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The Number of Isotropic Subspaces in a Finite Geometry

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Matematica. — *The Number of Isotropic Subspaces in a Finite Geometry.* Nota di VERA PLESS, presentata (*) dal Socio B. SEGRE.

1. INTRODUCTION.—Let V be an n-dimensional vector space over a finite field $K = GF(q)$ on which a non-degenerate symmetric (orthogonal case) or skew symmetric (symplectic case) form f is defined. It is of interest to determine the number, denoted by $\sigma_{n,k}$, of isotropic subspaces of dimension k in such a space. The possibilities are the following: (1) V orthogonal, characteristic of $K \neq 2$; (2) V symplectic, characteristic of K arbitrary; (3) characteristic of $K = 2$ and f a non-alternating, symmetric (= skew symmetric) form on V. By applying Segre's results [3, Theorem 1, p. 4] for the number of linear subspaces of a fixed dimension contained in a quadric in projective space, we can get the answer for the first possibility. Let υ be the dimension of a maximal isotropic subspace of V. If the characteristic of K is not 2, and n is even, then there are two types of orthogonal geometries. For one of these $\upsilon = n/2$ and we define ε to be 1 in this case, for the other $\upsilon = (n/2) - 1$ and we let $\varepsilon = -1$ here. If n is odd, there is one type of geometry and $\upsilon = (n - 1/2)$. By the results mentioned [3, Theorem 1, p. 4] for the first possibility we get

$$\sigma_{n,k} = \frac{(q^{n-k} - \varepsilon q^{n/2-k} + \varepsilon q^{n/2} - 1) \prod_{i=1}^{k-1} (q^{n-2i} - 1)}{\prod_{i=1}^{k} (q^i - 1)}$$, $k \geq 2$ if n is even, V orthogonal, and characteristic of $K \neq 2$

and

$$\sigma_{n,k} = \frac{\prod_{i=0}^{k-1} (q^{2(\nu-i)} - 1)}{\prod_{i=1}^{k} (q^i - 1)}$$ if n is odd, V orthogonal, and characteristic of $K \neq 2$.

The solutions for possibility (2) and possibility (3) for n odd are formally the same when expressed in terms of ν. For possibility (2), $\nu = n/2$ and for possibility (3) with n odd, $\nu = (n - 1/2)$. For these cases,

$$\sigma_{n,k} = \frac{\prod_{i=0}^{k-1} (q^{2(\nu-i)} - 1)}{\prod_{i-1}^{k} (q^i - 1)} .$$

(*) Nella seduta del 13 novembre 1965.

The only remaining case is possibility (3) with n even ($\nu = n/2$) and for this we obtain

$$\sigma_{n,k} = \frac{(q^{n-k} - 1) \prod_{i=1}^{k-1} (q^{n-2i} - 1)}{\prod_{i=1}^{k} (q^{i} - 1)}, \qquad k > 2.$$

From these equations it is easy to compute the number of isotropic subspaces of maximal dimension ν for each type of geometry. For possibility (1) [3, Cor. 1, p. 4],

$$\sigma_{n,\nu} = 2 \prod_{i=1}^{\nu-1} (q^{\nu-i} + 1) \qquad \text{for } n \text{ even, } \varepsilon = 1,$$

$$\sigma_{n,\nu} = \prod_{i=0}^{\nu-1} (q^{\nu+1-i} + 1) \qquad \text{for } n \text{ even, } \varepsilon = -1,$$

$$\sigma_{n,\nu} = \prod_{i=0}^{\nu-1} (q^{\nu-i} + 1) \qquad \text{for } n \text{ odd.}$$

For possibility (2) and possibility (3) for n odd,

$$\sigma_{n,\nu} = \prod_{i=0}^{\nu-1} (q^{\nu-i} + 1) .$$

For possibility (3) with n even,

$$\sigma_{n,\nu} = \prod_{i=1}^{\nu-1} (q^{\nu-i} + 1) .$$

If K is a finite field of characteristic 2, the bilinear form associated with a non-singular quadric in projective space is alternating (and degenerate for n odd). Further, a subspace contained in a quadric is isotropic (with respect to the bilinear form) but the converse does not hold. We mention these facts to distinguish this situation from the 2 characteristic cases considered here.

I wish to thank Professor Alex Rosenberg for informing me about Segre's results [3] and also for much kind advice.

II. PROOFS OF CASES (2) AND (3).—Our notation and terminology are mainly as in Chapter III of [1]. Let σ_n denote the number of non-zero isotropic vectors in V.

Case (2). Let V be a symplectic space. Hence n is even and it is well known that V is the orthogonal sum of hyperbolic planes. Every isotropic vector x is contained in exactly $\sigma_{(n-2),(k-1)}$ isotropic subspaces of dimension k. This is so since x is contained in a hyperbolic plane P and $\langle x\rangle^* = \langle x\rangle \perp P^*$ where P^* has the same type of geometry as V.

Hence $\sigma_n = \frac{\sigma_{n,k}(q^k - 1)}{\sigma_{(n-2),(k-1)}}$ which gives the recurssion

$$\sigma_{n,k} = \frac{\sigma_n \, \sigma_{(n-2),(k-1)}}{(q^k - 1)} .$$

As is known, $\nu = n/2$ and $\sigma_n = q^{2\nu} - 1$.

Hence $\sigma_{n,1} = \frac{(q^{2\nu} - 1)}{(q - 1)}$

$$\sigma_{n,2} = \frac{(q^{2\nu} - 1)(q^{2(\nu-1)} - 1)}{\prod_{i=1}^{2}(q^i - 1)}$$

$$\vdots$$

$$\sigma_{n,k} = \frac{\prod_{i=0}^{k-1}(q^{2(\nu-i)} - 1)}{\prod_{i=1}^{k}(q^i - 1)} .$$

This proof is similar to the one in [3] for the number of linear subspaces contained in a quadric in projective space.

Case (3): Let V be a space on which a non-alternating form f is defined where the characteristic of K is 2. Let N be the set of isotropic vectors in V. Then N is a subspace of dimension $n - 1$ [2] and we let $\langle h \rangle = N^*$.

We distinguish between the cases where n is odd and where n is even. When n is odd, N is a non-singular symplectic space, $V = N \perp \langle h \rangle$, and $\nu = (n - 1/2)$. Applying case (2) to N we have

$$\sigma_{n,k} = \frac{\prod_{i=0}^{k-1}(q^{2(\nu-i)} - 1)}{\prod_{i=1}^{k}(q^i - 1)} \quad \text{for V.}$$

Let n be even and let C be any complement of $\langle h \rangle$ in V. Clearly C is of odd dimension and can be shown to be non-singular. Let $\bar{\sigma}_{(n-1),k}$ denote the number of isotropic subspaces of dimension k in C.

To show that:

$$\sigma_{n,k} = \bar{\sigma}_{(n-1),(k-1)} + \bar{\sigma}_{(n-1),k} + \bar{\sigma}_{(n-1),(k-1)}(q^{n-2k} - 1).$$

Proof: It is easy to see that $\bar{\sigma}_{(n-1),(k-1)}$ is the number of k dimensional isotropic subspaces of V which contain $\langle h \rangle$. The number of k dimensional isotropic subspaces of V which are contained in C is $\bar{\sigma}_{(n-1),k}$ by definition.

Let W be a k dimensional subspace of V which is not contained in C and which does not contain $\langle h \rangle$. Then W must be of the form $\langle h + x \rangle \perp U$ where x is in $U^* \cap C \cap N$, $x \notin U$, and U is a $k - 1$ dimensional isotropic subspace of C. For a fixed U there are $(q^{n-2k} - 1)$ distinct such subspaces. Since the subspaces of this form are distinct for distinct U, there are $\bar{\sigma}_{(n-1),(k-1)}\,(q^{(n-2k)} - 1)$ k dimensional isotropic subspaces of V which do not contain $\langle h \rangle$ and are not contained in C. Q.E.D.

Since we know $\bar{\sigma}_{(n-1),(k-1)}$ and $\bar{\sigma}_{(n-1),k}$ from case (3) for n odd,

$$\sigma_{n,k} = \frac{(q^{n-k} - 1)\prod_{i=1}^{k-1}(q^{n-2i} - 1)}{\prod_{i=1}^{k}(q^{i} - 1)}, \qquad k \geq 2$$

and clearly

$$\sigma_{n,1} = \frac{(q^{n-1} - 1)}{(q - 1)}.$$

REFERENCES.

[1] E. ARTIN, *Geometric Algebra*, Interscience Tracts in Pure and Applied Mathematics No. 3, Interscience, New York 1957.

[2] V. PLESS, *On Witt's Theorem for Non-Alternating Symmetric Bilinear Forms over a Field of Characteristic 2*, Proc. American Mathematical Society 15, December 1964, 979–983.

[3] B. SEGRE, *Le geometrie di Galois*, «Annali di Matematica Pura ed Applicata», ser. 4ª, *48*, 1–96 (1959).

SUNTO. — Per i vari tipi di polarità in uno spazio di Galois si determina il numero – già noto per le quadriche – degli spazi autoconiugati di data dimensione.