Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Robert Carroll

Some differential problems related to spectral theory in several operators

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **39** (1965), n.3-4, p. 170–174.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1965_8_39_3-4_170_0>

Matematica. — *Some differential problems related to spectral theory in several operators.* Nota (*) di ROBERT CARROLL (**), presentata dal Socio Straniero A. WEINSTEIN.

1. We will show that for a wide class of differential equations with operator coefficients a Green's operator can be constructed by spectral considerations and consequently the Cauchy problem can be solved for suitable data. The results generalize some theorems of [1; 2] in certain respects and apply to a much wider class of problems. Some of the results have been sketched in a lecture [4] at the Séminaire de Mathématiques Supérieures, Université de Montréal, été 1965, and will appear in detail in the mimeographed proceedings of this seminar to be ready some time next year; additional details can be found in [1; 2; 3; 5] and with these as references we are also able to furnish in this note the demonstrations of the other theorems announced here whose proofs are not be found in [1; 4].

2. We consider the differential operator

$$\mathrm{M}(u) = u^{(m)} + \sum_{j=0}^{m-1} \mathrm{P}_j(t, \Lambda_k, b_l)\, u^{(j)} \tag{2.1}$$

where $u^{(j)} = d^j u/dt^j$, P_j is a polynomial in (Λ_j, b_l) of degree p_j in the Λ_k, the Λ_k $(k = 1, \cdots, \mathrm{N})$ are self-adjoint (densely defined) positive operators in a separable Hilbert space H $((\Lambda_k u, u) \geq c_k \| u \|^2)$, the $a_k = \Lambda_k^{-1}$ and b_l $(l = 1, \cdots, \mathrm{L})$ are commuting bounded operators in H, $a_k \Lambda_s \subset \Lambda_s a_k$ and $b_l \Lambda_s \subset \Lambda_s b_l$, and $t \to u(t)$ takes values in H with $0 \leq t \leq \mathrm{T} < \infty$. Let A be the Banach algebra generated by the a_k, b_l, and the identity e and let $\sigma_{\mathrm{A}} \subset \mathbf{C}^{\mathrm{N+L}}$ be the joint spectrum of the generators a_k, b_l, defined as the image of the carrier space Φ_{A} under the map $\gamma : \Phi_{\mathrm{A}} \to \mathbf{C}^{\mathrm{N+L}}$ where $\varphi(x) = \hat{x}(\varphi)$ and $\gamma(\varphi) = (\hat{a}_1(\varphi), \cdots, \hat{a}_{\mathrm{N}}(\varphi), \hat{b}_1(\varphi), \cdots, \hat{b}_{\mathrm{L}}(\varphi))$. We recall that σ_{A} is compact and homomorphic to Φ_{A}. The spectral variables corresponding to a_k are denoted by z_k and similarly $b_l \sim z'_l$; we write $\lambda_k = 1/z_k$ $(k = 1, \cdots, \mathrm{N})$ and note that $c_k \leq \lambda_k \leq \infty$ or $0 \leq z_k \leq c_k^{-1}$.

We suppose that there is a compact set $\Xi \subset \mathbf{C}^{\mathrm{N+L}}$ with $\sigma_{\mathrm{A}} \subset \Xi$ or *not* and $z_k \geq 0$ real in Ξ $(k = 1, \cdots, \mathrm{N})$, such that for any polynomial $p(z, z')$ one has $\| p(a, b) \| \leq c \sup | p(z, z') |$ for $(z, z') \in \Xi$ (we write $p(a, b) = \Gamma p(z, z')$). Such a set will be called a p–spectral set and we will describe in various ways the construction (always possible) of such sets (as small as we can make them) and the nature of the constant c which intervenes; moreover our

(*) Pervenuta all'Accademia il 6 settembre 1965.

(**) Research supported by NSF grant GP 4575.

p–spectral sets will be made polynomially convex (p–convex) for reasons to appear. A set Ξ is p–convex if for any $\zeta \notin \Xi$ there is a polynomial p such that $p(\zeta) = 1$ and $|p(\eta)| < 1$ for $\eta \in \Xi$. It is well known that σ_A is p–convex for example (see [6] for Banach algebras, with suitable precautions concerning the false theorem 1 on p. 86, and [6; 7] for functional calculi).

The spectral form of the Green's operators is obtained formally as

$$g(t, \tau, \lambda, z') = \exp[-(t-\tau) P(\lambda, z')] \tag{2.2}$$

$$P(\lambda, z') = \begin{pmatrix} 0 & -1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & -1 \\ P_0 & P_1 & \cdots & P_{m-1} \end{pmatrix} \tag{2.3}$$

where $P_j = P_j(\lambda, z')$ is assumed independent of t. The case of time dependent coefficients can be handled as well of course (cf. [8]) but we will not spell out the details since it will be routine after seeing the present cases. The characteristic roots of $-P(\lambda, z')$ are the roots of

$$Q(\zeta, \lambda, z') = \zeta^m + \sum_{j=0}^{m-1} P_j(\lambda_k, z_l') \zeta^j \tag{2.4}$$

and we set $\psi(\lambda, z') = \max \operatorname{Re} \zeta_k(\lambda, z')$.

We denote by $\sigma_A(b)$ the joint spectrum of the b_l in $\mathbf{C}^L$, i.e. the image of Φ_A under the map $\alpha : \Phi_A \to \mathbf{C}^L$ given by $\alpha(\varphi) = (\hat{b}_1(\varphi), \cdots, \hat{b}_L(\varphi))$; evidently $\sigma_A(b) = pr_L \sigma_A$ where pr_L is the projection onto the space $\mathbf{C}^L$. $M(u)$ will be called Ξ hyperbolic if $\psi(\lambda, z') \leq c|\lambda| + c_1$ for z' bounded and $\psi(\lambda, z') \leq c_2$ for real $\lambda_k \geq c_k$ and $z' \in \sigma_A(b)$ ($|\lambda| = (\Sigma|\lambda_i|^2)^{1/2}$); $M(u)$ will be called Ξ parabolic if $\psi(\lambda, z') \leq -c|\lambda|^h + c_1$ for real $\lambda_k \geq c_k$ and $z' \in \sigma_A(b)$ where $0 < h \leq \max p_{k/(m-k)}$ $(0 \leq k \leq m-1)$. In order to complete the definition of g later we shall have to consider also the sets $V_j = \{(z, z') \in \Xi, z_j = 0\}$, $1 \leq j \leq N$. Finally we define $\tilde{C}(\Xi)$ to be the uniform closure of the polynomials on Ξ and $\tilde{M}(\Xi)$ to be the bounded Baire functions obtained from $\tilde{C}(\Xi)$ by taking iterated pointwise bounded sequential limits (i. e. $f_n \in \tilde{C}(\Xi)$, $|f_n| \leq c$, and $f_n(\zeta) \to f(\zeta)$ for each $\zeta \in \Xi$ implies $f \in \tilde{M}(\Xi)$ and repetitions give all $\tilde{M}(\Xi)$). The continuous maps $\delta_{xy} : f \to (\Gamma(f) x, y)_H : \tilde{C}(\Xi) \to \mathbf{C}$ define by Hahn-Banach a family of measures ν_{xy} on Ξ with $(\Gamma(f) x, y)_H = \int f d\nu_{xy}$ and, using the Lebesgue bounded convergence theorem, one can extend Γ as a homomorphism (algebraic) to $\tilde{M}(\Xi)$ with values in $\mathfrak{L}(H)$ ($\mathfrak{L}(H)$ denotes bounded operators). We use the same symbol $\Gamma : P(\Xi) \to A$, $\Gamma : \tilde{C}(\Xi) \to A$, and $\Gamma : \tilde{M}(\Xi) \to \mathfrak{L}(H)$.

First consider the parabolic case and define g by (2.2) for $t > \tau$ with $g(\tau, \tau, \lambda, z')$ defined as 1 on $\Xi - \cup V_j$ and zero on $\cup V_j$. Using the inequality

$$|g| \leq c(1 + (t-\tau)^{1/p} |\lambda|)^{p(m-1)} \exp(-c_1(t-\tau)|\lambda|^h \tag{2.5}$$

where $p \geq \max p_j$ one shows that $g \in \tilde{\mathrm{M}}(\Xi)$ (i.e. the elements of g belong to $\tilde{\mathrm{M}}(\Xi)$). The crucial fact is that a function holomorphic in a neighbourhood of the polynomially convex Ξ belongs to $\tilde{\mathrm{C}}(\Xi)$ (see [9]) and here one considers functions $g_\varepsilon(t, \tau, \lambda, z') = \exp\left[-(t-\tau)\,\mathrm{P}\left(\frac{1}{z+\varepsilon}, z'\right)\right]$, $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_N)$, for $t > \tau$ which tend to g pointwise boundedly as $\varepsilon \to 0$. Defining $\mathfrak{L}_w(\mathrm{H})$ as $\mathfrak{L}(\mathrm{H})$ with the weak operator topology and $\vec{\mathrm{H}} = \mathrm{H}^m$ we obtain (see [4]).

THEOREM 1.—*Let* M (u) *be* Ξ *parabolic with* Ξ *a p–convex p–spectral set for* A *as indicated. Then there exists a weak Green's operator* $\mathrm{G}(t, \tau) = \Gamma g$ *with* $(t, \tau) \to \mathrm{G}(t, \tau) \in \mathrm{C}^0(\mathfrak{L}_w(\vec{\mathrm{H}}))$ *for* $t \geq \tau$, $t \to \mathrm{G}(t, \tau)$ *as well as* $\tau \to \mathrm{G}(t, \tau)$ *belong to* $\mathrm{C}'(\mathfrak{L}_w(\vec{\mathrm{H}}))$ *for* $t > \tau$, *and* $\mathrm{G}_t + \mathrm{PG} = 0$ *formally with* $\mathrm{G}(\tau, \tau) = \mathrm{I}$.

In the general hyperbolic case g will usually not be bounded (see however [1; 2; 4] and remark 2) and we can treat this as a special case of a Ξ–correct problem where M (u) Ξ–correct means $\psi(\lambda, z') \leq c$ for $\lambda_k \geq c_k$ real and $z' \in \sigma_{\mathrm{A}}(b)$. Then one has an estimate $|g| \leq c_1 (1 + |\lambda|)^{p(m-1)}$ and the index of correctness is defined as the smallest number μ such that for $\lambda_k \geq c_k$ real, $z' \in \sigma_{\mathrm{A}}(b)$, and $0 \leq \tau \leq t \leq \mathrm{T} < \infty$, there holds $|g| \leq c(1 + |\lambda|)^\mu$, Let $s > \mu$ and define $\mathcal{K}(t, \tau, \lambda, z') = |||\lambda|||^{-s} g(t, \tau, \lambda, z')$ where $|||\lambda||| = (\Sigma \lambda_i^2)^{1/2}$ (thus $|||\lambda||| = |\lambda|$ in Ξ). Then consider the functions $\mathcal{K}_\varepsilon = \mathcal{K}(t, \tau, 1/z + \varepsilon, z')$, $\varepsilon = (\varepsilon_1, \cdots, \varepsilon_N)$, where one chooses a branch of $|||1/z + \varepsilon|||^{-s}$ so as to deal with $\mathcal{K}_\varepsilon$ holomorphic in a neighborhood of Ξ. Then $\mathcal{K}_\varepsilon \in \tilde{\mathrm{C}}(\Xi)$, $\mathcal{K}_\varepsilon$ is continuous in (t, τ, λ, z') for $\lambda < \infty$, $|\mathcal{K}_\varepsilon| \leq c$ for $(z, z') \in \Xi$, and $\mathcal{K}_\varepsilon \to \mathcal{K}$ as $\varepsilon \to 0$ where $\mathcal{K}$ is defined as $\mathcal{K} = |||\lambda|||^{-s} g$ for $t \geq \tau$, $\lambda < \infty$ and $\mathcal{K} = 0$ for $t \geq \tau$ and $(z, z') \in \cup \mathrm{V}_j$. Hence $\mathcal{K} \in \tilde{\mathrm{M}}(\Xi)$ and one defines $\mathrm{K}(t, \tau) = \Gamma\mathcal{K} \in \mathfrak{L}(\vec{\mathrm{H}})$. To check continuity note that if $(t_n, \tau_n) \to (t, \tau)$ then $\mathcal{K}(t_n, \tau_n, \lambda, z') \to \mathcal{K}(t, \tau, \lambda, z')$ pointwise boundedly in Ξ and hence by bounded convergence $\mathrm{K}(t_n, \tau_n) \to \mathrm{K}(t, \tau)$ in the weak operator topology (note that $|||\lambda|||^{-s} = 0$ on $\cup \mathrm{V}_j$ when checking continuity at (τ, τ)). Next observe that $|\partial\mathcal{K}/\partial t| = |-\mathrm{P}\mathcal{K}| \leq c |\lambda|^p |\mathcal{K}|$ and we take $p = 2\nu$ even now for convenience. Define an operator $\Omega = \Gamma(|||\lambda|||^{-2}) = \Gamma([\Sigma\, 1/z_i^2]^{-1})$ which evidently makes sense (consider $|||1/z + \varepsilon|||^{-2}$ to conclude that $|||\lambda|||^{-2} \in \tilde{\mathrm{M}}(\Xi)$). Then if $\vec{x} = \Omega^\nu \vec{y}$ and $\Delta\mathrm{K} = \mathrm{K}(t + \Delta t, \tau) - \mathrm{K}(t, \tau)$ we have, using the Fubini-Tonelli theorems

$$(2.6) \qquad (\Delta\mathrm{K}\vec{x}, \vec{z})_{\mathrm{H}} = (\Delta\mathrm{K}\,\Omega^\nu \vec{y}, \vec{z}) = \int_\Xi \Delta\mathcal{K}\, |||\lambda|||^{-2\nu}\, d\nu_{\vec{y}\vec{z}}$$

$$= -\int_t^{t+\Delta t} \left(\int_\Xi \mathrm{P}\mathcal{K}\, |||\lambda|||^{-2\nu}\, d\nu_{\vec{y}\vec{z}} \right) d\eta$$

$$= -\int_t^{t+\Delta t} (\Gamma(\mathrm{P}\, |||\lambda|||^{-p})\, \mathrm{K}(\eta, \tau)\vec{y}, \vec{z})\, d\eta\,.$$

Setting $g = [\Sigma\, 1/z_i^2]^{-1}$ we note that $\Gamma(\pi z_k^2 g^{-1})\,\Gamma(g)\,x = 0$ implies $x = 0$ and hence $y = \Gamma(g)\,x = \Omega x = 0$ implies $x = 0$, i.e. Ω is $1 - 1$. We can now state (cf. again [1; 4]).

THEOREM 2.—*Let* $M(u)$ *be* Ξ *correct with index* μ, $s > \mu$ $p = 2\nu$ *even, and* Ξ *a p-convex p-spectral set for* A *as indicated. Then* $(t, \tau) \rightarrow$ $\rightarrow K(t, \tau) \in C^0(\mathfrak{L}_w(\vec{H}))$ *with* $t \rightarrow K(t, \tau)$ (*and* $\tau \rightarrow K(t, \tau)$) *belonging to* $C'(\mathfrak{L}_w(\vec{D}(\Omega^{-\nu}), \vec{H}))$ ($D(\Omega^{-\nu})$ *has the graph topology*).

Remark 1.—Evidently to solve a homogeneous Cauchy problem one considers initial data $\vec{x}$ of the form $\vec{x} = S\Omega^{\nu}\vec{y}$ where S is defined as $\Gamma(|||\lambda|||^{-s})$ with the same choice of branch as before. Then $\vec{w} = K(t, \tau)\,\Omega^{\nu}\vec{y}$ satisfies $\vec{w}' + P(\Lambda, b)\,\vec{w} = 0$ formally with $\vec{w}(\tau) = \vec{x}$.

Remark 2.—Theorem 3 of [1] can be generalized considerably by using spectral sets Ξ instead of uniform algebras (see [4] for details). This is a case where g is bounded on Ξ and it requires a much more delicate argument to show $g \in \tilde{M}(\Xi)$ (see [1; 4]).

3. We give here two theorems about spectral sets which are proved in detail in [4]. The first idea is to construct Ξ of the form $\Xi = \pi\,[0, c_k^{-1}] \times J$ where $J \subset \mathbf{C}^L$ is a suitable compact p-convex set in $\mathbf{C}^L$ containing $\sigma_A(b)$ such that $\|p(z, b)\| \leq c \sup |p(z, z')|$ for $z' \in J$ (z fixed). We give two ways of finding such J, each with certain merits, one way based on the Weil integral [10] and the other on the Arens functional calculus [7]. Then one can shows that Ξ is p-convex and use the spectral theorem for self-adjoint operators to obtain the desired estimates.

THEOREM 3.—*There exist p-spectral p-convex sets* Ξ *for* A *with* $z_i \geq 0$ *in* Ξ $(i = 1, \cdots, N)$ *and* $\sigma_A \subset \Xi$.

An improvement of this in one direction (with a loss of knowledge in another) can be obtained by recalling that if $\Sigma \|b_i\|^2 \leq 1$ then there is a Hilbert space $\tilde{H} \supset H$ and permutable unitary operators U_i $(i = 1, \cdots, L)$ with $\pi b_i^{n_i} = pr\, \pi U_i^{n_i}$, $n_i \geq 0$, where $b = pr\, U$ means $bx = PUx$ for $P : \tilde{H} \rightarrow H$ the orthogonal projection (see [11]). Then if J is the joint spectrum in $\mathbf{C}^L$ of the U_i considered as generators (with the identity) of a Banach algebra B in $\mathfrak{L}(\tilde{H})$, i.e. $J = \sigma_B$, it follows that the desired domination occurs with $c = 1$. One can then prove (since we can suppose $\Sigma \|b\|^2 \leq 1$ without loss of generality).

THEOREM 4.—*p-spectral p-convex sets* Ξ *for* A *can be found such that* $\|p(a, b)\| \leq \sup |p(z, z')|$ *for* $(z, z') \in \Xi$ *with* $z_i \geq 0$ *in* Ξ $(i = 1, \cdots, N)$.

One expects sets Ξ of some kind to exist; the point of the theorems is to describe small ones, to find Ξ with $z_i \geq 0$, and to estimate the constants.

REFERENCES.

[1] R. CARROLL, *On the spectral determination of the Green's operator*, *to appear*.

[2] R. CARROLL, *Sur la définition spectrale d'un opérateur de Green*, «Comptes Rendus Acad. Sci. Paris», *257*, 1663–1665 (1963).

[3] R. CARROLL, *Problems in linked operators*. II., to appear in «Math. Annalen».

[4] R. CARROLL, *Quelques problèmes différentiels abstraits*, Séminaire de Math. Sup., Université de Montréal, été 1965.

[5] R. CARROLL, *Problèmes d'opérateurs reliés*, «Comptes Rendus Acad. Sci. Paris», *259*, 1687–1689 (1964).

[6] I. GELFAND, D. RAIKOV, G. ŠILOV, *Communtative normed rings*, Moscow 1960.

[7] R. ARENS, *The analytic functional calculus in communtative algebras*, «Pacific Jour. of Math.», *11*, 405–429 (1961).

[8] I. GELFAND, G. ŠILOV, *Certain questions in the theory of differential equations*, Moscow 1958.

[9] R. GUNNING, H. ROSSI, *Analytic functions of several complex variables*, Prentice–Hall, Englewood Cliffs 1965.

[10] B. FUKS, *Introduction to the theory of analytic functions of several complex variables*, Moscow 1962; Special topics in the theory of analytic functions of several complex variables, Moscow 1963.

[11] S. BREHMER, *Über vertauschbare Kontraktionen des Hilbertschen Raumes*, «Acta Sci. Math. Szeged», *22*, 106–111 (1961).