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A new approach to the theory of algebraic numbers. Nota II

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NOTE PRESENTATE DA SOCI

Logica matematica. — *A new approach to the theory of algebraic numbers*. Nota II di ABRAHAM ROBINSON (*) presentata (**) dal Socio B. SEGRE.

RIASSUNTO. — Questa Nota II continua lo sviluppo (iniziato nella Nota I) di una teoria degli anelli di Dedekind, D, usufruendo dei metodi dell' « analisi non-standard ». In particolare si mostra che ogni elemento di D possiede una rappresentazione come prodotto di elementi primi di *D. Questa rappresentazione è unica, a prescindere dall'ordine dei fattori e dalla loro eventuale sostituzione con elementi ad essi associati.

1. In a previous paper (ref. [1]), we considered a Dedekind domain D containing at least one proper ideal (i.e., ideal different from D and zero), and we introduced an *enlargement* *D of D. Defining the *monad* μ of *D by $\mu = \cap_{\nu}^{*} J_{\nu}$, where J_{ν} varies over the proper ideals of D, we then investigated the quotient ring $\Delta = {}^{*}D/\mu$. We showed that the canonical mapping φ from *D to Δ maps all *internal* ideals in *D on principal ideals in Δ; and that any finite set of non-zero elements in Δ possesses a greatest common divisor. Continuing our investigation, we shall arrive at a detailed understanding of the laws of divisibility and factorization in Δ. The reader is referred to ref. [1] for some definitions and results used in the present paper.

2. Let J be an internal proper ideal in *D.

THEOREM.—$\varphi(J) = \Delta$ *if and only if all the internal prime divisors of* J *are non-standard.*

Proof.—The condition is necessary. For suppose *P divides J where P is a prime ideal in D. Then $P \neq D$ so there exists an element $a \in D - P$. Now if $\varphi(J) = \Delta$ then $\varphi({}^{*}P) = \Delta$ and so $\varphi(a) \in \varphi({}^{*}P)$. But then $a \in ({}^{*}P, \mu) = {}^{*}P$, contrary to the fact that $a \notin {}^{*}P$, by transfer from D to *D.

The condition is also sufficient. We established in ref., section 3, (iii), that there exists an internal proper ideal in *D, to be called here J_0, such that $J_0 \subset \mu$. Let $J_1 = (J, J_0)$; then J_1 does not possess any standard prime divisors. Let $J_2 = J : J_1$. Then $J_2 \subset \mu$ since J_2 is divisible by all ${}^{*}P_{\nu}^{n}$ where P_{ν} is any prime ideal in D and n is any finite natural number. Also, $(J, J_2) = {}^{*}D$, and so $(J, \mu) = {}^{*}D$, $\varphi(J) = \Delta$, as asserted.

3. Let J be an internal proper ideal in *D. We know that $\varphi(J) = 0$ if and only if J is divisible by all ${}^{*}P_{\nu}^{n}$, P_{ν} any prime ideal in D, n any finite natural number. $\varphi(J)$ will be called a zero divisor if $\varphi(J) \neq 0$ and if there exists an element $a \neq 0$ in Δ such that $\varphi(J)\, a = 0$.

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THEOREM.—*In order that* φ (J) *be a zero divisor, it is necessary and sufficient that* $\varphi(J) \neq 0$ *and that there exist a prime ideal* P *in* D *such that* J *is divisible by all finite powers of* $^*P$.

Proof.—The conditions are necessary. For suppose that $\varphi(J)$ is a zero divisor. Then $\varphi(J) \neq 0$, by definition. Suppose that J is not divisible by any infinite power of any $^*P_\nu$, where P_ν varies over the prime ideals of D, but that $\varphi(J)$ is a zero divisor. Then $\varphi(J)\, a = 0$ for some $a \neq 0$ in Δ, $a = \varphi(b)$ say, where $b \in {}^*D$, $b \neq 0$. Let $J_1 = (b)$, so that J_1 is an internal ideal in $^*D$. Then $\varphi(J)\,\varphi(J_1) = \varphi(J J_1) = 0$. It follows that for any prime ideal P_ν in D, $J J_1$ is divisible by all finite powers of $^*P_\nu$, and hence is divisible by some infinite power of $^*P_\nu$. Writing $\mathrm{ord}_Q(A)$ for the exponent of the highest power of an ideal Q, by which an ideal $A \neq 0$ is divisible, we then have $\mathrm{ord}_{^*P_\nu}(J J_1) = m_\nu$ where m_ν is an infinite natural number and $\mathrm{ord}_{^*P_\nu}(J) = n_\nu$, where n_ν is finite. But $\mathrm{ord}_Q(AB) = \mathrm{ord}_Q(A) + \mathrm{ord}_Q(B)$ for any ideals Q, $A \neq 0$, $B \neq 0$, and so $\mathrm{ord}_{^*P_\nu}(J_1) = \mathrm{ord}_{^*P_\nu}(J J_1) - \mathrm{ord}_{^*P_\nu}(J) = m_\nu - n_\nu$ is infinite. This implies $\varphi(J_1) = 0$, $a = 0$, contrary to assumption.

The conditions (taken together) are also sufficient. Suppose that $\varphi(J) \neq 0$ and that there exists a prime ideal P in D such that $\mathrm{ord}_{^*P}(J) = n$ is infinite. Let $J_0 \subset \mu$ be the ideal which was introduced in Section 2 above and let $\mathrm{ord}_{^*P}(J_0) = m$, so that m is infinite. Then the ideal $J_1 = J_0 : {}^*P^m$ is not divisible by $^*P$. Hence, J_1 is not a subset of μ although $J J_1 \subset \mu$, $\varphi(J J_1) = 0$. Choose $b \in J_1 - \mu$, so that $a = \varphi(b) \neq 0$. Then $\varphi(J)\, a \subset \varphi(J J_1)$ and so $\varphi(J)\, a = 0$, $\varphi(J)$ is a zero divisor.

4. Let J be an internal proper ideal in $^*D$, such that $\varphi(J)$ is different from the zero ideal and is not a zero divisor.

THEOREM.—$\varphi(J)$ *is a prime ideal in* Δ *if and only if* $\mathrm{ord}_{^*P_\nu}(J) = 0$ *for all prime ideals* P_ν *in* D *except one*, P, *for which* $\mathrm{ord}_{^*P} = 1$.

Proof.—The condition is sufficient. Suppose that $J = {}^*P J_1$ where J_1 does not have any standard prime divisors (so that $J_1 = {}^*D$ or J_1 is the product of non standard prime ideals). Then $\varphi(J) = \varphi({}^*P)\,\varphi(J_1) = \varphi({}^*P)$. Let $ab \in \varphi({}^*P)$ where $a = \varphi(a_0)$, $b = \varphi(b_0)$. Then $\varphi(a_0)\,\varphi(b_0) = \varphi(a_0 b_0) \in \varphi({}^*P)$ and so $a_0 b_0 \in ({}^*P, \mu) = {}^*P$. Since $^*P$ is prime it follows that one of the factors a_0 or b_0 belongs to $^*P$, e.g. $a_0 \in {}^*P$. Hence $\varphi(a_0) \in \varphi({}^*P)$, $a \in \varphi({}^*P)$. On the other hand, $\varphi({}^*P) \neq \Delta$, by Section 2, and $\varphi({}^*P) \neq 0$. This shows that $\varphi(J) = \varphi({}^*P)$ is prime.

The condition is also necessary. For suppose that $J = {}^*P_1^m\, {}^*P_2^n Q$ where P_1 and P_2 are prime ideals in D, equal or different, and Q is not divisible by either P_1 or P_2. Since $\varphi(J)$ is not a zero divisor m and n must be finite. Now $\varphi(J) = (\varphi({}^*P_1))^m (\varphi({}^*P_2))^n \varphi(Q)$. If $\varphi(J)$ were prime we should then conclude that either $\varphi({}^*P_1) = \varphi(J)$ or $\varphi({}^*P_2) = \varphi(J)$ or $\varphi(Q) = \varphi(J)$. In the first case $\varphi({}^*P_1) \subset \varphi({}^*P_1\, {}^*P_2)$ and so ${}^*P_1 \subset ({}^*P_1\, {}^*P_2, \mu) = {}^*P_1\, {}^*P_2$, which is impossible. A similar argument applies to the remaining two cases. This completes the proof.

As shown in the course of the proof, $\varphi(J) = \varphi({}^*P)$ if $^*P$ is the unique prime divisor of J. Thus, as P ranges over all the prime ideals of D, $\varphi({}^*P)$

ranges over all the prime ideals $\varphi(J)$ in Δ which are mentioned in the theorem. Moreover, if P_1 and P_2 are distinct prime ideals in D, then $\varphi({}^*P_1) \neq \varphi({}^*P_2)$. For if $a \in P_1 - P_2$ then $a \in {}^*P_1 - {}^*P_2$ and so $\varphi(a) \in \varphi({}^*P_1)$. However, $\varphi(a) \in \varphi({}^*P_2)$ would imply $a \in ({}^*P_2, \mu) = {}^*P_2$, contrary to assumption. Thus, φ provides a multiplicative bijection between the standard prime ideals of *D and the prime ideals of Δ which are not zero divisors and which are images of internal ideals in *D.

5. The monad μ is not an internal set in the sense of Non-standard Analysis. For suppose it were. Then $\mathrm{ord}_{{}^*P}(\mu) = n_P$ would be infinite for all prime ideals P in D. Choose one such $P = P_0$ and define $\mu_0 = \mu : P_0$. Then $\mu_0 \supset \mu$, $\mu_0 \neq \mu$. On the other hand, μ_0 would still be divisible by all finite powers of *P for any prime ideal P in D, and so $\mu_0 \subset \mu$, $\mu_0 = \mu$. This contradiction shows that μ is not internal.

It follows that if we regard $\Delta = {}^*D/\mu$ as a set of subsets of *D then Δ cannot be an internal entity; for in that case, μ also would be an internal entity.

Let J be a principal ideal in Δ, $J = (a)$. We know that an ideal in Δ is principal if and only if it is the image under φ of an internal ideal in *D. If $J = \Delta$, then a is a unit in Δ and, conversely, if a is a unit in Δ then $J = \Delta$. If a is a zero divisor in Δ, so that $ab = 0$ in Δ for $b \neq 0$ then J is a zero divisor for in that case $Jb = 0$. Conversely, if J is a zero divisor, then a is a zero divisor.

Suppose now that $J = (a)$, $a \neq 0$, where a is neither a unit nor a zero divisor. As usual, a is said to be *prime* if a is divisible only by units (invertible elements) and by elements associated with a.

THEOREM.—J *is prime if and only if a is prime.*

Proof.—Let J be prime. If $a = bc$ then $bc \in J$, hence either $b \in J$ or $c \in J$. If $b \in J$, $b = da$ for some $d \in \Delta$, then $dac = a$, $a(dc - 1) = 0$. Since a is not a zero divisor, we conclude that $dc - 1 = 0$, c is a unit, b is associated with a. A similar argument applies if $c \in J$. Hence, a is prime. On the other hand, suppose that a is prime and that $bc \in J$ but $b \notin J$, $c \notin J$. Let $(a, b) = (d)$ so that d is a greatest common divisor of a and b. Since d divides a but is not in J it must be a unit, $(d) = \Delta$. Similarly $(a, c) = \Delta$, and so $(a, b)(a, c) = \Delta$. But $(a, b)(a, c) = (a^2, ac, ba, bc) = J$ and so $J = \Delta$, contrary to assumption. This proves our assertion.

6. Let $a \in \Delta$, $a \neq 0$, and let p be a prime element of Δ. If $p^n \mid a$ but $p^{n+1} a$ for some finite natural number $n \geq 0$ then we set $\mathrm{ord}_p(a) = n$. If $p^n \mid a$ for all finite natural numbers n then we set $\mathrm{ord}_p(a) = \infty$. Notice that p^n is not defined for infinite n.

For a and p as above, set $(a) = A$, $(p) = P$ in Δ. Then there exists an internal ideal J in *D and a prime ideal Q in D such that $A = \varphi(J)$, $P = \varphi({}^*Q)$. Suppose that $p^n \mid a$ for some finite n. Then $P^n \mid A$ and so $\varphi({}^*Q^n) \mid \varphi(J)$, $\varphi(J) \subset \varphi \subset ({}^*Q^n)$, $J \subset ({}^*Q^n, \mu) = {}^*Q^n$, ${}^*Q^n \mid J$. Conversely, if ${}^*Q^n \mid J$ then $P^n \mid A$ and so $p^n \mid a$. We conclude that if $\mathrm{ord}_{{}^*Q}(J)$ is an infinite natural number then ord_p

$(a) = \infty$, and conversely; while if $\mathrm{ord}_{*Q}(J)$ is finite then $\mathrm{ord}_p(a)$ is finite and conversely, and in this case $\mathrm{ord}_p(a) = \mathrm{ord}_{*Q}(J)$.

THEOREM.—*For* $a, b, p \in \Delta$, $a \neq 0$, $b \neq 0$, p *prime*:

$$\mathrm{ord}_p(ab) = \mathrm{ord}_p(a) + \mathrm{ord}_p(b)$$

In this connection, the sum on the right hand side is defined to be ∞ if at least one of $\mathrm{ord}_p(a)$, $\mathrm{ord}_p(b)$ is ∞.

For, introducing A, P, J, Q as above and setting $(b) = B$, $B = \varphi(K)$, where K is an internal ideal in $^*D$, suppose next that $\mathrm{ord}_p(a)$ and $\mathrm{ord}_p(b)$ are finite. Then $\mathrm{ord}_{*Q}(J)$ and $\mathrm{ord}_{*Q}(K)$ also are finite and

$$\mathrm{ord}_{*Q}(JK) = \mathrm{ord}_{*Q}(J) + \mathrm{ord}_{*Q}(K)$$

applies in $^*D$. But

$$(ab) = AB = \varphi(J)\,\varphi(K) = \varphi(JK)$$

and so $\mathrm{ord}_p(ab) = \mathrm{ord}_{*Q}(JK)$. This establishes the assertion for finite $\mathrm{ord}_p(a)$, $\mathrm{ord}_p(b)$. The remaining cases can be disposed of in a similar way.

The fact that $\mathrm{ord}_p(a) = \infty$ if and only if $\mathrm{ord}_{*Q}(J)$ is infinite also shows that $a \in \Delta$, $a \neq 0$, is a zero divisor if and only if $\mathrm{ord}_p(a) = \infty$ for at least one prime element p of Δ and a is a unit if and only if $\mathrm{ord}_p(a) = 0$ for all prime elements p of Δ.

7. If p and q are prime elements of Δ then $\mathrm{ord}_p(q) = 1$ if q is associated with p, and $\mathrm{ord}_p(q) = 0$ in the alternative case. From every class of associated primes select one, π, calling it a representative prime. Now let $a \in \Delta$, $a \neq 0$, a not a zero divisor and possessing only a finite number of distinct representative primes $\pi_1, \cdots, \pi_j$, $j \geq 0$ as divisors. Thus, $\mathrm{ord}_{\pi_i}(a) = n_i$ is a positive integer for $\pi_1, \cdots, \pi_i$ and $\mathrm{ord}_p(a) = 0$ for any prime p not associated with one of these.

Consider the product $b = \Pi_{i=1}^{j} \pi_i^{n_i}$. b is a divisor of a. For since $\pi_1^{n_i} | a$, we have $a = \pi_1^{n_1} a_1$ where $\mathrm{ord}_{\pi_1}(a_1) = 0$, $\mathrm{ord}_{\pi_i}(a_1) = n_i$, $i = 2, \cdots, j$, by the theorem of Section 6. Continuing in this way we show that $b | a$, $a = \varepsilon b$. Then $\mathrm{ord}_{\pi_i}(\varepsilon) = 0$, $i = 1, \cdots, j$ and more generally $\mathrm{ord}_p(a) = 0$ for all prime elements p of Δ. This shows that ε is a unit. Thus, we have represented a as a product of powers of distinct representative primes multiplied by a unit. It is not difficult to see that this representation is unique.

Now suppose in addition that a is an element of D regarded as a subset of Δ. Let $(a) = P_1^{n_1} \cdots P_j^{n_j}$ be the representation of (a) as a product of powers of distinct prime ideals in D, so that $^*(a) = {}^*P_1^{n_1} \cdots {}^*P_j^{n_j}$ in $^*D$. By Section 4 and 5 there exist representative primes $\pi_1, \cdots, \pi_j$ of Δ such that $\varphi({}^*P_i) = (\pi_i)$, $i = 1, \cdots, j$. Then $\mathrm{ord}_{\pi_i}(a) = n_i$, $i = 1, \cdots, j$ in Δ and $\mathrm{ord}_p(a) = 0$ for prime elements of Δ not associated with $\pi_1, \cdots, \pi_j$. Hence, $a = \varepsilon \pi_1^{n_1} \cdots \pi_j^{n_j}$, where ε is a unit. Thus, the multiplicative system H generated by the units and prime elements of Δ contains all non-zero elements of D. For every element a of H there exists an internal ideal J in $^*D$ such that $\varphi(J) = (a)$. For if $a = \varepsilon p_1^{n_1} \cdots p_j^{n_j}$ where ε is a unit and $p_1, \cdots, p_j$ are prime elements of Δ, then there exist prime ideals $P_1, \cdots, P_j$ in D such that $\varphi({}^*P_i) = (P_i)$, $i = 1, \cdots, j$. Then the ideal J defined by $J = {}^*P_1^{n_1} \cdots {}^*P_j^{n_j}$ satisfies $\varphi(J) = \varphi({}^*P_1^{n_1}) \cdots \varphi({}^*P_j^{n_j}) = (a)$, as required. The elements of H will be called *Prüfer–finite*.

THEOREM.—*Let* $a \in \Delta$, $a \neq 0$. *Then* a *is Prüfer-finite if and only if there exists an element* $b \in D$, $b \neq 0$ *such that* $a \mid b$ *in* Δ.

Proof.—The condition is sufficient. For let $b \in D$, $b \neq 0$, so that b can be written as $b = \varepsilon \pi_1^{n_1} \cdots \pi_j^{n_j}$ where $\pi_1, \cdots, \pi_j$ are representative primes and ε is a unit. Then $a \mid b$ implies that a is not a zero-divisor and that it can be written in the form $a = \eta \pi_1^{m_1} \cdots \pi_j^{m_j}$ where η is a unit and $0 \leq m_i \leq n_i$, $i = 1, \cdots, j$. Hence $a \in H$.

The condition is also necessary. For suppose $a \in H, a = \varepsilon p_1^{n_1} \cdots p_j^{n_j}$ where ε is a unit and $p_1, \cdots, p_j$ are prime elements of Δ. Then $(p_i) = \varphi({}^*P_i)$, $i = 1, \ldots, j$ where $P_1, \cdots, P_j$ are prime ideals in D, and so $(a) = \varphi({}^*P_1^{n_1} \cdots \cdots {}^*P_j^{n_j})$. Choose $b_i \in P_i$, $b_i \neq 0$, then $b = b_1^{n_1} \cdots b_j^{n_j}$ is different from zero and is contained in ${}^*P_1^{n_1} \cdots {}^*P_j^{n_j}$ and hence in (a). Thus, $a \mid b$, as required.

8. In addition to the elements of H, we may consider also elements of Δ which, while not zero divisors, are divisible by an infinite number of distinct representative primes. Let a be such an element of Δ. Then the function $\mathrm{ord}_P(a)$ takes finite values only. Moreover, if b is a second element of this kind and $\mathrm{ord}_\pi(a) = \mathrm{ord}_\pi(b)$ for all representative primes π then a and b must be associated. For let J and K be internal ideals in *D such that $(a) = \varphi(J)$, $(b) = \varphi(K)$. Let R be the set of internal prime ideals P in *D such that $\mathrm{ord}_P(J) = \mathrm{ord}_P(K) = n_P > 0$. Let $Q = \prod_{P \in R} P^{n_P}$. Then $Q \mid J$, $Q \mid K$, $J = QJ'$, $K = QK'$. Also, R includes all standard prime divisors of J and K and so $\varphi(J') = \varphi(K') = \Delta$. Hence, $\varphi(J) = \varphi(Q) = \varphi(K)$, $(a) = (b)$, a and b are associated.

Now let $f(\pi)$ be any function from the representative primes into the finite natural numbers. We claim that there exists an element $a \in \Delta$ such that $\mathrm{ord}_\pi(a) = f(\pi)$ for all representative primes π. Indeed, for any prime ideal P in D there is a unique representative prime π such that $\varphi({}^*P) = (\pi)$, and we write $\pi = g(P)$, $f(\pi) = f(g(P)) = h(P)$. Consider the relation $R(x, y)$ which holds if x is a prime ideal in D and y is an ideal in D, $y \neq 0$, such that $\mathrm{ord}_x(y) = h(x)$. Then R is concurrent. It follows that there exists an internal ideal $J \neq 0$ in *D such that $\mathrm{ord}_{{}^*P}(J) = h(P)$ for all prime ideals P in D. Let $a \in \Delta$ such that $(a) = \varphi(J)$. Then $\mathrm{ord}_\pi(a) = f(\pi)$ for any representative prime element π—as required.

BIBLIOGRAPHY.

[1] A. ROBINSON, *A new approach to the theory of algebraic numbers*, « Rend. Acc. Naz. Lincei », *40*, 227–225 (1966).