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Gaetano Fichera

On the compactness of the base operator in the theory of intermediate problems

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RENDICONTI

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NOTE DI SOCI

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Analisi matematica. — *On the compactness of the base operator in the theory of intermediate problems.* Nota (*) del Corrisp. GAETANO FICHERA.

RIASSUNTO. – Si prova un teorema relativo a particolari operatori di base non compatti.

Let T be a positive (1) compact (linear) operator (PCO) in the complex Hilbert space S. S is supposed to be separable and infinite dimensional. Let $\{\mu_k\}$ be the sequence of the eigenvalues of T. We suppose that this sequence is ordered in the usual manner, i.e.,

$$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_k \geq \cdots$$

each eigenvalue of T appearing in the sequence as many times as its multiplicity. From now on, when we consider the sequence of the eigenvalues of a PCO, we suppose that it is ordered according to the above criterion.

In [4], [5], [6] it is defined as "*base operator*" for the operator T (see [1], [7]) any positive compact linear operator T_0 such that, if we denote by $\{\sigma_k^{(0)}\}$ the sequence of its eigenvalues, we have

$$\sigma_k^{(0)} \geq \mu_k. \qquad (k = 1, 2, \cdots). \tag{1}$$

(*) Pervenuta all'Accademia il 29 agosto 1966.

(1) For simplicity we assume the term *positive* as *strictly positive*, i.e., $(Tu, u) > 0$ for any $u \neq 0$. We recall that any (linear) positive operator is hermitian, i.e., $(Tu, v) = (u, Tv)$ for any pair of vectors of S.

Conditions (1) are satisfied if

$$T_0 \geq T . \tag{2}$$

Let us assume that (2) *holds*. Then it has been shown (see [1], [3], [6], [7]) that it is possible to construct a sequence $\{T_n\}$ of PCO's (*intermediate operators*) satisfying the following conditions:

i) Let $\{\sigma_k^{(n)}\}$ $(k = 1, 2, \cdots)$ be the sequence of the eigenvalues of T_n, then

$$\sigma_k^{(0)} \geq \sigma_k^{(n)} \geq \sigma_k^{(n+1)} \geq \mu_k ;$$

ii) $T_0 - T_n$ is degenerate, i.e., its range is finite dimensional;

iii) $\lim_{n \to \infty} \sigma_k^{(n)} = \mu_k$ $(k = 1, 2, \cdots)$;

iv) It is possible to compute the $\sigma_k^{(n)}$'s in terms of the eigenvalues $\sigma_k^{(0)}$ and the eigenvectors of T_0.

It follows that, in order to apply the *method of intermediate problems* for the upper approximations of the μ_k's, the base operator T_0 must be known, *in the sense that all its eigenvalues and eigenvectors must be considered as data of the problem.*

No doubt this is a serious limitation in the applicability of the method of intermediate problems.

Let T be an integral operator in the Hilbert space $\mathfrak{L}^2(0, 1)$

$$Tu = \int_0^1 k(x, y)\, u(y)\, dy ;$$

$k(x, y)$ is a kernel which belongs to $\mathfrak{L}^2[(0, 1) \times (0, 1)]$, which is hermitian (i.e., $k(x, y) = \overline{k(y, x)}$) and of "*positive type*".

In [4] (p. 39), [5] (p. 332) and [6] (pp. 139–140) I made the following remark: in general, there is no known base operator, T_0, which can be used in applying the method of intermediate problems for the upper approximation of the eigenvalues of T.

Weinstein, in his paper [10] (p. 191), writes: *Fichera states, see* [42] (2), *that there is in general no known base problem for a compact positive integral operator. This statement is incorrect, as will be shown in a forthcoming paper by Weinstein.*

More recently, in the volume of the Abstracts of the Brief Scientific Communications to the Moscow International Congress of Mathematicians (August 16–26, 1966; see fasc. 5—*Functional Analysis*, section 5—pp. 30–31) the following Abstract is included.

(2) In the References List of WEINSTEIN's paper [10], paper [42] corresponds to paper [4] of the References List of the present note.

WEINSTEIN ALEXANDER—ON THE EXISTENCE OF A BASE PROBLEM FOR EIGENVALUES.

Let K *be a symmetric compact positive operator in a Hilbert space,* $0 < (Ku, u) \leq c(u, u)$. *Let* G *be a compact symmetric operator with known or prescribed positive eigenvalues and eigenvectors. Let* I *be the identity. Then* $K = G + cI + (K - cI - G)$. *As* $K - cI - G$ *is negative,* $G + cI$ *is a base operator and the method of intermediate problems applies (see for instance S. H. Gould, Variational Methods, An Introduction to Weinstein's Theory of Intermediate Problems, second edition, 1966). Fichera's statement (Linear Elliptic Differential Systems and Eigenvalue Problems, Springer 1965, p. 139) that for a given* K *there is in general no known base problem is incorrect and is due to the limitation to compact base operators.*

The aim of this note is to show that the condition of compactness for the base operator cannot be so easily relaxed. Otherwise the method of intermediate problems could give completely useless results. That is exactly what happens if the base operator $G + cI$, proposed by Weinstein, is used.

Let us first recall the following lemma.

I.—*Let* G *be a hermitian* (3) *positive operator of the Hilbert space* S. *Let* D *be a degenerate hermitian operator. The minimal subspace of* S *containing all the eigenvectors of* $H = G - D$ *corresponding to negative eigenvalues is finite dimensional.*

Since H is a hermitian operator, it is well known that we can decompose the space S as direct sum of two mutually orthogonal subspaces S^+ and S^- each of them being an invariant subspace for H. Moreover the restriction H^+ of H to S^+ is a positive operator in S^+ and the restriction H^- of H to S^- is a negative operator in S^- (4). In this context positiveness and negativeness must not be understood in the strict sense. It follows that any eigenvector of H corresponding to a negative eigenvalue must belong to S^-. Let P^- be the orthogonal projector of S onto S^-. We have $H^- = P^-GP^- - P^-DP^-$. Let R^- be the range of P^-DP^- and Π the orthogonal projector of S^- onto $S^- \ominus R^-$. Since we have in $S^-: P^-GP^- \leq P^-DP^-$, it follows $0 \leq \Pi P^-GP^-\Pi \leq \Pi P^-DP^-\Pi = 0$. On the other hand we have $P^-GP^- = \Pi P^-GP^-\Pi + (P^- - \Pi)P^-GP^-(P^- - \Pi) + (P^- - \Pi)P^-GP^-\Pi + \Pi P^-GP^-(P^- - \Pi)$. Since $P^- - \Pi$ projects S^- onto the finite dimensional space R^-, it follows that P^-GP^- is degenerate, i.e., H^- is degenerate. From this the proof follows.

Let now T be an arbitrary PCO and let c be a constant such that $(Tu, u) \leq c(u, u)$ (5). Let T_0 be a not necessarily compact base operator for

(3) The term *hermitian* is used in this paper instead of the term *symmetric*, used by Weinstein. It is well known that any hermitian operator is bounded.

(4) See [8] p. 180, [9] p. 274.

(5) According to our previous notation we use the letter T instead of K used by Weinstein.

T of the kind considered by Weinstein. Thus we assume $T_0 = G + \sigma_0 I > T$, where G is a PCO and σ_0 a nonnegative constant. We have compactness for T_0 when and only when $\sigma_0 = 0$. The operator T_0 admits the following spectral decomposition

$$T_0 u = \sum_{k=1}^{\infty} \sigma_k^{(0)} (u , u_k) u_k$$

where $\sigma_k^{(0)} = \gamma_k + \sigma_0$ and $\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_k \geq \cdots$ denote the eigenvalues of G and $u_1 , u_2 , \cdots , u_k , \cdots$ the corresponding eigenvectors.

Let us now consider the sequence of intermediate operators T_n

$$T_0 \geq T_n \geq T_{n+1} > T ,$$

$$T_n = T_0 - D_n ;$$

D_n being constructed using the Aronszajn device (see [1], [7]) or some alternative method (see [6], p. 135). In what follows the method of construction of D_n is irrelevant. What matters is the fact that D_n is degenerate. According to our previous notations, $\sigma_1^{(n)}$ denotes the greatest eigenvalue of T_n and—in general—$\sigma_k^{(n)}$ denotes the maximum of the quadratic form $(T_n u , u)$ under the conditions $(u , u) = 1$ and $(u , u_1^{(n)}) = \cdots = (u , u_{k-1}^{(n)}) = 0$, where $u_s^{(n)}$ $(s = 1 , 2 , \cdots)$ is eigenvector of T_n corresponding to $\sigma_s^{(n)}$.

From the above lemma it follows that $G - D_n$ has at most a finite number of linearly independent eigenvectors corresponding to negative eigenvalues. Then the greatest eigenvalue of $G - D_n$ is nonnegative and, in general, we have that the maximum of $((G - D_n) u , u)$ under the conditions $(u , u) = 1$, $(u , u_1^{(n)}) = \cdots = (u , u_{k-1}^{(n)}) = 0$ is nonnegative. It follows that

$$\sigma_k^{(n)} \geq \sigma_0 .$$

Then

II.—*If* T_0 *is not compact, i.e., if* $\sigma_0 > 0$, *the upper approximations given by the method of intermediate problems are never less than* σ_0.

Assuming the base operator proposed by Weinstein, i.e., $T_0 = G + cI$, since $\sigma_0 = c \geq \frac{(Tu , u)}{(u , u)}$, the best information that the method of intermediate problems is able to give is that $\mu_k \leq c$.

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