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A mean value theorem in Generalized Axially Symmetric Potential Theory

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Matematica. — *A mean value theorem in Generalized Axially Symmetric Potential Theory* (*). Nota di RICHARD J. WEINACHT, presentata (**) dal Socio straniero A. WEINSTEIN.

1. INTRODUCTION.—In this note we give a mean value theorem for solutions of the equation of generalized axially symmetric potential theory (GASPT) for arbitrary real k

$$U_{xx} + U_{yy} + \frac{k}{y} U_y = 0 \tag{1}$$

introduced by A. Weinstein in 1948 [1] (see also Weinstein [2]). Although mean value theorems have been considered in GASPT (Weinstein [1], Diaz and Weinstein [3], Huber [4]) and generalizations thereof (Kapilevich [5], Weinacht [6]) the result here differs in that a weighted mean value is taken over circles *not* intersecting the singular line $y = 0$ and also the point at which the mean value is attained is not at the center of the circle.

Our result is made possible by a certain generalization to E_n, n–dimensional Euclidean space, of a result of Fichera [7] for function harmonic in a torus in E_3 (see Section 3).

2. NOTATIONS.—Let

$$H = \{ (x ; y) : y > 0 \} \cdot$$

For an arbitrary point (x_0 , y_0) in H introduce (η , ψ) as bipolar coordinates

$$\begin{cases} x = x_0 + \dfrac{y_0 \sin \psi}{\cosh \eta - \cos \psi} \\ y = \dfrac{y_0 \sinh \eta}{\cosh \eta - \cos \psi} \end{cases} \tag{2}$$

with (x_0 , y_0) as a pole. Here

$$0 < \eta < + \infty \quad , \quad - \pi < \psi \leq \pi$$

and as $\eta \to + \infty$ the point (x , y) tends to (x_0 , y_0). The curves η = constant are circles with center at $(x_0 , y_0 \coth \eta)$ and radius $y_0 \operatorname{csch} \eta$. Such circles will be denoted by $C (\eta ; x_0 , y_0)$, or shoter, $C (\eta)$ and the corresponding open disks enclosed by $C (\eta)$ will be denoted by $B (\eta)$.

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3. THE MEAN VALUE THEOREM.

THEOREM: *Let* U *be a solution of* (1) *in a region* G *contained with its closure in the half-space* H. *Then for an arbitrary point* (x_0, y_0) *in* G *and any disk* B (η) *wich is contained with its boundary* C (η) *in* G *we have*

$$U(x_0, y_0) = \int_{C(\eta)} \Phi(P)\, U(P)\, ds. \tag{3}$$

Here

$$\Phi(P) \equiv \frac{\sinh^{\frac{k-1}{2}} \eta\, (\cosh - \cos\psi)^{\frac{2-k}{2}}}{2\sqrt{2}\,\pi y_0 \tilde{Q}(\cosh\eta)},$$

$\tilde{Q} \equiv \tilde{Q}^{(k-1)/2}_{-1/2}$ is a general spherical harmonic of the second kind (see (6) below), s denotes arc length and P is on C (η).

The theorem was obtained from the viewpoint of Weinstein's spaces of " fractional dimension " [1] as follows. Let $U = U(x, y)$ be a solution of (1) and write x_1 for x. For k a non-negative integer the function w defined by

$$w(x_1, x_2, \cdots, x_{k+2}) = U(x_1, \sqrt{x_2^2 + \cdots + x_{k+2}^2})$$

is harmonic in $(k+2)$–dimensional Euclidean space and is symmetric about the x_1 axis. In particular for $k = 1$ we obtain the classical axially symmetric harmonic functions to which is applicable the theorem of Fichera [7] for functions harmonic in a torus with the x_1 axis as the axis of the torus. Because of the axial symmetry Fichera's theorem reduces to a mean value theorem over a circle of the type presented here. In fact for $k = 1$ our result reduces to that of Fichera for axially symmetric harmonic functions. By a generalization of Fichera's result to functions harmonic in a torus in E_n and accounting for the axial symmetric we arrive at (3). We emphasize that the result remains valid for arbitrary real k, including Tricomi's equation ($k = 1/3$).

4. PROOF OF THE THEOREM.—Without loss of generality we may assume $x_0 = 0$ and for ease of notation write $y_0 = a$ so that we consider $(x_0, y_0) = (0, a)$. With these agreements the introduction of the corresponding bipolar coordinates as in (2) and the function u in terms of a solution $U = U(x, y)$ of (1)

$$u(\eta, \psi) = U\left(\frac{a \sin\psi}{\cosh\eta - \cos\psi}, \frac{a \sinh\eta}{\cosh\eta - \cos\psi}\right)$$

leads to the differential equation

$$\frac{\partial}{\partial\eta}\left[\frac{\sinh^k \eta}{(\cosh\eta - \cos\psi)^k} \frac{\partial u}{\partial\eta}\right] + \frac{\partial}{\partial\psi}\left[\frac{\sinh^k \eta}{(\cosh\eta - \cos\psi)^k} \frac{\partial u}{\partial\psi}\right] = 0$$

which is the equation of GASPT in bipolar coordinates. If by v is denoted the function

$$v(\eta, \psi) \equiv \sinh^{\frac{k-1}{2}} \eta\, (\cosh\eta - \cos\psi)^{-k/2}\, u(\eta, \psi)$$

then v satisfies

$$\frac{\partial^2 v}{\partial \eta^2} + \frac{\partial^2 v}{\partial \psi^2} + \coth \eta \frac{\partial v}{\partial \eta} + \left[\frac{1}{4} - \frac{(k-1)^2}{4 \sinh^2 \eta}\right] v = 0$$

and further if $\rho = \cosh \eta$ and V is defined by

$$V(\rho) \equiv \int_{-\pi}^{\pi} v(\cosh^{-1} \rho, \psi)\, d\psi \tag{4}$$

then V is a solution of the equation of general spherical harmonics (see, e.g., [8])

$$(1-\rho^2) V'' - 2 \rho V' + \left[-\frac{1}{4} - \frac{(k-1)^2}{4(1-\rho^2)}\right] V = 0. \tag{5}$$

For linearly independent solutions of (5) we take [8 ; p. 136, 138]

$$\tilde{Q}^{\mu}_{-1/2}(\rho) \equiv \sqrt{\frac{\pi}{2}}\, \rho^{-\mu-1/2} (\rho^2 - 1)^{\mu/2} F\left(\frac{2\mu+1}{4}, \frac{2\mu+3}{4}; 1; \rho^{-2}\right) \tag{6}$$

and

$$P^{\mu}_{-1/2}(\rho) \equiv \frac{1}{\Gamma(1-\mu)} \left(\frac{\rho-1}{\rho+1}\right)^{-\mu/2} F\left(\frac{1}{2}, \frac{1}{2}; 1-\mu; \frac{1-\rho}{2}\right) \tag{7}$$

where $\mu = \frac{1}{2}(1-k)$ for $k \geq 1$ and $\mu = \frac{1}{2}(k-1)$ for $k \leq 1$, F denotes the hypergeometric function and Γ the gamma function. Note that in our considerations ρ is real and $\rho > 1$ in which domain the above functions are well-defined. The $\tilde{Q}$ function used here is real valued for real ρ.

Since [8; p. 151] the Wronskian

$$W[\tilde{Q}^{\mu}_{-1/2}, P^{\mu}_{-1/2}] = \frac{1}{\Gamma\left(\frac{1}{2}-\mu\right)} \frac{1}{(1-\rho^2)}$$

we see that these functions are indeed linearly independent. Moreover, from the definition of $\tilde{Q}$, as $\rho \to +\infty$

$$\tilde{Q}^{\mu}_{-1/2}(\rho) = \sqrt{\frac{\pi}{2}}\, \rho^{-1/2} + o(\rho^{-1/2}) \tag{8}$$

and from the analytic continuation of the hypergeometric series [9; p. 63]

$$P^{\mu}_{-1/2}(\rho) = \frac{1}{\Gamma\left(\frac{1}{2}-\mu\right)} \sqrt{\frac{2}{\pi}}\, \rho^{-1/2} \log \rho + o(\rho^{-1/2}) \tag{9}$$

and by direct examination of V in (4)

$$V(\rho) = 2\pi\, \rho^{-1/2} U(0, a) + o(\rho^{-1/2}). \tag{10}$$

Hence, from (8), (9) and (10), in the representation

$$V(\rho) = A\tilde{Q}^{\mu}_{-1/2}(\rho) + BP^{\mu}_{-1/2}(\rho)$$

the constant B must be zero and $A = 2\sqrt{2\pi}\,U(0, a)$. Thus using the fact [8; p. 137] that $\tilde{Q}^{\mu}_{\nu}(\rho) = \tilde{Q}^{-\mu}_{\nu}(\rho)$ and changing the variable of integration in (4) to that of arc length we arrive at (3). This completes the proof of the theorem.

5. CONCLUDING REMARKS.—The peripheral mean value theorem obtained here can be generalized to equations related to (1) and "solid" mean value theorems can be obtained. These results and consideration of a converse will be presented in a subsequent paper.

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