_n - \beta_n \in \mathfrak{a}_n$. Similarly $\gamma \cdot \alpha_n$ is the $n\text{—}th$ component of $\gamma \cdot a \,,\, \gamma \in R$, a as above and $\deg(\gamma \cdot a) = \deg(a) \leq n \Rightarrow \gamma \cdot \alpha_n \in \mathfrak{a}_n$.

LEMMA 2: $\mathfrak{a}_n \subseteq \mathfrak{a}_{n+1}$.

Proof: $\alpha_n \in \mathfrak{a}_n \,,\, \exists\, a = \sum_{i=0}^{n} \alpha_i x_i \in A$. Take now $s(a) = \sum_{i=0}^{n} \alpha_i x_{i+1} \in A$ (since A is an intule!), $\deg(s(a)) \leq n + 1$ and α_n is the $n + 1$ component of $s(a)$.

LEMMA 3: *Let $A \subseteq B$; A , B intules in $\mathfrak{A} \Rightarrow \mathfrak{a}_n \subseteq \mathfrak{B}_n$ for all n.*

Proof: Trivial

LEMMA 4: *Let $A \subseteq B$; A , B intules in $\mathfrak{A}$. Suppose $\mathfrak{a}_n = \mathfrak{B}_n$ for all $n \Rightarrow A = B$.*

Proof: Let $b \in B$ and $\deg(b) = 0 \Rightarrow b = \beta_0 \in \mathfrak{B}_0 = \mathfrak{a}_0$ but $\mathfrak{a}_0 = \{ \alpha_0 \mid \alpha_0 = 0\text{-}th$ component of some $a \in A$, $\deg(a) = 0 \} \Rightarrow \alpha_0 = a \in A \Rightarrow \mathfrak{a}_0 \subseteq A \Rightarrow b \in A$.

Assume now that if $c \in B$ and $\deg(c) = n - 1 \Rightarrow c \in A$. Let $b \in B$ and $\deg(b) = n \Rightarrow b = \sum_{i=0}^{n} \beta_i x_i$. We know then that the $n\text{—}th$ component of b namely $\beta_n \in \mathfrak{B}_n = \mathfrak{a}_n \Rightarrow \exists\, a \in A \ni a = \sum_{i=0}^{n} \alpha_i x_i$ with $\alpha_n = \beta_n \Rightarrow a - b = \sum_{i=0}^{n-1} (\alpha_i - \beta_i) x_i \in B$ (since $A \subseteq B$) and $\deg(a - b) \leq n - 1 \Rightarrow a - b \in A$ by hypothesis $\Rightarrow b \in A$. Now the proof continues analogous to the proof mentioned above.

PROPOSITION 10: *If A is an analytic intule in a intule-noetherian ring* $\mathfrak{A}$. $\Rightarrow \exists\, n \ni d^{(i)} A = d^{(n)} A$ *for all* $i \geq n$.

Proof: By proposition 8 *b*. we can construct the chain of intules: $A \subseteq dA \subseteq d^{(2)} A \subseteq \cdots$; using then proposition 9 we get the desired result.

Definition 10:

$$r(A) = \text{radical of the intule } A = \left\{ \alpha \,\middle|\, \begin{array}{l} \alpha \in R \\ \alpha^n \cdot \mathfrak{A} \in A \text{ for some } n \end{array} \right\} \text{ i.e. } r(A) \text{ is}$$

the radical of the submodule A.

Definition 11: An intule A is called primary if and only if for $a \in \mathfrak{A}$, $\beta \in R$; $a \notin A$, $a \cdot \beta \in A \Rightarrow \beta \in r(A)$ i.e. if A is primary as a submodule.

Definition 12: An intule is called irreducible if and only if from $A = B \cap C$; B, C intules $\Rightarrow B = A$ or $C = A$.

PROPOSITION 11: *Let* $\mathfrak{A}$ *be intule-noetherian.*

A *irreducible intule* $\Rightarrow$ A *primary intule.*

Proof: Suppose A not primary $\Rightarrow \exists\, a \in \mathfrak{A}$ and $\beta \in R \ni a \cdot \beta \in A$, $a \notin A$ and $\beta^n \cdot \mathfrak{A} \notin A$ for all $n = 1, 2, \cdots$

Let $A : \beta^n = \{c \mid c \cdot \beta^n \in A\}$ this is clearly a submodule and since $s(c) \cdot \beta^n = s(c \cdot \beta^n) \in A$, it is also an intule. Since we assume the ascending chain condition in $\mathfrak{A} \Rightarrow \exists\, m \ni A : \beta^m = A : \beta^{m+i}$ for all $i = 1, 2, \cdots$.

Assertion: $A = \langle A, a \rangle \cap \langle A, \beta^m \rangle$. This is obviously a contradiction with the irreducibility of A. To prove the assertion, we observe that if

$$c \in \langle A, a \rangle \cap \langle A, \beta^m \rangle \Rightarrow c = a' + \sum_{i=0}^{n} \alpha_i s^{(i)}(a) = a'' + b \cdot \beta^m \text{ where } a', a'' \in A,$$

$\alpha_i \in R$, $b \in \mathfrak{A}$. Multiplying both sides by $\beta : c \cdot \beta = a' \cdot \beta + \sum_{i=1}^{n} \alpha_i s^{(i)}(\beta \cdot a) =$ $= a'' \cdot \beta + b \cdot \beta^{m+1}$. Since $\beta \cdot a$, $a' \cdot \beta$, $a'' \cdot \beta \in A$ and A is an intule $\Rightarrow b \cdot \beta^{m+1} \in A \Rightarrow$ $\Rightarrow b \in A : \beta^{m+1} = A : \beta^m \Rightarrow b \cdot \beta^m \in A \Rightarrow c \in A$.

PROPOSITION 12: $\mathfrak{A}$ *intule-noetherian.*

A *intule* $\Rightarrow A = \bigcap_{\text{finite}} Q_i$ *where* Q_i *is irreducible intule.*

The proof goes completely analogous to the proof of the analogous proposition for submodules.

Definition 13: A representation of an intule $A = \bigcap_{i=1}^{n} Q_i$ into primary intules is called irredundant if

$$(1)\ \bigcap_{\substack{i=1 \\ i \neq j}}^{n} Q_i \not\subset Q_j \qquad \text{and} \quad (2)\ r(Q_i) \neq r(Q_j) \text{ if } i \neq j.$$

These $r(Q_i)$ are called the associated prime ideals of A.

THEOREM II.—*Let* $\mathfrak{A}$ *be intule-noetherian. Then every intule can be represented irredundantly as an intersection of primary intules and the associated prime ideals are uniquely determined.*

Proof: The first part of the theorem is proved with help of propositions 11 and 12 it follows the same pattern as the existence proof for irredundant decompositions for submodules in noetherian modules. The proof of the second statement is entirely independent of the fact that we are dealing with intules and is therefore the same as for submodules. (See [2] pp. 252).

Remark: Theorem II. holds also for inteals, if one replaces in the definitions and proofs the word intule by inteal and appropriately $\langle\ \rangle$ by $*$.

4. DETULES AND DETEALS.—There exists a co-theory to the theory of intules and inteals in a given *s–d*–ring.

Replacing the mapping s by the mapping d in definition 1. we obtain a new object, which we call " d e t u l e ". A detule which is also an ideal, we call " d e t e a l ". The " detule generated by the set E " is obtained by replacing in definition 3. s by d, we denote it by the symbol $\rangle \mathrm{E} \langle$. The duals of proposition 1. and 2. (i.e. replacing intule (resp. inteal) by detule (resp. deteal) and $\langle\ \rangle$ by $\rangle\ \langle$ hold true. The duals of proposition 3 and its corollaries hold also. This is seen by dualizing the proofs given in this text and using the productformula 1.3 instead of formula (F). The " deteal generated by the set E " is therefore also defined. We denote it by E_*. The dual of proposition 4 holds, but instead of the dual of proposition 5 we have:

PROPOSITION 5_*: $\mathrm{E} \subseteq \mathrm{R} \Rightarrow$ a. $\rangle \mathrm{E} \langle = [\mathrm{E}]$,
b. $\mathrm{E}_* = (\mathrm{E}) = \mathrm{E}^*$.

Proof:

(a) $a \in \rangle \mathrm{E} \langle \Rightarrow a = \Sigma \gamma_i d^{(n_i)}(e_i) = \Sigma \gamma_i e_i \in [\mathrm{E}]$, $\gamma_i \in \mathrm{R}$, $e_i \in \mathrm{E} \Rightarrow \rangle \mathrm{E} \langle \subseteq [\mathrm{E}]$, but always $[\mathrm{E}] \subseteq \rangle \mathrm{E} \langle$.

(b) $a \in (\mathrm{E}) \Rightarrow a = \Sigma a_i e_i$; $a_i \in \mathfrak{A}$, $e_i \in \mathrm{E} \Rightarrow da = \Sigma da_i e_i \in (\mathrm{E}) \Rightarrow \mathrm{E}_* = \rangle (\mathrm{E}) \langle = (\mathrm{E})$, and proposition 5 tells us that $(\mathrm{E}) = \mathrm{E}^*$.

PROPOSITION 6_*:

(a) A *is detule* $\Rightarrow d$A *is detule.*

(b) *Let* A *be subset of* $\mathfrak{A}$ *then:*

A *is deteal* $\Longleftrightarrow$ A *is intule and detule* $\Longleftrightarrow$ A *is intule and* $\mathrm{A} = d\mathrm{A} \Longleftrightarrow \mathrm{A} = \mathrm{A}_0^* \Longleftrightarrow \mathrm{A} = (\mathrm{A} \cap \mathrm{R})^* \Longleftrightarrow \mathrm{A} = (\mathrm{A} \cap \mathrm{R}) \Longleftrightarrow \mathrm{A} = (\mathrm{A} \cap \mathrm{R})_*$.

Proof:

(a) is obvious.

(b) We give a cyclic proof. (1) Let A be be a deteal. $a = \sum_{i=1}^{n} \alpha_i x_i \Rightarrow \alpha_n = d^{(n)}(a) \in \mathrm{A} \Rightarrow \alpha_n \cdot x_n \in \mathrm{A} \Rightarrow \sum_{i=0}^{n-1} \alpha_i x_i \in \mathrm{A}$. Repeat! $\Rightarrow \alpha_i \in \mathrm{A}$ for $i = 1, 2, \cdots, n \Rightarrow s(a) = \sum_{i=0}^{n} \alpha_i x_{i+1} \in \mathrm{A} \Rightarrow \mathrm{A}$ is intule and A is detule by definition. (2) From A detule $\Rightarrow d\mathrm{A} \subseteq \mathrm{A}$ and from A intule $\mathrm{A} \subseteq d\mathrm{A}$ by proposition 8b. $\Rightarrow \mathrm{A} = d\mathrm{A}$. (3) If A is intule and $\mathrm{A} = d\mathrm{A} \Rightarrow \mathrm{A}_i(\mathrm{A}) = \mathrm{A}_i(d\mathrm{A}) = \mathrm{A}_{i+1}(\mathrm{A})$ by proposition 7 d., therefore $\mathrm{A}_i = \mathrm{A}_0$ and $\mathrm{A}_0 \subset \mathrm{A}$ by proposition 8c. $\Rightarrow \mathrm{A}_0^* = \langle \mathrm{A}_0 \rangle \subseteq \mathrm{A}$. Now if

$a \in \mathrm{A}$ and $a = \sum_{i=0}^{n} \alpha_i x_i \Rightarrow \alpha_i \in \mathrm{A}_i = \mathrm{A}_0 \Rightarrow \mathrm{A} \subseteq \mathrm{A}_0^* \Rightarrow \mathrm{A} = \mathrm{A}_0^*$. (4) If $\mathrm{A} = \mathrm{A}_0^* \Rightarrow$ $\Rightarrow \mathrm{A}_0 \subseteq \mathrm{A} \cap \mathrm{R}$ and if $\alpha \in \mathrm{A} \cap \mathrm{R} \Rightarrow \alpha = \alpha_0 \in \mathrm{A}_0 \Rightarrow \mathrm{A}_0 = \mathrm{A} \cap \mathrm{R} \Rightarrow \mathrm{A} = (\mathrm{A} \cap \mathrm{R})^*$. (5) and (6) follow from proposition 5_* by taking for $\mathrm{E} = \mathrm{A} \cap \mathrm{R}$. (7) The cycle closes with the trivial statement that from $\mathrm{A} = (\mathrm{A} \cap \mathrm{R})_*$ follows A is deteal.

COROLLARY 1: *There is a* 1—1 *correspondence between the deteals in* $\mathfrak{A}$ *and the ideals in R established by*:

$$\mathrm{A}\ (\textit{deteal in } \mathfrak{A}) \to \bar{\mathrm{A}} = \mathrm{A} \cap \mathrm{R}\ (\textit{ideal in } \mathrm{R}),$$

$$\bar{\mathrm{A}}\ (\textit{ideal in } \mathrm{R}) \to \bar{\mathrm{A}}_*\ (\textit{deteal in } \mathfrak{A}).$$

Proof: To show that this is a 1—1 correspondence we must show:

(1) A deteal $\Rightarrow (\mathrm{A} \cap \mathrm{R})_* = \mathrm{A}$ which is true by proposition 6_*b.

(2) From $\bar{\mathrm{A}}$ ideal in R should follow: $\bar{\mathrm{A}} = \bar{\mathrm{A}}_* \cap \mathrm{R}$. Now $\bar{\mathrm{A}}_* = (\bar{\mathrm{A}})$ by proposition 5_*, so for $a \in \bar{\mathrm{A}}_* \Rightarrow a = \Sigma\, a_i e_i$, $e_i \in \bar{\mathrm{A}}$. Let $a_i = \Sigma\, \alpha_{ik} x_k \Rightarrow$ $\Rightarrow a = \Sigma\, \alpha_{ik} e_i x_k$, but $a \in \mathrm{R} \Rightarrow a = \Sigma\, \alpha_{i0} e_i \Rightarrow a \in \bar{\mathrm{A}}$ (ideal !) $\Rightarrow \bar{\mathrm{A}} \supseteq \bar{\mathrm{A}}_* \cap \mathrm{R}$ and naturally $\bar{\mathrm{A}} \subseteq \bar{\mathrm{A}}_* \cap \mathrm{R}$.

COROLLARY 2: *Each ideal of* R *is the contraction of its ideal-extension to the over-ring* $\mathfrak{A}$.

Proof: This was just proved at the end of the proof to corollary 1.

COROLLARY 3: *A a subset of* $\mathfrak{A}$ *is equal to the ideal-extension of its contraction if and only if* A *is a deteal.*

Proof: A deteal $\Rightarrow (\mathrm{A} \cap \mathrm{R}) = \mathrm{A}$ by proposition 6_*. If $\mathrm{A} = (\mathrm{A} \cap \mathrm{R})$ then A is deteal by the same proposition.

The detule-structure is in general non-noetherian, however, if the minimal condition holds in R, then it holds also for detules.

In a further "Nota II" we shall study *d*–homomorphism, *s*–homomorphism, *s*–*d*–homomorphism and *s*–rings. We shall introduce a product in the detule-intule structure. Quotients will also be defined and further structure theorems can then be proved. Furthermore we propose to exhibit some examples.

LITERATURE

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