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# On the existence, uniqueness, stability and approximation of solutions of Prandtl's system for the nonstationary boundary layer

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**Analisi matematica.** — *On the existence, uniqueness, stability and approximation of solutions of Prandtl's system for the nonstationary boundary layer.* Nota di OLGA OLEINIK, presentata (*) dal Corrisp. G. FICHERA.

RIASSUNTO. — Vengono provate l'esistenza, l'unicità e la stabilità delle soluzioni del sistema di Prandtl per lo strato limite non stazionario bidimensionale sotto certe condizioni per i dati. Si dànno due schemi, alle differenze finite, convergenti per l'approssimazione di queste soluzioni.

A system of equations for the boundary layer was suggested by Prandtl in 1904. This boundary layer system is very important for many problems of mechanics concerning the flow of a viscous fluid (with small viscosity) around a body. However there are only a few mathematical papers about Prandtl's system, (see, for example, [1], [2], [3]). In this paper we consider the question of existence and uniqueness of solutions of the Prandtl system, and also the behaviour of these solutions when $t \to \infty$, for two dimensional nonstationary flow of incompressible fluid. We will also outline a proof of the convergence of finite-difference approximate solutions to the solution of the Prandtl system.

The Prandtl system of equations for a nonstationary flow of viscous incompressible fluid has the form

$$u_t + u\,u_x + vu_y = -p_x + \nu u_{yy}\,, \qquad u_x + v_y = 0, \tag{1}$$

where $u\,,v$ are the components of the velocity, $p(t\,,x)$ is the pressure, $\nu = \text{const}$, and where we assume that the density $\rho = 1$. This system is considered in the domain $\mathfrak{D}\,\{0 \le t \le t_0\,, 0 \le x \le x_0\,,\ 0 \le y < \infty\}$ under the conditions:

$$u\,|_{t=0} = u_0(x\,,y)\quad,\quad u\,|_{y=0} = 0\quad,\quad v\,|_{y=0} = v_0(t\,,x)\quad,\quad u|_{x=0} = u_1(t\,,y), \tag{2}$$

$$\lim_{y\to\infty} u(t\,,x\,,y) = U(t\,,x), \tag{3}$$

where as usual $U(t\,,x)$ is a given component of the exterior flow velocity, related by Bernoulli's law to the pressure $p(t\,,x)$ by $-p_x = U_t + UU_x$. By physical considerations it is necessary that $u > 0$ for $y > 0$ and $u_y > 0$ for $y = 0$. We will also suppose that $u_0 > 0$ and $u_1 > 0$ for $y > 0$, and $u_{0y} > 0\,, u_{1y} > 0$ for $y \ge 0$; $U(t\,,x) > 0$.

We remark that the Prandtl system of the boundary layer for axial-symmetric two-dimensional and three-dimensional flows can also be similarly

(*) Nella seduta del 22 giugno 1966.

investigated by the method of this paper. For these cases we assume $u|_{x=0} = 0$ in conditions (2) which is required by physical reasoning.

We prove under some natural conditions on the smoothness of the functions $u_0$, $u_1$, $v_0$, $p$, $U$ and some natural compatibility conditions for $u_0$, $u_1$, $v_0$ that there exists a solution of the problem (1), (2), (3) in the domain $\mathfrak{D}$ for any $x_0 \leq \infty$ when $t_0$ is sufficiently small or for any $t_0 \leq \infty$ when $x_0$ is sufficiently small. It is well-known that the solution of the problem (1), (2), (3) sometimes does not exist in $\mathfrak{D}$ for certain (larger) $x_0$ and $t_0$ because then the separation of the boundary layer appears. We show also that the solution of the problem (1), (2), (3) depends continuously on the given functions $u_0$, $u_1$, $v_0$, $p$, $U$ and that this solution is stable as $t \to \infty$. If the functions $u_1$, $v_0$, $p$, $U$ have limits when $t \to \infty$, then the flow becomes steady there; i.e., the velocity component $u(t, x, y)$ tends to a limit which is the velocity component $u(t, x)$ of the corresponding stationary flow.

In order to study the problem (1), (2), (3) we introduce the new independent variables

$$\tau = t \quad , \quad x = \xi \quad , \quad \eta = u(t, x, y)$$

and a new unknown function

$$w = u_y .$$

Then for $w$ we have the equation

$$L(w) \equiv \nu w^2 w_{\eta\eta} - w_\tau - \eta w_\xi + p_x w_\eta = 0. \tag{4}$$

The domain $\mathfrak{D}$ maps into the domain $\Omega\{0 \leq \tau \leq t_0, 0 \leq \xi \leq x_0, 0 \leq \eta < < U(\tau, \xi)\}$.

The conditions (2), (3) on the boundary of $\mathfrak{D}$ correspond to the following conditions on the boundary of $\Omega$:

$$w|_{\tau=0} = u_{0y} \equiv w_0(\xi, \eta), \; w|_{\xi=0} = u_{1y} \equiv w_1(\tau, \eta), \; w|_{\eta=U(\tau,\xi)} = 0, \tag{5}$$

$$l(w) \equiv \nu w w_\eta - v_0 w - p_x = 0 \quad \text{for } \eta = 0. \tag{6}$$

Let us remark that the quasilinear equation (4) is of elliptic-parabolic type. Linear elliptic-parabolic equations were considered in G. Fichera's papers [4], [5], (see also [6]).

The solution of the non linear problem (4), (5), (6) is obtained as a limit of functions $w^n$, $(n = 1, 2, \cdots)$ which are defined in $\Omega$ as the solutions of the equations

$$L_n(w^n) \equiv \nu (w^{n-1})^2 w^n_{\eta\eta} - w^n_\tau - \eta w^n_\xi + p_x w^n_\eta = 0 \tag{7}$$

with the boundary conditions

$$w^n|_{\tau=0} = w_0 \;, \; w^n|_{\xi=0} = w_1 \;, \; w^n|_{\eta=U(\tau,\xi)} = 0, \tag{8}$$

$$l_n(w^n) \equiv \nu w^{n-1} w^n_\eta - v_0 w^{n-1} - p_x = 0 \quad \text{for } \eta = 0. \tag{9}$$

We let $w^0$ be a smooth extension of the function $w^*$ which we will define later, $w^0 = 0$ for $\eta = U(\tau, \xi)$, $w^0 > 0$ for $\eta = 0$. Let us suppose that the function $w^{n-1}$ with bounded fourth derivatives in $\Omega$ is defined such that $w^{n-1} > 0$ for $\eta = 0$. We now construct $w^n$ as follows. We define a boundary value problem of Neumann type for second order elliptic equations with a small parameter $\varepsilon$ in a special domain; the solutions $w^{\varepsilon, n}$ of these problems tend, when $\varepsilon \to 0$, to the solution $w^n$ of the problem (7), (8), (9).

Let us suppose that $x_0$ and $t_0$ are finite. Let G be an infinitely differentiable bounded domain in the plane $\xi, \eta$, let the cylinder $[0, t_0] \times G$ contain $\Omega$, and let the boundary $\sigma$ of the domain G contain the segment $[-2\delta, x_0 + 2\delta]$ of the axis $\xi$, where $\delta > 0$ is a small number. Let us suppose that in some neighborhood of the point, which is the intersection of $\sigma$ with $\xi = 0$, the boundary $\sigma$ is situated on the straight line $\eta = \eta_1 = \text{const}$. Let us consider the simply connected infinitely differentiable domain Q, with boundary S, which coincides with the cylinder $[-1, t_0 + 1] \times G$ when $-1 \leq \tau \leq$ $\leq t_0 + 1$ and which is contained in the cylinder $[-2, t_0 + 2] \times G$. We denote by $\Omega_1$ the points of Q for which $\tau \geq 0$ and $\xi \geq 0$ or for which $\tau \geq t_0$. Let us extend in a smooth way the coefficient $P_x$ in (7) and the functions $v_0$ and $p_x$ in (9) for all values of $\xi$ and $\tau$. We introduce the notation: $S_1 = \{\tau = 0 \, , \, 0 \leq \xi \leq x_0 \, , \, 0 \leq \eta \leq U(0, \xi)\}$ , $S_2 = \{0 \leq \tau \leq t_0 \, , \, \xi = 0 \, ,$ $0 \leq \eta \leq U(\tau, 0)\}$ and $S_0 = \{0 \leq \tau \leq t_0 \, , \, 0 \leq \xi \leq x_0 \, , \, \eta = 0\}$.

Let us suppose that there exits a smooth function $w^*$, defined in the closure of $Q - \Omega$, which satisfies the following conditions:

$$w^* = w_0 \quad \text{on} \quad S_1 \quad , \quad w^* = w_1 \quad \text{on} \quad S_2 \, ,$$

$$L(w^*) = O(\xi^4) \tag{10}$$

for $\xi \leq 0$ and $\tau \geq 0$ in a neighborhood of $S_2$ (three dimensional);

$$L(w^*) = O(\tau^4) \tag{11}$$

for $\xi \geq 0$ , $\tau \leq 0$ in a neighborhood of $S_1$;

$$l(w^*) = O(\xi^4) \tag{12}$$

on S in a neighborhood of the segment $[0, t_0]$ of $\tau$-axis, and

$$l(w^*) = O(\tau^4) \tag{13}$$

on S in a neighborhood of the segment $[0, x_0]$ of $\xi$-axis. We can suppose that $w^*$ has continuous derivatives of sixth order in the closure of $Q - \Omega_1$ and $w^*$ is an infinitely differentiable function outside of some neighborhood of $S_1$ and $S_2$. Such a function $w^*$ can be constructed, if $w_0, w_1, v_0, p_x$ are sufficiently smooth and, in addition, if $w_0, w_1, v_0$ satisfy some compatibility conditions on the axis of $\tau, \xi, \eta$ and on $\eta = U(0, \xi)$ in accordance with the equation (4) and the boundary conditions (5), (6). The function $w^*$ can be represented by a truncated Taylor's series (with respect to $\tau$ in a neighborhood of $S_1$, with respect to $\xi$ in a neighborhood of $S_2$), using for the definition of the coefficients the procedure which is usually used in the Cauchy–Kovalevsky Theorem. These coefficients can be expressed in terms of $w_0$, $w_1$ and their derivatives. In this way the condition (10) and (11) will be satisfied, and the condition (12) and (13) can be satisfied due to the compatibility conditions.

Let us denote by $\sigma_\delta$ the points of $\sigma$ which do not belong to the segment $[-2\delta, x_0 + 2\delta]$ of $\xi$-axis, and let $S^\delta = [-1, t_0 + 1] \times \sigma_\delta$. In Q we consider the elliptic operator

$$L^\varepsilon(w) \equiv \varepsilon(w_{\tau\tau} + w_{\xi\xi} + w_{\eta\eta}) + a_1 w_{\tau\tau} + a_2 w_{\xi\xi} + a_3 w_{\eta\eta} + \nu (w^{n-1})^2_\varepsilon w_{\eta\eta} -$$

$$- w_\tau - \eta w_\xi + (p_x)_\varepsilon w_\eta - 2(a_1 + \varepsilon) w ,$$

where $\varepsilon > 0$. The nonnegative infinitely differentiable functions $a_1$, $a_2$, $a_3$ are positive for $\tau < -1/2$ and for $\tau > t_0 + \delta$; $a_3$ is positive in the $\delta$-neighborhood of $S^\delta$; $a_2$ is positive at all points of this neighborhood of $S^\delta$, which do not belong to the plane $\xi = 0$; and at the rest of the points of Q the functions $a_1$, $a_2$, $a_3$ are equal to zero. (The number $\delta$ is taken so small that $a_1$, $a_2$, $a_3$ are equal to zero in $\Omega$). The function $w^{n-1} = w^*$ in $Q - \Omega_1$ and is a smooth extension of this function in $\Omega_1 - \Omega$. We denote by $(f)_\varepsilon$ the average (e.g., see [7]) function for $f$ with radius $\varepsilon$ and a nonnegative infinitely differentiable kernel. Let us consider in the domain Q the boundary value problem for the elliptic equation

$$L^\varepsilon(w) = (\mathcal{F})_\varepsilon \tag{14}$$

with boundary condition on S

$$\frac{\partial w}{\partial \bar{n}} = (\Phi)_\varepsilon , \tag{15}$$

where $\bar{n}$ is the internal normal direction on S. The function $\mathcal{F}$ in (14) is defined in Q in the following way: $\mathcal{F} = L(w^*) + a_1 w^*_{\tau\tau} + a_2 w^*_{\xi\xi} + a_3 w^*_{\eta\eta} - 2 a_1 w^*$ in $Q - \Omega_1$; $\mathcal{F} = 0$ in $\Omega$; and in $\Omega_1 - \Omega$ the function $\mathcal{F}$ is any sufficiently smooth extension of this function, (with bounded derivatives up to and including order 4).

The function $\Phi$ in (15) is equal to $\frac{1}{\nu}\left(v_0 + \frac{p_x}{w^{n-1}}\right)$ on $S_0$; $\Phi = \frac{\partial w^*}{c\bar{n}}$ on the intersection S with the boundary of $Q - \Omega_1$; on the rest of S the function $\Phi$ is any sufficiently smooth extension of this function. It is easy to see that $\mathcal{F}$ has bounded fourth derivatives in Q and is an infinitely differentiable function outside of some neighborhood of $\Omega$. The function $\Phi$ has bounded fourth derivatives on S and is an infinitely differentiable function outside some neighborhood of $S_0$ due to the properties of the function $w^*$. The boundary value problem (14), (15) has a unique infinitely differentiable solution $w^{\varepsilon,n}$ in Q, since the coefficients in (14), the functions $(\mathcal{F})_\varepsilon$ and $(\Phi)_\varepsilon$, and the boundary S of Q are infinitely differentiable (see, for example, [8], [9]).

One can prove that all derivatives of $w^{\varepsilon,n}$ up to the order 4 are uniformly bounded with respect to $\varepsilon$. Therefore there exists a convergent subsequence $w^{\varepsilon_k,n}$, where $\varepsilon_k \to 0$, and the limit function $w^n$ satisfies the equation (7) and the boundary condition (9). Using the maximum principle, one can prove that $w^n = w^*$ in $Q - \Omega_1$ if $w^{n-1} = w^*$ in $Q - \Omega_1$. Therefore $w^n$ satisfies the boundary conditions (8) for $\xi = 0$ and for $\tau = 0$. On the surface $\eta = U(\tau, \xi)$ $w^n_\tau + \eta w^n_\xi - p_x w^n_\eta \equiv \frac{\partial w^n}{\partial l} = 0$, where the vector $l$ is tangent to the surface $\eta = U(\tau, \xi)$, and $w^n = 0$ on this surface when $\tau = 0$ or when $\xi = 0$. Thus $w^n = 0$ for $\eta = U(\tau, \xi)$.

*Remark.*—Taking into account the existence of the smooth solution $w^n$ of the problem (7), (8), (9), one can give simpler (than the construction of $w^{\varepsilon,n}$) methods for the construction of approximations to $w^n$. In particular, the finite-difference method can be used.

In order to prove the existence of a solution of the problem (4), (5), (6) it is sufficient to prove that the function $w^n$ and their derivatives up to the order 2 are uniformly bounded with respect to $n$.

LEMMA 1. — *There exists a constant* $\tau_0 > 0$ *such that for all* $n$, $(n = 1, 2, \cdots)$, *the inequality*

$$H_1(\tau, \xi, \eta) \geq w^n \geq h_1(\tau, \xi, \eta) \tag{16}$$

*holds in that part of* $\Omega$ *for which* $\tau \leq \tau_0$, *where* $H_1$ *is a bounded function in* $\Omega$ *and the continuous function* $h_1$ *is positive for* $\eta < U(\tau, \xi)$, $\tau \leq \tau_0$. *There exists a constant* $\xi_0 > 0$ *such that, for all n, the inequality*

$$H_2(\tau, \xi, \eta) \geq w^n \geq h_2(\tau, \xi, \eta) \tag{17}$$

*holds in that part of* $\Omega$ *for which* $\xi \leq \xi_0$, *where* $H_2$ *is bounded in* $\Omega$ *and the continuous function* $h_2$ *is positive for* $\eta < U(\tau, \xi)$ *and* $\xi \leq \xi_0$.

In order to estimate the derivatives of $w^n$ of the first and second order, we consider the function $W^n = w^n e^{\alpha\eta}$ where $\alpha > 0$ is a constant to be chosen later. We introduce the function

$$\Phi_n = (W^n_\tau)^2 + (W^n_\xi)^2 + W^n_\eta (W^n_\eta - 2\psi^n) + K_0 + K_1 \eta,$$

where

$$\psi^n = \left[\frac{1}{\nu} v_0 + \frac{1}{\nu W^{n-1}} p_x + \alpha W^n\right] \chi(\eta),$$

$\chi(\eta)$ is a smooth function, $\chi(\eta) = 1$ for $\eta \leq \frac{\delta_0}{2}$ and $\chi(\eta) = 0$ for $\eta > \delta_0$, $\delta_0 = \frac{1}{2} \min U(\tau, \xi)$, and $K_0$ and $K_1$ are some positive constants. We also introduce the function

$$\mathcal{F}_n = (W^n_{\tau\tau})^2 + (W^n_{\xi\xi})^2 + (W^n_{\tau\xi})^2 + W^n_{\xi\eta}(W^n_{\xi\eta} - 2\psi^n_\xi) +$$

$$+ W^n_{\tau\eta}(W^n_{\tau\eta} - 2\psi^n_\tau) + g(\eta)(W^n_{\eta\eta})^2 + N_0 + N_1 \eta,$$

where $N_0$, $N_1$ are some positive constants and $g(\eta)$ is a smooth function which satisfies $g(0) = 0$, $g'(0) = 0$, $g > 0$ for $\eta > 0$.

One can prove that constants $\alpha$, $K_0$, $K_1$, $N_0$, $N_1$ in $\Phi_n$ and $\mathfrak{F}_n$ can be chosen independently of $n$ in such a way that $\Phi_n$ and $\mathfrak{F}_n$ are uniformly bounded with respect to $n$ in $\Omega$ when $\tau \leq \tau_1$ or $\xi \leq \xi_1$, where $\tau_1$ and $\xi_1$ are some positive constants which do not depend on $n$. Using the equation which $w^n - w^{n-1}$ satisfies, one can show the uniform convergence of $w^n$ in $\Omega$ when $\tau \leq \tau_1$ or $\xi \leq \xi_1$. Therefore we have the following result.

THEOREM 1.—*The solutions $w^n$ of the problem* (7), (8), (9) *converge in the domain $\Omega$ when $\tau \leq \tau_1$ or $\xi \leq \xi_1$ to the function $w$, which is a solution of the problem* (4), (5), (6) *and $w > 0$ for $\eta < U(\tau, \xi)$.*

The uniqueness of the solution of the problem (4), (5), (6), follows from the maximum principle.

Going back to independent variables $t, x, y$ and functions $u$ and $v$, existence and uniqueness for the problem (1), (2), (3) follow easily. We recall that we assumed the smoothness of the functions $u_0$, $u_1$, $v^0$, $p$, $U$, and also the compatibility conditions which we gave in terms of the existence of a function $w^*$, defined above.

THEOREM 2.—*There exists a unique solution $u, v$ of the problem* (1), (2), (3) *in any domain $\mathfrak{D}$ such that $t_0 \leq \tau_1$ or $x_0 \leq \xi_1$; $\tau_1$ and $\xi_1$ are positive constants which can be defined by means of the data of the problem* (1), (2), (3). *The solution of the problem* (1), (2), (3) *has the following properties: $u > 0$ for $y > 0$; $u_y \geq 0$ for $y \geq 0$; the derivatives $u_t$, $u_x$, $u_y$, $u_{yy}$ are continuous and bounded in $\mathfrak{D}$. In addition,* $\dfrac{u_{yy}}{u_y}$ and $\dfrac{u_{yyy}\, u_y - u_{yy}^2}{u_y}$ *are bounded in $\mathfrak{D}$.*

The function $u(t, x, y)$ is given by means of the integral

$$y = \int_0^{u(t,x,y)} \frac{ds}{w(t,x,s)}, \tag{18}$$

where $w$ is the solution of the problem (4), (5), (6); the function $v$ can be found from the equation $u_x + v_y = 0$, using the condition $v|_{y=0} = v_0$. Proofs of Theorems 1 and 2 are given in [10] and the uniqueness of the solution of the problem (1), (2), (3) has been proved by another method in [11].

For the approximate solution of the problem (4), (5), (6), and therefore for the problem (1), (2), (3), we can suggest two finite-difference schemes. The first is implicit, the second is explicit.

For the first scheme, let $h$ be the mesh width of the lattice and let $(\tau, \xi, \eta) = (mh, lh, kh)$ be an arbitrary lattice point; $m, l, k = 0, 1, 2, \cdots$ Let us denote $w(mh, lh, kh)$ by $w_{m,l,k}$. Instead of (4) we consider the finite-difference equation at every interior lattice point of $\Omega$

$$(\nu\, w_{m-1,l,k}^2 + Mh)\left(\frac{w_{m,l,k+1} - 2w_{m,l,k} + w_{m,l,k-1}}{h^2}\right) - \tag{19}$$

$$- \left(\frac{w_{m,l,k} - w_{m-1,l,k}}{h}\right) - \eta\left(\frac{w_{m,l,k} - w_{m,l-1,k}}{h}\right) + p_x\left(\frac{w_{m,l,k} - w_{m,l,k-1}}{h}\right) = 0,$$

where the constant $M \geq \max |p_x|$. Instead of (5), the boundary conditions for $w_{m,l,k}$ are

(20) $w_{m,l,k} = w_0$ for $m = 0$ , $w_{m,l,k} = w_1$ for $l = 0$, and $w_{m,l,k} = 0$

at the lattice points of $\Omega$ for which the distance to the boundary $\eta = U(\tau, \xi)$ is less than or equal to $h$. Instead of (6), on the boundary $k = 0$ we have the condition

$$\nu w_{m-1,l,0} \left( \frac{w_{m,l,1} - w_{m,l,0}}{h} \right) - v_0 w_{m-1,l,0} - p_x = 0. \tag{21}$$

Let us suppose that for $\tau \leq (m-1) h$ we have found the solution of the system of equations (19) with conditions (20), (21). Then we can find a solution $w_{m,l,k}$ of equations (19), (20), (21) for $\tau = mh$ as a solution of a linear algebraic system of equations. By the same method which we used to prove Lemma 1 we can get a priori estimates of the form (16), (17) for solutions of (19), (20), (21) when $mh \leq \tau_0$ or $lh \leq \xi_0$. The existence of a solution of the linear algebraic equations (21), (19), (20) for $\tau = mh$ follows from the fact that the corresponding homogeneous system has only the trivial solution. One can prove the latter using a maximum principle and the condition $M \geq \max |p_x|$. For the difference $z_{m,l,k} = w_{m,l,k} - w$, where $w$ is the solution of the problem (4), (5), (6), which exists by Theorem 1, we can write equations of the form (19), (20), (21) with functions occurring on the right-hand-sides which are $O(h)$. Using the maximum principle, one can prove that $z_{m,l,k} \to 0$ when $h \to 0$ and $\tau \leq \tau_0$ or $\xi \leq \xi_0$. Moreover $|w_{m,l,k} - w| = O(h)$, when $h \to 0$. An approximate solution $u$ for the problem (1), (2), (3) can be obtained from the integral (18) with $w_{m,l,k}$ instead of $w$.

In an analogous way we can also prove the convergence of the following explicit finite-difference scheme. For $w_{m,l,k} = w(mh, l\sigma, k\sigma)$, $\sigma > 0$, $h > 0$, we consider in $\Omega$ the finite-difference equations

$$(\nu w^2_{m-1,l,k} + M\sigma) \left( \frac{w_{m-1,l,k+1} - 2 w_{m-1,l,k} + w_{m-1,l,k-1}}{\sigma^2} \right) -$$

$$- \left( \frac{w_{m,l,k} - w_{m-1,l,k}}{h} \right) - \eta \left( \frac{w_{m-1,l,k} - w_{m-1,l-1,k}}{\sigma} \right) + p_x \left( \frac{w_{m-1,l,k} - w_{m-1,l,k-1}}{\sigma} \right) = 0$$

which can be easily solved with respect to $w_{m,l,k}$ if we have $w_{m-1,l,k}$. The boundary conditions are: $w_{m,l,k} = w_0$ for $m = 0$, $w_{m,l,k} = w_1$ for $l = 0$, $w_{m,l,k} = 0$ at the lattice points of $\Omega$ for which the distance to the boundary $\eta = U(\tau, \xi)$ is less than or equal to $\sigma$ and

$$\nu w_{m-1,l,0} \left( \frac{w_{m,l,1} - w_{m,l,0}}{\sigma} \right) - p_x - v_0 w_{m-1,l,0} = 0 \text{ for } \eta = 0.$$

The solutions $w_{m,l,k}$ of these equations tend to the solution $w$ of the problem (4), (5), (6), if $\tau \leq \tau_0$ or $\xi \leq \xi_0$, and also $\frac{h}{\sigma^2} < M_1$, where $M_1$ is defined by the data of the problem (4), (5), (6). For example, when $t_0 \leq \tau_0$, we may take $M_1 = \frac{1}{1+2\nu \max H_1^2}$ and when $x_0 \leq \xi_0$, we may take $M_1 = \frac{1}{1+2\nu \max H_2^2}$. Moreover $|w_{m,l,k} - w| \leq M_2(h+\sigma)$, where the constant $M_2$ also depends only on the data of the problem (4), (5), (6).

Finally we consider the behavior of the solutions of the problem (1), (2), (3) when $t \to \infty$, i.e., stability. We suppose that for $t \to \infty$ the given functions $p(t,x)$, $U(t,x)$, $v_0(t,x)$ tend, uniformly with respect to $x$, to $\tilde{p}(x)$, $\tilde{U}(x)$, $\tilde{v}_0(x)$, respectively. The function $u_1(t,y)$ for $t > t_1$ is not to depend on $t$ and therefore $u_1(t,y) = \tilde{u}_1(y)$ for $t > t_1$. We suppose also that the solution $\tilde{u}, \tilde{v}$ of Prandtl's system for the stationary flow

(22) $$uu_x + vu_y = -p_x + \nu u_{yy}, \quad u_x + v_y = 0$$

in the domain $\tilde{\mathfrak{D}}\{0 \leq x \leq x_0,\ 0 \leq y < \infty\}$ with the conditions

(23) $$u|_{y=0} = 0, \quad v|_{y=0} = \tilde{v}_0(x), \quad u|_{x=0} = \tilde{u}_1(y), \quad \lim_{y\to\infty} u(x,y) = \tilde{U}(x)$$

exists, and also that this solution has the following properties: $u_y > 0$ for $y \geq 0$; $u$ and $u_y$ have bounded first derivatives with respect to $x$ and $y$ in $\tilde{\mathfrak{D}}$, and the derivative $u_{yyy}$ exists in $\tilde{\mathfrak{D}}$. The solution of the problem (22), (23) is obtained in [3]. We assume in addition that there exists a solution of the problem (1), (2), (3) with the properties stated in Theorem 2. These assumptions are fulfilled if the data of these problems are sufficiently smooth, the compatibility conditions are satisfied, and $x_0 > 0$ is sufficiently small. Under these conditions

$$\lim_{t\to\infty} u(t,x,y) = \tilde{u}(x,y)$$

for any $x, y$ in $\tilde{\mathfrak{D}}$. If $p(t,x)$, $U(t,x)$, $v_0(t,x)$ do not depend on $t$ for $t > t_1$, then

$$|u(t,x,y) - \tilde{u}(x,y)| \leq Ce^{-\gamma t}$$

for $y < y_1$ and $0 \leq x \leq x_0$, where $\gamma$ is any positive constant and C depends on $\gamma$ and $y_1$. The proof of these stability results is given in [12]. The method is based on the use of the solutions of the problem (4), (5), (6), and the corresponding problem for the stationary flow.

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## REFERENCES.

[1] H. WEYL, *On the differential equations of the simplest boundary layer problems*, «Ann. Math.», Ser. 2, *43*, n. 2, pp. 381–407 (1942).

[2] K. NICKEL, *Einige Eigenschaften von Lösungen der Prandtlischen Grenzschicht Differentialgleichungen*, «Arch. Rat. Mech. and Analysis», *2*, n. 1, pp. 1–31 (1958).

[3] O. A. OLEINIK, *On equations of a boundary layer*. Seminari 1962–63 Istituto Naz. Alta Matem. *1*, Ediz. Cremonese, Roma, pp. 372–387.

[4] G. FICHERA, *Sulle equazioni differenziali lineari ellittico–paraboliche del secondo ordine*, « Atti Acc. Naz. Lincei, Memorie », VIII, 5, fasc. 1/1, pp. 1–30 (1956).

[5] G. FICHERA, *On a unified theory of boundary value problems for elliptic–parabolic equations of second order*, Boundary problems in differential equations, Madison (1960), pp. 97–120, Russian translation « Matematica », 7, n. 6 (1963).

[6] O. A. OLEINIK, *Alcuni risultati sulle equazioni lineari e quasi lineari ellittico–paraboliche a derivate parziali*, « Rend. Acc. Naz. Lincei », *40*, maggio 1966.

[7] S. L. SOBOLEV, *Applications of Functional Analysis in Mathematical Physics*, 7, « Trans. of Math. Monographs », Amer. Math. Soc., Providence, R. I. (1963).

[8] C. MIRANDA, *Equazioni alle derivate parziali di tipo ellittico*, Berlin, Springer (1955).

[9] S. AGMON, A. DOUGLIS, L. NIRENBERG, *Estimates near the boundary for solutions of elliptic partial differential equations, satisfying general boundary conditions*. I, « Commun. Pure Appl. Math. », *12*, n. 4, pp. 623–727 (1959).

[10] O. A. OLEINIK, *On the mathematical theory of the boundary layer for the nonstationary flow of an incompressible fluid*, « Prikladn. Matem. and Mech. », *30*, n. 5 (1966) (in russo).

[11] K. NICKEL, *Ein Eindeutigkeitssatz für instationäre Grenzschichten*, « Math. Zeitschr. », *83*, pp. 1–7 (1964).

[12] O. A. OLEINIK, *On the stability of solutions of the boundary layer system for the nonstationary flow of an incompressible fluid*, « Prikladn. Math. and Mech. », *30*, n. 3 (1966) (in russo).