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A new approach to the red-shift law, as a consequence of a new theory of relativity. Nota II

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Astronomia. — *A new approach to the red-shift law, as a consequence of a new theory of relativity.* Nota II (*) di GIUSEPPE ARCIDIACONO e NICOLA VIRGOPIA, presentata dal Corrisp. M. CIMINO.

4. FANTAPPIÈ'S RELATIVITY.—In 1954 L. Fantappiè pointed out that the Lorentz group on which relativistic physics is based, is a limiting case —when the radius of the space-time tends to infinity—of a new ten-parameter group, which represents the motions within itself of a four dimensional spacetime with constant curvature, namely De Sitter's space-time. But, while in general relativity this space-time is obtained under the assumption that the universe is without matter (i.e. the density of matter can be neglected), Fantappiè obtained his results by basing the reasoning on the theory of groups and not on general relativity.

The problem then arose of developing a new theory of relativity parallel to restricted relativity. In this new theory a new universal constant r (radius of the space-time) other than the old one c (velocity of light), must be taken into account. To develop this theory, without using differential geometry, but using the technique of groups (i.e. methods like the ones used in restricted relativity), it was noted that for a general observer a curved space must look as if it were flat, and that besides it was necessary to distinguish between the " absolute universe " (with constant curvature) in which the physical events happen, and the " relative universe " (tangential space) in which the fenomena are seen [1].

From this point of view a double scale for times and distances, analogous to those of Milne, was introduced, and therefore Fantappiè's relativity could be developed using the methods of elementary projective geometry.

In this theory in fact the double time-scale is given by the projective invariant τ of the non-euclidean geometry

$$\tau = \frac{t_0}{2} \log \frac{1 + \frac{t}{t_0}}{1 - \frac{t}{t_0}} \tag{4.1}$$

where $t_0 = r/c$. To compare (4.1) with (1.2) we notice that while in Milne's theory the coincidence of the two scales takes place for $t = t_0$ in our case the two scales coincide for $t = 0$. Taking the transformation:

$$\tau \to \tau - t_0 \quad , \quad t \to t - t_0$$

equation (4.1) becomes:

$$\tau = t_0 + \frac{1}{2} t_0 \log \frac{t}{t_0} - \frac{1}{2} t_0 \log \left(2 - \frac{t}{t_0}\right)$$

(*) Pervenuta all'Accademia il 16 agosto 1965.

which can be written as follows

$$(4.2) \qquad \tau = t_0 + t_0 \log \frac{t}{t_0} + \mathrm{K}(t)$$

with $\mathrm{K}(t) = -\frac{1}{2} t_0 \log \left[\frac{t}{t_0}\left(2 - \frac{t}{t_0}\right)\right]$.

As we can see, (4.2) and (1.2) coincide except for the K (t) term. In figures 1 and 2 are plotted the graphs of (1.2) and (4.2) respectively.

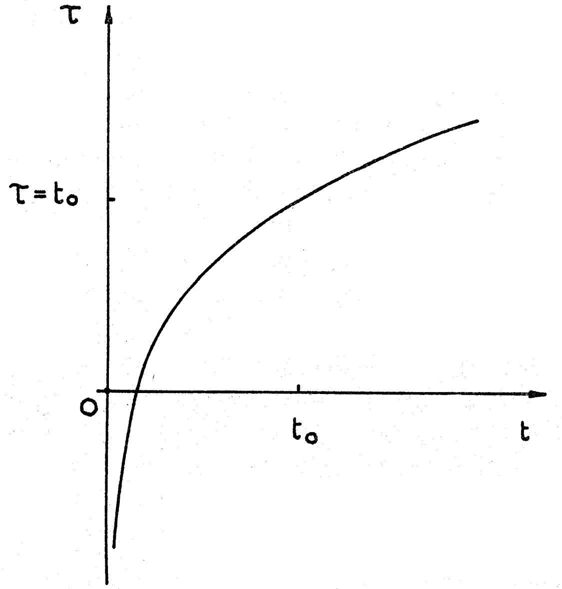

Fig. 1. – The double scale-time in Milne's theory.

The motions within itself of De Sitter's space-time are represented by the collineations which change within itself the absolute quadric of Cayley-Klein:

$$(4.3) \qquad c^2 t^2 - x^2 - y^2 - z^2 = r^2$$

and the transformations of Fantappiè's group turn out to be the following:

a) *Spatial translations*. In the case of two observers O, O′ separated from each other by a length T, the transformations are:

$$(4.4) \qquad \begin{cases} x' = \dfrac{x + \mathrm{T}}{1 - \dfrac{\alpha}{r} x} & y' = \dfrac{y\sqrt{1+\alpha^2}}{1 - \dfrac{\alpha}{r} x} \\[2ex] z' = \dfrac{z\sqrt{1+\alpha^2}}{1 - \dfrac{\alpha}{r} x} & t' = \dfrac{t\sqrt{1+\alpha^2}}{1 - \dfrac{\alpha}{r} x} \end{cases}$$

in which we have set $\alpha = \mathrm{T}/r$. For $r \to \infty$ (relativistic limit), they reduce to the ordinary spatial translations on the x–axis:

$$x' = x + \mathrm{T} \quad , \quad y' = y \quad , \quad z' = z \quad , \quad t' = t\,.$$

b) *Time-translations.* In the case of two observers at the same position but separated by the time interval T_0 (or in the case of the same observer in two different instants), we have instead:

$$r' = \frac{r\sqrt{1-\gamma^2}}{1+\frac{c}{r}\gamma t} \quad , \quad t' = \frac{t+T_0}{1+\frac{c}{r}\gamma t} \tag{4.5}$$

where $\gamma = cT_0/r$, and $\boldsymbol{r}$ is the vector whose components are x, y, z. In this case also, (4.5) reduces to the ordinary time-translations for $r \to \infty$:

$$x' = x \quad , \quad y' = y \quad , \quad z' = z \quad , \quad t' = t + T_0 \, .$$

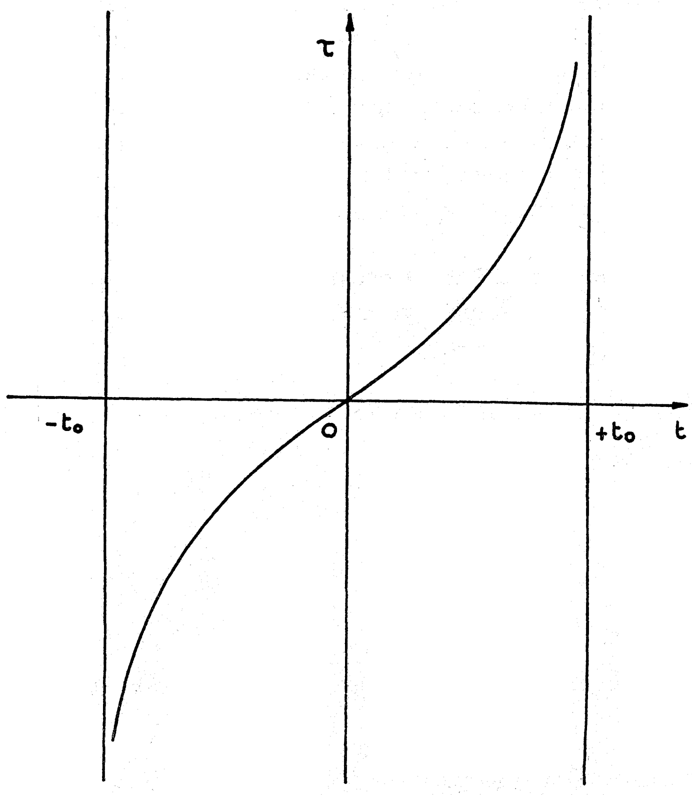

Fig. 2. – The double scale-time in Fantappiè's relativity.

The last equation of (4.5) gives a "law for adding durations", which is analogous to that of the velocities valid in special relativity:

$$d = \frac{d_1 + d_2}{1+\frac{c^2}{r^2} d_1 d_2} \tag{4.6}$$

from which it follows that if $d_1 = r/c$ and d_2 is arbitrary, then $d = r/c$.

In Fantappiè's relativity there is a limiting duration r/c which may be interpreted as the apparent age of the universe, also because for $T_0 = \pm r/c$, i.e. for $\gamma = \pm 1$, the first of (4.5) gives $\boldsymbol{r}' = 0$ whatever $\boldsymbol{r}$ may be, and therefore the universe reduces to a point.

In such a way a cosmology based on the theory of groups is obtained, in which is found the double time-scale just as in Milne's theory and also the origin of the universe from a point (as in the evolutionary theories). A cosmological perfect principle is also valid (as in the stationary theories), in the sense that the physical laws are invariant against the group and therefore remain the same for every observer.

5. VELOCITY-DISTANCE LAW IN FANTAPPIÈ'S RELATIVITY.—In Fantappiè's theory (which, as we have seen, may be considered as the long distance physics—in space and time—because its relative corrections to the physical laws become appreciable only for distances of the order of the constant r and for durations of the order of r/c), unlike what occurs in special relativity (which may be considered as high velocity physics), the velocity of a body depends essentially also upon its space-time distance from the observer O.

Let us try in fact to see what occurs in the case of spatial translations. For this, let us take—for simplicity—the bidimensional case (x, t). By differentiation from (4.4), we get:

$$dx' = \frac{\left(1 - \frac{\alpha}{r} x\right) + (x + T)\frac{\alpha}{r}}{\left(1 - \frac{\alpha}{r} x\right)^2} dx = \frac{1 + \alpha^2}{\left(1 - \frac{\alpha}{r} x\right)^2} dx$$

$$dt' = \frac{\frac{\alpha}{r} t \sqrt{1 + \alpha^2}}{\left(1 - \frac{\alpha}{r} x\right)^2} dx + \frac{\sqrt{1 + \alpha^2}}{1 - \frac{\alpha}{r} x} dt .$$

If we divide member by member and put $dx'/dt' = V'$ and $dx/dt = V$, we have:

$$V' = \frac{(1 + \alpha^2) V}{\frac{\alpha}{r} Vt \sqrt{1 + \alpha^2} + \left(1 - \frac{\alpha}{r} x\right)\sqrt{1 + \alpha^2}}$$

i.e. the formula:

$$V' = \frac{V \sqrt{1 + \alpha^2}}{1 - \frac{\alpha}{r}(x - Vt)} . \tag{5.1}$$

It follows therefore that the velocity of a far galaxy depends also on position in space and time. For the special case $V = 0$, we have also $V' = 0$; this means that if a galaxy is at rest in respect to O, it will also be at rest in respect to O'.

More interesting results may be obtained by considering the effect due to a time translation of length T_0. In fact by differentiation of (4.5), we get:

$$dr' = \frac{\sqrt{1-\gamma^2}}{1+\frac{c}{r}\gamma t}\,dr - \frac{r\sqrt{1-\gamma^2}}{\left(1+\frac{c}{r}\gamma t\right)^2}\,\frac{c}{r}\gamma dt$$

$$dt' = \frac{1+\frac{c}{r}\gamma t-\frac{c}{r}\gamma t-\gamma^2}{\left(1+\frac{c}{r}\gamma t\right)^2}\,dt = \frac{1-\gamma^2}{\left(1+\frac{c}{r}\gamma t\right)^2}\,dt$$

and putting as usual $dx'/dt' = V'$ and $dx/dt = V$, it follows:

(5.2) $$V' = \frac{1}{\sqrt{1-\gamma^2}}\left[V+\frac{c}{r}\gamma(Vt-x)\right]$$

namely

(5.3) $$V\left(1+\frac{c}{r}\gamma t\right)-\frac{c}{r}\gamma x = V'\sqrt{1-\gamma^2}$$

which gives the link between V and V′ in the case of an observer who observes a galaxy in two different instants separated by the interval T_0.

If besides we take the maximum possible interval, i.e. $T_0 = r/c$ (apparent age of the universe), the second side of (5.3) becomes zero and we get the velocity-distance relation valid in Fantappiè's relativity:

(5.4) $$V = \frac{c}{r}\,\frac{1}{1+\frac{c}{r}t} = H(t)\,x$$

where we have put:

$$H(t) = \frac{c}{r}\,\frac{1}{1+\frac{c}{r}t}.$$

As a consequence the function $R(t)$ is obtained by integration of $H(t) = \dot{R}/R$ which gives:

(5.5) $$R(t) = 1 + \frac{c}{r}t.$$

We may therefore conclude that:

$$\frac{d}{dt}H(t) = \frac{\ddot{R}R-\dot{R}^2}{R^2} = -\left(\frac{\dot{R}}{R}\right)^2 = -H^2(t)$$

and the " deceleration parameter " will be zero:

(5.6) $$q = -\frac{\ddot{R}}{RH^2} = 0.$$

From (5.4) we can say that in the " space of contemporaneity " of the observer O (namely, for $t = 0$), the velocity-distance relation

(5.7) $$V = \frac{c}{r}x \qquad \left(\text{in general: } \mathbf{V} = \frac{c}{r}\,\mathbf{r}\right)$$

holds, and this is strictly linear. In effect we see a galaxy far off in space as it is in time and therefore in (5.4) we must put $ct = x$ (in general $ct = |\mathbf{r}|$). Finally, taking into account that when we observe the events in the past the time is negative, we have:

$$V = \frac{c}{r} \frac{x}{1 - \frac{x}{r}} \qquad \left(\text{in general: } \mathbf{V} = \frac{c}{r} \frac{\mathbf{r}}{1 - \frac{|\mathbf{r}|}{r}}\right), \tag{5.8}$$

from which it follows that the velocity of light is reached only for $x = \frac{r}{2}$.

We have obtained in such a way the velocity-distance relation starting from a theory of classical type based on the group of transformations.

In the table are given the values of the velocities and distances derived from formula (5.8), the linear law $V = (c/r)\,x$ (deduced from De Sitter's model), and the law $V = c \tanh \sigma (x/c)$ (proposed by H. Nariai [2]).

V \ x	$(1/4)\,r$	$(1/2)\,r$	$(3/4)\,r$	r	∞
$\frac{c}{r} \frac{x}{1 - \frac{x}{r}}$	$0.333\ c$	c	$3\ c$	∞	0
$(c/r)\,x$	$0.250\ c$	$0.500\ c$	$0.750\ c$	c	∞
$c \tanh \sigma \frac{x}{c}$	$0.245\ c$	$0.462\ c$	$0.635\ c$	$0.762\ c$	c

The difference between the linear law of De Sitter and the one deduced from Fantappiè's group is due to the fact that in this last case the plane-representation of De Sitter's space-time is used; this is equivalent to a change of the two space and time scales.

The law (5.4) reduces to De Sitter's law when one places oneself in the contemporaneous space of the observer, i.e. for $t = 0$.

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[1] G. Arcidiacono, *Sul significato fisico della teoria di relatività finale*, « Rend. Acc. Lincei », ser. VIII, vol. XX (1956).

[2] H. Nariai, *On some linear equivalence in Kinematic Relativity*, « Sc. Rep. Tohoku University », ser. 1ª, vol. XXXVII, n. 2, p. 243.