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Divisors on equisingular hypersurfaces

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Matematica. — *Divisors on equisingular hypersurfaces* (*). Nota (**) di SHREERAM SHANKAR ABHYANKAR, presentata dal Socio straniero O. ZARISKI.

SUNTO. — Si stabilisce un caso particolare di un teorema (da dimostrarsi altrove) che caratterizza localmente il luogo singolare - supposto $(n-1)$-dimensionale - di un'ipersuperficie algebroide di dimensione n, nell'ipotesi che questa risulti equisingolare lungo quello nel senso di Zariski.

Let R be an n-dimensional ($n > 1$) algebroid hypersurface having a singular point at its origin M, over an algebraically closed ground field k. Assume that the singular locus H of R is $(n-1)$-dimensional and has a simple point at M. Also assume that R is equisingular along H at M in the sense of Zariski [1]. Recently we came across a rather interesting property embodied in the following theorem.

THEOREM. H *is the only* $(n-1)$*-dimensional subvariety* H′ *of* R *which has a simple point at* M *and is set-theoretically complete intersection of* R (with another hypersurface R′).

The assertion being obvious when R is (analytically) reducible, we suppose that R is irreducible. The theorem will be proved in all generality elsewhere. Here we only prove it in the special case in which H′ is assumed to be set-theoretically complete intersection of R with a hypersurface R′ *having a simple point at* M.

Let e be the multiplicity of R at M. We can choose local coordinates $(Z, X_1, \cdots, X_n)$ so that R′ is given by $Z=0$, and R is given by $f(Z)=0$ where $f(Z)$ is a monic polynomial of degree e in Z with coefficients in the power series ring $S=k[[X_1, \cdots, X_n]]$. Let N be the maximal ideal in S. Since H′ is contained in R′ and H′ has a simple point at M, we can find $x' \in N$ with $x' \notin N^2$ such that H′ is given by $Z=0=x'$. Since H′ is the complete set-theoretic intersection of R with R′, we must have $f(0)= d'x'^c$ where d' is a unit in S and c is a positive integer.

Since R is equisingular along H at M, we can choose [1] a basis $(x, x_2, \cdots, x_n)$ of N such that H is given by $f(Z)=0= x$, and the Z-discriminant of $f(Z)$ equals dx^b where d is a unit in S and b is a positive integer.

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(1) The concept of equisingularity has been introduced and studied by ZARISKI in the paper: *Studies in equisingularity*. II, «American Journal of Mathematics», vol. 87 (1965), pp. 972–1006. We shall freely use the results, especially the discriminant criterion, given in this paper.

Let $z_1, \cdots, z_e$ be the roots of $f(z)$. Then [1]

$$z_1 = r + \sum_{i=m}^{\infty} r_i x^{i/e} \tag{1}$$

where $r \in N$, m is a positive integer with $m > e$ and $m \neq 0 \mod e$, and r_m, $r_{m+1}, r_{m+2}, \cdots$ are elements in S with $r_m \notin N$. Let $S^* = S[x^{1/e}]$ and $P = x^{1/e} S^*$. Then S^* is an n–dimensional regular local ring, $(x^{1/e}, x_2, \cdots, x_n)$ is a basis of the maximal ideal in S^*, P is a nonzero principal prime ideal in S^*, and

$$(z_1 - z_i) S^* = P^{b_i} \qquad \text{for} \quad 1 < i \leq e \tag{2}$$

where b_i is a positive integer.

Suppose, if possible, that $H' \neq H$. Then $x' S \neq x S$ and hence $x' S = P_1 \cdots P_u$ where $P_1, \cdots, P_u$ $(1 \leq u \leq e)$ are distinct nonzero principal prime ideals in S^* with $P_j \neq P$ for $1 \leq j \leq u$. Now $(-1)^e z_1 \cdots z_e = f(0) = d' x'^c$, and hence, upon relabelling $P_1, \cdots, P_u$ suitably, we have

$$z_1 S^* = P_1^{a_1} \cdots P_v^{a_v}$$

where $v, a_1, \cdots, a_v$ are positive integers with $v \leq u$. There exists an S–automorphism g of S^* such that $g(P_v) = P_1$. Now $g(z_1) = z_w$ for some w with $1 \leq w \leq e$, and $z_1 - z_w \in P_1$; consequently by (2) we must have $w = 1$; therefore g must be the identity automorphism and hence $v = 1$. If $u < e$ then there would exist a nonidentical S–automorphism h of S^* such that $h(P_1) = P_1$; now $h(z_1) = z_i$ for some i with $1 < i \leq e$, and $z_1 - z_i \in P_1$ which would contradict (2). Therefore $u = e$.

If $x' S + N^2 \neq x S + N^2$ then we can find elements $y_3, \cdots, y_n$ in S such that $(x, x', y_3, \cdots, y_n)$ is a basis of N; now $(x^{1/e}, x', y_3, \cdots, y_n)$ is a basis of the maximal ideal in S^* and hence in particular $x' S^*$ is a prime ideal in S^*; this is a contradiction because $u = e > 1$. Therefore $x' S + N^2 = x S + N^2$.

Let $S' = k[[x_2, \cdots, x_n]]$ and let N' be the maximal ideal in S'. Since $x' S + N^2 = x S + N^2$ and $x' S \neq x S$, by the Weierstrass preparation theorem, we have $x' S = (x + t^*) S$ with $0 \neq t^* \in N'$. Since $x' S^* = P_1 \cdots P_u$ with $u = e$, we must have $t^* = t'^e$ with $0 \neq t' \in N'$. It follows that $P_1 = (x^{1/e} + t) S^*$ with $0 \neq t \in N'$; (namely, $t = t_1 t'$ where t_1 is an e^{th} root of 1).

Now

$$z_1 = s (x^{1/e} + t)^{a_1} \tag{3}$$

where s is a unit in S^*. We have

$$s = s_0 + s_1 x^{1/e} + s_2 x^{2/e} + \cdots \tag{4}$$

where s_0 is a unit in S', and $s_1, s_2, \cdots$ are elements in S'. By (3) and (4) we get

$$z_1 = q_0 + q_1 x^{1/e} + q_2 x^{1/e} + \cdots \tag{5}$$

where $q_0, q_1, q_2, \cdots$ are elements in S' with $q_1 \neq 0$. (1) and (5) lead to a contradiction. Therefore $H' = H$.