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Jerzy Herzsberg

Extension of the Notion «Comportamento Associato»

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Geometria. — *Extension of the Notion* «Comportamento Associato». Nota (*) di JERZY HERSZBERG, presentata dal Socio B. SEGRE.

1\. The notion of associated behaviour (*comportamento associato*) was introduced first by B. Segre in his paper on resolution of singularities [3]. Some further results were given by the same author in [4] and related topics were also considered in [5] and [1].

Let V be an algebraic primal in S_d. For simplicity we take non-homogeneous coordinates $x_1, \cdots, x_d$ and the equation of V is then given by $f(x) = 0$, where $f(x)$ is a polynomial in $x_1, \cdots, x_d$.

We write

$$g(x) \sim \{f(x)\}_r,$$

if there exists an identity of the form

$$g(x) = a(x) f(x) + \sum_{s \le r} b_s(x)\, \omega^{(s)} f(x),$$

where $a(x)$, $b_s(x)$ are polynomials in $x_1, \cdots, x_d$, and $\omega^{(s)}$ denotes some differential operator of the form $\frac{\partial^s}{\partial x_1^{\lambda_1} \cdots \partial x_d^{\lambda_d}}$ with $\lambda_1 + \cdots + \lambda_d = s$.

The primal W of S_d whose equation is $g(x) = 0$ is said to have *associated bahaviour* at 0 of index r with V. If the multiplicity of V at 0 is m $(m > r)$ and if the multiplicity of W at 0 is precisely $m - r$, then the behaviour is said to be *regular*.

The primal W which has associated behaviour with V plays a fundamental role in connection with Segre's methods of resolution of singularities, as pointed out by the author in [3]. However, one of the conditions that a subvariety may be a base of a dilatation is that the subvariety is non-singular. If V has a subvariety C such that each poit of C is s-ple on V and no point of V is s_1-ple, where $s_1 > s$, then it is possible that C itself has singularities. Thus, we have to perform preliminary dilatations on C, in order to resolve the singularities of V. Here, however, C is *not* a primal and it appears that it is essential to extend the notion of associated behaviour to varieties situated in a space of arbitrary dimension.

In this paper we estend the definition of associated behaviour and show that such associated primals exist and have properties similar to those shown by B. Segre in [4].

2\. Let V be an irreducible algebraic variety of dimension d situated in a projective space of dimension n over the complex number field K, where $n \ge d + 1$. For simplicity we take non-homogeneous coordinates $x_1, \cdots, x_n$.

(*) Pervenuta all'Accademia il 14 luglio 1965.

Then the variety V is given by a prime ideal $\mathfrak{p}$ in the polynomial ring $K[x_1, \cdots, x_n]$. Suppose C is a subvariety of V such that

i) C is not at infinity,
ii) C itself has no multiple points.
iii) each point of C is s-ple on V, where $s > 1$,
iv) no point of V has multiplicity s_1, where $s_1 > s$.

If, in general, $F = F_\alpha + F_{\alpha+1} + \cdots + F_\beta$, where $F_\alpha \neq 0$ and F_i is a form of degree i, then we say that the *order* of F is α and F_α is called the *subform* of F. We write $o(F) = \alpha$.

Let the basis of the ideal $\mathfrak{p}$ be $f_1, \cdots, f_m$. Suppose C passes through the origin o and suppose further that $o(f_i) = 1$ for $i = 1, 2, \cdots, \rho$ and $o(f_i) > 1$ for $i > \rho$. Let Φ_i be the subform of f_i. If $\Phi_1, \cdots, \Phi_\varrho$ are linearly dependent, then we may replace $f_1, \cdots, f_\varrho$ by suitable linear combinations of them which have the following properties:

i) $o(f_i) = 1$ for $i = 1, 2, \cdots, r$, where $r < \rho$,
ii) $o(f_i) > 1$ for $i = r + 1, \cdots, \rho$,
iii) $\Phi_1, \cdots, \Phi_r$ are linearly independent.

Now V is of dimension d. Hence $r \leq d$. We assert that $r < d$. For suppose that $r = d$. Then $\Phi_1, \cdots, \Phi_d$ can be taken as the uniformising parameters of V at o and thus o is a simple point of V. This is a contradiction to our hypothesis and thus the assertion is proved. Of course, it is possible that $r = 0$, i. e. $o(f_i) > 1$ for all i.

We now proceed in the same way as above and arrange the basis of $\mathfrak{p}$ so that the following properties are satisfied:

i) $o(f_j) = n_{i+1}$ for $j = r_i + 1, \cdots, r_{i+1}$ and $i = 0, 1, \cdots, M - 1$, $r_0 = 0$, $r = r_1$,
ii) Φ_j $(j = r_i + 1, \cdots, r_{i+1})$ are linearly independent for $i = 0, 1, \cdots, M - 1$ and $r_M = m$,
iii) $1 \leq n_1 < n_2 < \cdots < n_M$.

Let $\mathfrak{a}_0$ be the ideal formed by the subforms Φ_i. Then $\mathfrak{a}_0$ is a homogeneous ideal and represents a cone K_0 which determines the multiplicity of V at o (cf. [2]). Thus K_0 is of order s when all its components are counted with the proper multiplicity.

We assumed that each point of C is of multiplicity s on V. Thus, if P is a general pont of C, we may transform the origin to P and obtain a homogeneous ideal in the same way as we obtained $\mathfrak{a}_0$. Hence, with each point of C we may associate a homogeneous ideal and we denote it by $\mathfrak{a}_0[P]$. The cone $K_0[P]$ represented by it is also of order s. Suppose that the basis of the ideal $\mathfrak{a}_0$ is given by the forms Φ_i, $i = i_1, i_2, \cdots, i_\mu$ and $i_1 < i_2 < \cdots < i_\mu \leq M$. Since each point of C has multiplicity s, it follows that f_i, has has the same multiplicity at each point of C for $i = i_1, i_2, \cdots, i_\mu$.

Let k be an integer such that $r_i + 1 \leq k \leq r_{i+1}$ and $k = i_j$ for some j, $1 \leq j \leq \mu$. Put

$$g_k(x) = a_k(x) f_k(x) + \sum_{1 \leq r \leq m_k} b_r^{(k)}(x)\, \omega^{(r)} f_k(x),$$

where m_k is an integer, $m_k \leq n_{k+1} - 1$ and $a_k(x)$, $b_r^{(k)}(x)$ are polynomials in $x_1, \cdots, x_n$.

Suppose $o[g_k(x)] = n_{k+1} - m_k$. Suppose further that we choose the polynomials $g_k(x)$ and the integers m_k so that $\lambda = n_{i+1} - m_i$ for $i = i_1, i_2, \cdots, i_\mu$ and $1 \leq \lambda \leq n$, where $n = \min\limits_{1 \leq i \leq M} (n_i)$.

Let

$$(1) \qquad g(x) = \sum_{i=1}^{M} \Phi_i(x)\, g_i(x),$$

where $\Phi_i(x) \in K[x]$.

We now give a definition.

DEFINITION. A primal W represented by the equation (1) is said to have *associated behaviour* with V along C of order λ.

We remark that here $1 \leq \lambda \leq n$ and n is possibly less than $s - 1$. Indeed, if C is a curve of order 9 in S_3 with a quadruple point at P, given by the intersection of two cubic surfaces, each having a double pount at P, but otherwise general, then $s = 4$, but the only choice for λ is $\lambda = 1$.

Furthermore, $g_i(x) = a_i(x) f_i(x)$ for $i = 1, 2, \cdots, r$, and we also remark that the polynomials $g_i(x)$ are so chosen that the corresponding primals have *regular* associated behaviour. We restrict our definition to this case only.

We now perform a dilatation with C as the base. We obtain a variety V_1 corresponding to V situated in an affine space A_N. Let C_1 be a subvariety of V_1, corresponding to C of V. Suppose C_1 is also s–ple on V_1 and C_1 is not at infinity. If P_1 is any point of C_1 we form a homogeneous ideal $a_1[P_1]$, as before. Since the multiplicity of V_1 at P_1 is s, hence $K_1[P_1]$ is of order s and thus $\mathfrak{a}_1[P_1]$ repesents a cone of order s.

Now $\mathfrak{a}_0$ is given by Φ_i, $i = i_1, i_2, \cdots, i_\mu$, where $1 \leq \mu \leq m$. Let the proper transform of f_i be $f_i^{(1)}$ and let $\Phi_i^{(1)}$ be the subform of $f_i^{(1)}$ for $i = 1, 2, \cdots, m$. Here $\Phi_i^{(1)}$ is not necessarily the proper transform of Φ_i. Then the polynomials $\Phi_i^{(1)}$, where $i = i_1, i_2, \cdots, i_\mu$ certainly occur among the basis of the ideal $\mathfrak{a}_1[P]$. Since the multiplicity of V_1 at P_1 is s, the cone $K_1[P_1]$ is of degree s and thus the multiplicities of f_i, $i = i_1, \cdots, i_\mu$, are not diminished.

But $f_i = 0$ represents a primal W_i and C is s_i–ple on it, say. Thus, after the transformation, C_1, is also s_i–ple on V_1. Hence, by the result of B. Segre [4] the transform $W_i^{(1)}$ of W_i has regular associated behaviour with V_1 along C_1.

We thus state

THEOREM I.—*Let* V *be a variety and let* W *have associated behaviour with* V *along* C. *Suppose* C *is s–ple on* V *and suppose we apply a dilatation with* C *as the base. If* V_1, W_1 *are the transforms of* V, W *respectively and if* C_1 *correponding to* C, *is s–ple on* V_1, *then* W_1 *has associated behaviour with* V_1 *along* C_1.

We may now apply a sequence of dilatations and repeating the argument we obtain

THEOREM II.—*With the notation of Theorem I, suppose we apply a sequence of dilatations which takes* V, W, C *into* V_i, W_i, C_i, *respectively. If* C_i *is s-ple on* V_i, *then* W_i *has associated behaviour with* V_i *along* C_i.

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SUNTO. — Si dà un'estensione della nozione di « comportamento associato » dovuta a B. Segre [4], e si stabilisce che anche per essa sussiste una proprietà d'invarianza di fronte alle dilatazioni.