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On the determination of persistent eigenvalues

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RENDICONTI

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Classe di Scienze fisiche, matematiche e naturali

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NOTE DI SOCI

Analisi matematica. — *On the determination of persistent eigenvalues.* Nota (*) del Socio Straniero ALEXANDER WEINSTEIN (**).

RIASSUNTO. — Discussione dei rapporti fra i tre metodi per la determinazione degli autovalori persistenti nella teoria dei problemi intermedi.

1. INTRODUCTION.—This paper deals with the determination of persistent eigenvalues in the method of intermediate problems. This method leading to lower bounds for eigenvalues was introduced by the author in 1935–37 [3] in the determination of the eigenvalues for clamped plates. Later other types of problems, always following the same scheme, were considered by several authors. The general scheme of intermediate problems is the following. First we have to introduce a solvable base problem which gives rough lower bounds for the eigenvalues of a given problem. Secondly, we have to define intermediate problems depending on a finite number of functions $p_1, p_2, \ldots, p_m$ which give improved intermediate eigenvalues. Thirdly, we have to devise methods for solving the intermediate problems theoretically and numerically in terms of the base problem. One of the characteristics of intermediate problems is that some of their eigenvalues may already appear as eigenvalues of the base problem in which they usually have a larger index. These are called *persistent eigenvalues*. There are at present *three procedures for the determination of per-*

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sistent eigenvalues each of which has its merits from either the theoretical or the computational point of view. Any exposition of the theory without a thorough discussion of these three procedures would be rather incomplete. The purpose of this note is to establish the relationships among the various methods, a question which has not been discussed up to now. The connection is the same regardless of the type of problems used so we shall use the oldest type of problems in as much as it appears in the theory of plates and in the new maximum-minimum theory. For completeness we shall have to recapitulate some of the older results in a somewhat different form. For the convenience of ordering the eigenvalues in an increasing sequence we consider a compact symmetric negative definite operator A on a Hilbert space H having the inner product (u, v). Let $\lambda_1 \leqq \lambda_2 \leqq \cdots$ be the eigenvalues and $u_1, u_2, \cdots$ the corresponding orthonormal set of eigenvectors for the equation

$$\mathrm{A}u = \lambda u. \tag{1}$$

We call (1) the base problem and denote by $\mathrm{R}(u)$ the Rayleigh quotient $(\mathrm{A}u, u)/(u, u)$. Let S be a subspace of H and let P denote the projection operator on S. Then the given problem would be the variational problem corresponding to

$$\min_{\mathrm{P}u=0} \mathrm{R}(u).$$

Let S_m be the subspace of S spanned by the m independent vectors $p_1, p_2, \cdots, p_m$ and let P_m be the corresponding projection operator. The problem

$$\begin{gathered} \mathrm{A}u - \mathrm{P}_m \mathrm{A}u = \lambda u \\ \mathrm{P}_m u = 0 \end{gathered} \tag{2}$$

is called the m^{th} intermediate problem. $\mathrm{A} - \mathrm{P}_m \mathrm{A}$ is a compact symmetric negative definite operator on the subspace defined by $\mathrm{P}_m u = 0$. Obviously any eigenvector of (2) corresponding to a non-zero eigenvalue satisfies $\mathrm{P}_m u = 0$.

2. NON-PERSISTENT EIGENVALUES.—Let us first briefly discuss as in [1] the non-persistent eigenvalues. In place of (2) we write the equation

$$\begin{gathered} \mathrm{A}u - \lambda u = a_1 p_1 + a_2 p_2 + \cdots + a_m p_m \\ (u, p_i) = 0 \qquad i = 1, 2, \cdots, m \end{gathered} \tag{3}$$

where we consider the coefficients a_i as parameters to be adjusted later. Clearly not all a_i vanish. The unique solution of (3) is given by

$$u = \sum_{i=1}^{m} a_i \mathrm{R}_\lambda p_i \tag{4}$$

where $\mathrm{R}_\lambda p_i$ denotes the resolvent

$$\mathrm{R}_\lambda p_i = \sum_{j=1}^{\infty} \frac{(p_i, u_j)}{\lambda_j - \lambda} u_j. \tag{5}$$

Using equations (3), (4) and (5) we obtain

$$\sum_{i=1}^{m} a_i \sum_{j=1}^{\infty} \left[\frac{\lambda_j (p_i , u_j)(p_k , u_j)}{\lambda_j - \lambda} - (p_i , p_k) \right] = 0 \qquad (k = 1 , 2 , \cdots, m). \tag{6}$$

Since not all $a_i = 0$ in this case, the determinant in (6) must be zero. Using Parseval's formula and omitting the non-zero factor λ we can write the determinant in the following compact form

$$W(\lambda) = \det \left\{ \sum_{j=1}^{\infty} \frac{(p_i , u_j)(p_k , u_j)}{\lambda_j - \lambda} \right\}. \tag{7}$$

Such a compact form for $W(\lambda)$ is not available in other types of intermediate problems. The multiplicity of the non-persistent eigenvalue λ is equal to the nullity of the matrix in the Weinstein determinant (7).

3. THE DISTINGUISHED CHOICE.—We now turn to the determination of persistent eigenvalues. Let us first discuss the method of the distinguished choice (or distinguished sequence) given by Weinstein [3]. This is the oldest method and it includes the special distinguished choice, or for short special choice, introduced later by Bazley [4]. *The essence of this method is to reduce the determination of a persistent eigenvalue to the problem of determining a non-persistent eigenvalue.* Let λ_* be a persistent eigenvalue of multiplicity μ and let $u^1 , u^2 , \cdots, u^\mu$ be an orthonormal basis for the eigenspace U_* of λ_*. Let $p_1 = p^1 , p_2 = p^2 , \cdots, p_\mu = p^\mu$ be a distinguished choice of vectors in S namely such that

$$\det (p^i , u^k) \neq 0. \qquad (i , k = 1 , 2 , \cdots, \mu). \tag{8}$$

This condition means that there is no non-zero vector in S_μ which is orthogonal to U_*. Such distinguished choices exist for every subspace provided the eigenvectors common to the given problem and the base problem are excluded. We start by considering the following intermediate problem of order μ

$$\begin{gathered} Au - \lambda u = a_1 p^1 + a_2 p^2 + \cdots + a_\mu p^\mu . \\ P_\mu u = 0 . \end{gathered} \tag{9}$$

It is easy to see that this problem does not admit the eigenvalue λ_*. In fact, the Fredholm conditions

$$\sum_{i=1}^{\mu} a_i (p^i , u^k) = 0 \qquad (k = 1 , 2 , \cdots, \mu) \tag{10}$$

yield by (8), $a_1 = a_2 = \cdots = a_\mu = 0$, which would imply $u = c_1 u^1 + c_2 u^2 + \cdots + c_\mu u^\mu$. However, since $P_\mu u = 0$, we again use (8) to get $u = 0$, which proves that λ_* is not persistent. Following [1] we shall now use

$$\begin{gathered} Au - P_\mu Au = \lambda_* u \\ P_\mu u = 0 \end{gathered} \tag{11}$$

as a new base problem and introduce additional orthogonality conditions. Let us consider the intermediate problem of order $\mu + s$

$$(12) \qquad \begin{gathered} Au - P_\mu Au - \lambda_* u = a_{\mu+1} p_{\mu+1} + a_{\mu+2} p_{\mu+2} + \cdots + a_{\mu+s} p_{\mu+s} \\ P_{\mu+s} u = 0 . \end{gathered}$$

This problem (12) may again admit λ_* as a persistent eigenvalue except in the case $\lambda_* = \lambda_1$. To solve this problem we proceed as in [3] or [5]. In view of (8) we can change the basis of $S_{\mu+s}$ so that the additional independent vectors $p_{\mu+1}, p_{\mu+2}, \cdots, p_{\mu+s}$ are all orthogonal to the subspace U_*. We use the same notation for the new basis, so that we have

$$(13) \qquad (p_{\mu+i}, u^k) = 0 \qquad (i, = 1, 2, \cdots, s \; ; \; k = 1, 2, \cdots, \mu).$$

Using (13) and Fredholm's orthogonality conditions we see that $(P_\mu Au, u^k) = 0$ for $k = 1, 2, \cdots, \mu$. Since $p^1, p^2, \cdots, p^\mu$ is *a* distinguished choice this means that $P_\mu Au = 0$, so instead of (12) we can write

$$(14) \qquad Au - \lambda_* u = a_{\mu+1} p_{\mu+1} + a_{\mu+2} p_{\mu+2} + \cdots + a_{\mu+s} p_{\mu+s} .$$

Since (13) holds, the general solution of (14) is

$$(15) \qquad u = \sum_{i=1}^{s} a_{\mu+i} R'_{\lambda_*} p_{\mu+1} + b_1 u^1 + b_2 u^2 + \cdots + b_\mu u^\mu$$

where R'_{λ_*} denotes the resolvent (5) with the terms corresponding to $u^1, u^2, \cdots, u^\mu$ omitted. In order to have a non-vanishing solution which is orthogonal to $S_{\mu+s}$ we must satisfy the equations

$$(16) \qquad (p_k, u) = 0 \qquad (k = 1, 2, \cdots, \mu + s).$$

Let us note that $a_{\mu+1}, a_{\mu+2}, \cdots, a_{\mu+s}$ cannot all be zero because then it would also follow by (8) and (16) that $b_1 = b_2 = \cdots = b_\mu = 0$. In view of (13) the equations (16) for $k = \mu + 1, \mu + 2, \cdots, \mu + s$ only contain the unknowns $a_{\mu+1}, a_{\mu+2}, \cdots, a_{\mu+s}$. For λ_* to be a persistent eigenvalue of (12) there must be some non-zero $a_{\mu+i}$ and therefore the determinant

$$(17) \qquad W_*(\lambda_*) = \det \{ (R'_{\lambda_*} p_{\mu+i}, p_{\mu+k}) \} = \det \left\{ \sum_{j=1}^{\infty}{}' \frac{(p_{\mu+i}, u_j)(p_{\mu+k}, u_j)}{\lambda_j - \lambda_*} \right\} \qquad (i, k = 1, 2, \cdots, s)$$

must be zero. The independent solutions for the $\{a_i\}$ are determined by the nullity ν of $W_*(\lambda_*)$. For each of the solutions, the orthogonality conditions (16) for $k = 1, 2, \cdots, \mu$ are satisfied in view of (8) by a proper choice of $b_1, b_2, \cdots, b_\mu$. The multiplicity of λ_* in (12) is ν. This has been proved by Weinstein [3, p. 45]. We give here a slightly modified proof. Denote by

$$(18) \qquad v^{(j)} = \sum_{i=1}^{s} a^{(j)}_{\mu+i} R'_{\lambda_*} p_{\mu+1} + \sum_{i=1}^{\mu} b^{(j)}_i u^i \qquad (j = 1, 2, \cdots, \nu)$$

the ν solutions of (12) corresponding to the ν independent solutions for $\{a_{\mu+i}\}$.

Suppose there exist $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(\nu)}$ not all zero, such that $\sum_{j=1}^{\nu} \alpha^{(j)} v^{(j)} = 0$. Since the vectors $R'_{\lambda_*} p_{\mu+i}$ are all orthogonal to U_*, (18) implies that both

$$\sum_{j=1}^{\nu} \alpha^{(j)} \sum_{i=1}^{s} a^{(j)}_{\mu+i} R'_{\lambda_*} p_{\mu+i} = 0 \tag{19}$$

and

$$\sum_{j=1}^{\nu} \alpha^{(j)} \sum_{i=1}^{\mu} b^{(j)}_i u^i = 0 .$$

Since the $R'_{\lambda_*} p_{\mu+i}$ are independent, it follows from (19) that $\sum_{j=1}^{\nu} \alpha^{(j)} a^{(j)}_{\mu+i} = 0$ for $i = 1, 2, \cdots, s$ which is a contradiction. *It is remarkable that* $W_*(\lambda_*)$ *is formally a Weinstein determinant.* Its order is only s since $p_1, p_2, \cdots, p_\mu$ do not appear. The value μ can be very large for higher eigenvalues (e.g. for the membrane problem) so that our procedure could be of interest for the computation of eigenvalues having large multiplicity.

4. THE GENERAL CHOICE.—We now drop the assumption that $p_1, p_2, \cdots, p_\mu$ is *a* distinguished sequence belonging to λ_* and consider the intermediate problem of order m (2). If λ_* is to be a persistent eigenvalue we must first satisfy the Fredholm conditions

$$\sum_{i=1}^{m} a_i (p_i, u^k) = 0 \qquad (k = 1, 2, \cdots, \mu). \tag{20}$$

If (20) is satisfied, the solution is given by

$$u = \sum_{i=1}^{m} a_i R'_{\lambda_*} p_i + \sum_{i=1}^{\mu} b_i u^i .$$

The condition $P_m u = 0$ gives us m equations and (20) gives μ equations so that we have $m + \mu$ equations for the $m + \mu$ unknowns $a_1, a_2, \cdots, a_m, b_1, b_2, \cdots, b_\mu$. In order to have non-trivial solutions, the following $m + \mu$ by $m + \mu$ determinant must be zero.

$$\begin{array}{c} \\ m \\ \mu \end{array} \begin{array}{c} \overbrace{\qquad\qquad\quad}^{m} \quad \overbrace{\qquad\quad}^{\mu} \\ \left|\begin{array}{c:c} (R'_{\lambda_*} p_i , p_k) & (p_i , u^k) \\ \hdashline (u^i , p_k) & 0 \end{array}\right| \end{array} \tag{21}$$

Let ν denote the nullity of (21). Then the multiplicity of λ_* is ν. Weinstein proposed this way of determining the persistency of λ_* [3, p. 33] but did not give a proof for its multiplicity. Such a proof was given by Bazley and Fox [6] and was used here in section 3 instead of the original proof of Weinstein. As a matter of fact, the proof of the result of Bazley and Fox can be easily reconstructed from the previous section. It is of interest to show the connection between this procedure and the method of the distinguished choice. Using a distinguished sequence $p_1 = p^1, p_2 = p^2, \cdots, p_\mu = p^\mu$ in the case $\mu < m$, we see that the order $m + \mu$ of (21) can be reduced to $m - \mu$. In fact (21) becomes

$$
\begin{array}{c|c:c:c|}
 & \mu & m-\mu & \mu \\
\hline
\mu & (R'_{\lambda_*} p^i, p^k) & (R'_{\lambda_*} p^i, p_k) & (p^i, u^k) \\
\hdashline
m-\mu & (R'_{\lambda_*} p_i, p^k) & (R'_{\lambda_*} p_i, p_k) & 0 \\
\hdashline
\mu & (u^i, p^k) & 0 & 0
\end{array}
$$

Obviously this determinant equals

$$[\det (p^i, u^k)]^2\, W_* (\lambda_*). \tag{22}$$

Since the first factor in (22) does not vanish we see, as in section 3, that the multiplicity of λ_* is the nullity of $W_* (\lambda_*)$.

5. Aronszajn's Rule.—The two methods in the previous sections allow us to compute one persistent eigenvalue at a time as the zeros of certain determinants which have to be formed for each λ_* separately. Aronszajn [7], however, has given a rule which determines *all* persistent eigenvalues from the zeros *and poles* of the Weinstein determinant (7) as follows. Let $\mu(\lambda_*)$

and $\mu_m(\lambda_*)$ denote the multiplicities of λ_* in the base problem and the m^{th} intermediate problem respectively. Then we have Aronszajn's rule

$$\mu_m(\lambda_*) = \mu(\lambda_*) + \Omega_{0m}(\lambda_*) \tag{23}$$

where $\Omega_{0m}(\lambda_*)$ is the order of the meromorphic function $W(\lambda) = W_{0m}(\lambda)$ at the point $\lambda = \lambda_*$. This rule which is of unsurpassed elegance has great theoretical value. However, unlike the two previous methods, it is difficult to apply in numerical computations, due to the fact that some elements of $W(\lambda_*)$ may be zero while others are infinite. Aronszajn's rule is based on the decomposition

$$W(\lambda) = W_{0m}(\lambda) = W_{01}(\lambda)\, W_{1m}(\lambda)$$

where W_{ik} is the Weinstein determinant which connects the i^{th} intermediate problem with the k^{th} intermediate problem and where $i = 0$ denotes the base problem. By iteration we also have

$$W_{0m}(\lambda) = W_{0\mu}(\lambda)\, W_{\mu m}(\lambda)\,. \tag{24}$$

All of these decompositions are valid only if $(p_1, p_k) = \delta_{ik}$ but of course Aronszajn's rule is independent of the basis chosen. Let us also note that in $W_{\mu m}(\lambda)$ the eigenvalues and eigenfunctions of the μ^{th} intermediate problem appear, whereas in $W_{0\mu}(\lambda)$ only the eigenvalues and eigenfunctions of the base problem appear. We now derive the connection between Aronszajn's rule and our rule of section 3. Again let $p_1 = p^1, p_2 = p^2, \cdots, p_\mu = p^\mu$ be a distinguished choice with respect to λ_*. By preserving the basis elements $p^1, p^2, \cdots, p^\mu$ the determinant $W_{0\mu}$ remains unchanged. We can replace $p_{\mu+1}, p_{\mu+2}, \cdots, p_m$ by new vectors (retaining the same notation) which are orthogonal to U_*. The effect of this will be to multiply W_{0m} and $W_{\mu m}$ by the same positive factor. Since $W_{0\mu}$ is unchanged we still have formula (24). We again consider the m^{th} intermediate problem (12) with $m = \mu + s$. Proceeding as in section 3 we can again replace (12) by (14). This means that in the determinant $W_{\mu m}$, instead of using (9), we can use $Au = \lambda_* u$ as the base problem. In this way the multiplicity of λ_* is given by the nullity of $W_*(\lambda_*)$. Now let us reconsider (24). In view of the choice of $p_{\mu+1}, p_{\mu+2}, \cdots, p_m$ it is clear that $W_* = W_{\mu m}$. Since $p^1, p^2, \cdots, p^\mu$ is a distinguished sequence, λ_* is not an eigenvalue of (9), and therefore by Aronszajn's rule $W_{0\mu}(\lambda)$ has a pole of order μ at $\lambda = \lambda_*$ or $\Omega_{0\mu}(\lambda_*) = -\mu$. Equation (24) means that $\Omega_{0m}(\lambda_*) = \Omega_{0\mu}(\lambda_*) + \Omega_{\mu m}(\lambda_*)$ so that if λ_* is to be a persistent eigenvalue of (12) the determinant $W_{\mu m}(\lambda) = W_*(\lambda)$ must have a zero at $\lambda = \lambda_*$. Furthermore the order of the zero determines the multiplicity of λ_*. This completes the connection between the methods.

For recent expositions of the theory of intermediate problems see the books by G. Fichera [8] and S. H. Gould [9]. Let us also note that Bazley and Fox [6] (see also [9]) have shown that some of the methods discussed in this paper apply to certain unbounded Schrödinger-type operators, thus opening an important new field for applications.

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