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Maxwell's Equations in a Deformed Body

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Elettromagnetismo. — *Maxwell's Equations in a Deformed Body.* Nota di J. B. WALKER, A. C. PIPKIN, and R. S. RIVLIN, presentata [(*)] dal Corrisp. G. FICHERA.

We consider a body which undergoes a static deformation in which a particle of the body initially at $\mathbf{X}$ in some reference configuration occupies the position $\boldsymbol{x}(\mathbf{X})$ in the deformed state. Let $\boldsymbol{e}$, $\boldsymbol{b}$, $\boldsymbol{d}$, $\boldsymbol{h}$, and $\boldsymbol{j}$ denote the electric, magnetic induction, electric displacement, magnetic intensity, and current density fields at the particle. Maxwell's equations in integral form are

$$\text{(1)}\qquad \begin{cases} \displaystyle\int_s \boldsymbol{e}\cdot d\boldsymbol{s} = -\frac{\partial}{\partial t}\int_a \boldsymbol{b}\cdot d\boldsymbol{a}, \\ \displaystyle\int_s \boldsymbol{h}\cdot d\boldsymbol{s} = \frac{\partial}{\partial t}\int_a \boldsymbol{d}\cdot d\boldsymbol{a} + \int_a \boldsymbol{j}\cdot d\boldsymbol{a}, \end{cases}$$

where $d\boldsymbol{s}$ is a vector element of a fixed closed curve s and $d\boldsymbol{a}$ is a vector element of area of a fixed surface, a, bounded by s. We suppose that the curve S and surface A in the body in its reference state become the curve s and surface a, respectively, in the deformed body, as a result of the deformation.

In Cartesian tensor notation, equations (1) may be re-written as

$$\text{(2)}\qquad \begin{cases} \displaystyle\int_s e_i\, ds_i = -\frac{\partial}{\partial t}\int_a b_i\, da_i, \\ \displaystyle\int_s h_i\, ds_i = \frac{\partial}{\partial t}\int_a d_i\, da_i + \int_a j_i\, da_i. \end{cases}$$

We have

$$\text{(3)}\qquad ds_i = x_{i,\mathrm{P}}\, d\mathrm{S}_\mathrm{P} \qquad \text{and} \qquad da_i = |\partial x/\partial \mathrm{X}|\, \mathrm{X}_{\mathrm{P},i}\, d\mathrm{A}_\mathrm{P}.$$

By introducing (3) into (2) and employing the notation

$$\text{(4)}\qquad \begin{cases} \mathrm{E}_\mathrm{P} = x_{i,\mathrm{P}}\, e_i \quad , \ \mathrm{H}_\mathrm{P} = x_{i,\mathrm{P}}\, h_i, \\ \mathrm{B}_\mathrm{P} = |\partial x/\partial \mathrm{X}|\, \mathrm{X}_{\mathrm{P},i}\, b_i \quad , \ \mathrm{D}_\mathrm{P} = |\partial x/\partial \mathrm{X}|\, \mathrm{X}_{\mathrm{P},i}\, d_i, \\ \mathrm{J}_\mathrm{P} = |\partial x/\partial \mathrm{X}|\, \mathrm{X}_{\mathrm{P},i}\, j_i, \end{cases}$$

we obtain

$$\text{(5)}\qquad \begin{cases} \displaystyle\int_\mathrm{S} \mathrm{E}_\mathrm{P}\, d\mathrm{S}_\mathrm{P} = -\frac{\partial}{\partial t}\int_\mathrm{A} \mathrm{B}_\mathrm{P}\, d\mathrm{A}_\mathrm{P}, \\ \displaystyle\int_\mathrm{S} \mathrm{H}_\mathrm{P}\, d\mathrm{S}_\mathrm{P} = \frac{\partial}{\partial t}\int_\mathrm{A} \mathrm{D}_\mathrm{P}\, d\mathrm{A}_\mathrm{P} + \int_\mathrm{A} \mathrm{J}_\mathrm{P}\, d\mathrm{A}_\mathrm{P}. \end{cases}$$

(*) Nella seduta dell'8 maggio 1965.

These are Maxwell's equations in integral form referred to the initial configuration of the body. We note that there is no distinction between the time partial derivative holding x_i fixed and the derivative holding X_A fixed, because the deformation is static. The relations (5) may be re-written in vector form as

$$(6)\qquad \begin{cases} \displaystyle\int_{\bar{S}} \mathbf{E}\cdot d\mathbf{S} = -\frac{\partial}{\partial t}\int_{\bar{A}} \mathbf{B}\cdot d\mathbf{A}, \\ \displaystyle\int_{\bar{S}} \mathbf{H}\cdot d\mathbf{S} = \frac{\partial}{\partial t}\int_{\bar{A}} \mathbf{D}\cdot d\mathbf{A} + \int_{\bar{A}} \mathbf{J}\cdot d\mathbf{A}, \end{cases}$$

where $\mathbf{E}$ denotes the vector with Cartesian components E_P and analogous meanings are attached to the other vectors in (6).

The Maxwell equations in differential form corresponding to (6) are

$$(7)\qquad \operatorname{Curl}\mathbf{E} = -\partial\mathbf{B}/\partial t \quad , \quad \operatorname{Curl}\mathbf{H} = \partial\mathbf{D}/\partial t + \mathbf{J}.$$

We employ a capital initial letter in "Curl" to indicate that it is taken with respect to the coordinates in the reference state.

Let ρ be the charge per unit volume in the deformed body. Let dv be an element of volume of the deformed body and v a domain in the deformed body bounded by a closed surface $\bar{a}$. Let $d\bar{\boldsymbol{a}}$ be a vector element of area of $\bar{a}$. Then, we have

$$(8)\qquad \int_{\bar{a}} \boldsymbol{d}\cdot d\bar{\boldsymbol{a}} = \int_{v} \rho\, dv \quad , \quad \int_{\bar{a}} \boldsymbol{b}\cdot d\bar{\boldsymbol{a}} = 0.$$

Now, let V be the domain in the reference state which becomes the domain v in the deformed body. Let $\bar{A}$ denote the boundary of V and let dV and $d\bar{\mathbf{A}}$ denote the elements in the reference state corresponding to the elements dv and $d\bar{\boldsymbol{a}}$ in the deformed state. We define $\hat{\rho}$ by

$$(9)\qquad \hat{\rho} = \rho\, dv/dV = \rho\, | \partial x/\partial X |.$$

Then $\hat{\rho}$ is the charge measured per unit volume in the reference state. Also, employing Cartesian tensor notation, we have (cf., (3)).

$$(10)\qquad d\bar{a}_i = | \partial x/\partial X |\, X_{P,i}\, d\bar{A}_P.$$

With (9), (10), and (4), the relations (8) yield

$$(11)\qquad \int_{\bar{A}} D_P\, d\bar{A}_P = \int_{V} \hat{\rho}\, dV \quad , \quad \int_{\bar{A}} B_P\, d\bar{A}_P = 0.$$

In vector notation these equations are

$$(12)\qquad \int_{\bar{A}} \mathbf{D}\cdot d\bar{\mathbf{A}} = \int_{V} \hat{\rho}\, dV \quad , \quad \int_{\bar{A}} \mathbf{B}\cdot d\bar{\mathbf{A}} = 0.$$

In differential form, these become

$$(13)\qquad \operatorname{Div}\mathbf{D} = \hat{\rho} \quad , \quad \operatorname{Div}\mathbf{B} = 0.$$

Again the capital initial letter in " Div " indicates that the divergence is taken with respect to the coordinates in the reference state.

From (6) and (12) we obtain the equation of conservation of electric charge in the form

$$\frac{\partial}{\partial t}\int_{V} \hat{\rho}\, dV + \int_{\bar{A}} \mathbf{J}\cdot d\bar{\mathbf{A}} = 0 , \tag{14}$$

with the corresponding differential form

$$\partial\hat{\rho}/\partial t + \operatorname{Div}\mathbf{J} = 0. \tag{15}$$

It follows from (6) that the tangential component of **E** is continuous across a surface A drawn in the undeformed body, and that the tangential component of **H** is continuous in the absence of surface currents. From (12) it follows that the normal component of **B** is continuous across such a surface and that the normal component of **D** is continuous in the absence of surface charge. It also follows from (14) that the normal component of **J** is continuous across a surface A drawn in the underformed body, in the absence of surface charge.

Equations (7), (13), and (15) bear the same relation to the usual Maxwell equations as do the Kirchoff equations of equilibrium to the Cauchy equations. We anticipate that in conjunction with the Kirchoff equations of equilibrium they will prove useful in discussing problems in magnetostriction and electrostriction in which is it not desired to make the usual assumption of infinitesimal strain. They may also be of interest in studying problems involving other electromagnetic effects in deformed bodies.

If we make the constitutive assumptions that d_i, b_i, and j_i, the electric displacement, magnetic induction, and current density fields, are functions of the electric and magnetic intensity fields, e_i and h_i, and of the deformation gradients $x_{i,P}$, then the constitutive equations must be expressible in the form [1]

$$\left\{\begin{aligned} D_R &= D_R(G_{PQ}, E_P, H_P) ,\\ B_R &= B_R(G_{PQ}, E_P, H_P) ,\\ J_R &= J_R(G_{PQ}, E_P, H_P) , \end{aligned}\right. \tag{16}$$

where

$$G_{PQ} = x_{i,P}x_{i,Q} , \tag{17}$$

and E_P and H_P are defined by (4). The generalization of the results given in this note to the dynamic case will be given elsewhere [2].

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References.

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