Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Knut Lage Sundet

On the existence of certain systems of plane elliptic curves: Condition for Cremonian reducibility to a Halphen-pencil

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **38** (1965), n.3, p. 362–368.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1965_8_38_3_362_0>

Geometria. — *On the existence of certain systems of plane elliptic curves: Condition for Cremonian reducibility to a Halphen-pencil.* Nota di KNUT LAGE SUNDET, presentata (*) dal Socio B. SEGRE.

1. INTRODUCTION.—Consider a system (φ) of order n having the base-points $P_1, P_2, \cdots, P_i$, of virtual multiplicities $r_1, r_2, \cdots, r_i$, making the virtual genus 1 and the virtual dimension 1 or zero. In [1] it is shown that, under a certain assumption and by applying a certain series of quadratic transformations $T_1, T_2, \cdots$ mapping (φ) into the systems $(\varphi_1), (\varphi_2), \cdots$ respectively, it is possible to reach a system (φ_s) of order $3r$ having the base-points $P_{s,1}$, $P_{s,2}, \cdots, P_{s,u}$ of virtual multiplicities r where $r = 1$ and $u = 8$ if the virtual dimension is 1 and $u = 9$ if the virtual dimension is zero. The assumption is that the condition I or II quoted in [3] does not hold. The system (φ) is reducible or does not have the assigned multiplicities as effective ones if the condition I or II is satisfied.

In the first section of this paper a condition for (φ) to be transformable into a Halphen-pencil by a Cremona transformation will be given. This condition, together with the condition for (φ) to be transformable into a pencil of cubics obtained in [3], make it possible—in the following sections—to obtain conditions for the reducibility of elliptic sysfems also in the cases when the virtual dimension is one or zero. The other cases of non-negative virtual dimension are treated, together with the existence of rational curves, in [1]. The lack of conditions for the existence of systems of plane algebraic curves is pointed out by B. Segre in [4].

It is to be noted that, as in my former paper [3], the set of points consisting of a point P and its adjacent points is regarded here as an algebraic curve of order zero having the multiplicity —1 at P. The neighbouring points have the multiplicities +1. Such a curve of order zero is referred to as a "pointline". "Pointlines" are counted among the rational curves.

2. CONDITIONS FOR A LINEAR SYSTEM TO BE TRANSFORMABLE INTO A HALPHEN-PENCIL BY A PLANE CREMONA TRANSFORMATION.—Consider the system (φ) mentioned in the Introduction. When the virtual dimension is zero, the system (φ_s) reached by the q-transformations $T_1, T_2, \cdots$ is of order $3r$ and given by the 9 base-points $P_{s,1}, P_{s,2}, \cdots, P_{s,9}$ of virtual multiplicities r provided I and II are not satisfied. As a common factor r of n and r_t $(t = 1, 2, \cdots, i)$ is invariant by q-transformations, r has to be greater than 1 if (φ_s) should be a possible Halphen-pencil.

(*) Nella seduta del 13 marzo 1965.

There is a cubic, C_s, passing through $P_{s,1}, P_{s,2}, \cdots, P_{s,9}$. As described by F. Enriques on p. 196 in [2], for instance $P_{s,9}$ has to be among the r^2 r–ple points in the g_r^{r-1} cut out on C_s by curves (φ_s') having $P_{s,1}, P_{s,2}, \cdots, P_{s,8}$ as base-points of multiplicities r. $P_{s,u}$ $(u = 1, 2, \cdots, 9)$ form then the base of a Halphen-pencil if $P_{s,9}$ is not also an $\left(\frac{r}{h}\right)$–ple point $(h \leq r)$ in the series cut out on C_s by curves of orders $\frac{3r}{h}$ having $P_{s,1}, P_{s,2}, \cdots, P_{s,8}$ as base-points of multiplicities $\frac{r}{h}$.

Let the Cremona transformation composed of the q–transformations $T_1, T_2, \cdots, T_s$ be denoted by T and the inverse of T by T′. Because of invariance by q–transformations, C_s is mapped by T′ into a curve, C_1, which is equal or a part of the curve C of order m having P_t $(t = 1, 2, \cdots, i)$ as base-points of virtual multiplicities v_t where $n = mr$ and $r_t = rv_t$. Consider further the curves (C_s') in (φ_s') passing through $P_{s,9}$. The curves in (C_s') passing through the $r - 2$ successively neighbouring points $P_{s,9,1}$, $P_{s,9,2}, \cdots, P_{s,9,r-2}$ of $P_{s,9}$ will—as supposed—also then pass through the next one, $P_{s,9,r-1}$. In order to examine the transformed system (C') of $C_s')$ by the transformation T′, let it be noted that (φ_s) together with $r - 1$ " pointlines " at $P_{s,9}$ form a system having the same virtual multiplicities at all the f–points of T′ as those of (C_s'). The " pointline " given by $P_{s,9}$ corresponds by T′ to a curve R of order n_1 completely determined by having P_t as base-points of virtual multiplicities, a_t, making the virtual genus O. Because of the invariance of the relations 1) in [3], the order of (C') is $n' = n + (r - 1) n_1$ and its virtual multiplicities at P_t are $R_t = r_t + (r - 1) a_t$. It is seen that (C') is cutting out a g_{r-1}^{r-2} on C_1.

Suppose $P_{s,9,u}$ $(u = 1, 2, \cdots, r - 1)$ are not f–points of T′. It is seen that $P_{s,9,1}$ is a point of intersection between the " pointline " $P_{s,9}$ and C_s. The point $P_{s,9,1}$ is then transformed into the point of intersection between C_1 and R, which is then a $(r - 1)$–ple point in the g_{r-1}^{r-2} cut out on C_1 by the system (C').

Suppose $P_{s,9,1}, P_{s,9,2}, \cdots, P_{s,9,k}$ are f–points of T′. The curves in the system (C_s') passing through $P_{s,9,1}, \cdots, P_{s,9,k}, \cdots, P_{s,9,r-1}$ will then, together the " pointlines " $P_{s,9,1}$, $P_{s,9,2}$ counted twice, $P_{s,9,k}$ counted 3 times, $\cdots$, $P_{s,9,k}$ counted k times, form a curve having the same multiplicities at the f–points of T′ as the general curve in (C_s'); and the number of intersections between this curve and C_s not belonging to the f–points is then $r - 1$, as the point $P_{s,9,k+1}$ absorbs $k + 1$ points of intersection between the composed curve and C_s when $k < r - 1$ and k points of intersection if $k = r - 1$. If $P_{s,9,r}$ is an f–point of T′, we may add $r - 1$ " pointlines " at this point and, provided $P_{s,9,r+1}$ is not an f–point of T′, the constructed curves in (C_s') get an intersection counted $r - 1$ times at at this point. If $P_{s,9,r+1}$ is an f–point, we may continue in the same way. Let it be noted that the " pointline " given by $P_{s,9}$, and thus degenerate into the " pointlines " $P_{s,9}, P_{s,9,1}, P_{s,9,2}, \cdots$, intersects C_s in the same point as the mentioned curves in (C'_s). This is then also the case with the transformed curves by T'. Hence:

The curve R given by the base-points P_t of virtual multiplicities a_t intersects the transformed curve C_1 of C_s in a point A which is then a $(r-1)$–ple point in the g_{r-1}^{r-2} cut out on C_1 by the system (C') of order $n+(r-1)\,n_1$ given by the base-points P_t of virtual multiplicities $R_t = r_t + (r-1)\,a_t$. The g_1^0 in the case $r=2$ indicates that every curve in the system (C') has to pass through A.

The point $P_{s,9}$ was not supposed to be j–ple in the series g_j^{j-1} on C_s defined by a system of order $3j$ having $P_{s,1}, P_{s,2}, \cdots, P_{s,8}$ as base-points of multiplicity j where $j = \binom{r}{h}$ $(h < r)$. Neither is then A an $(j-1)$–ple point in the g_{j-1}^{j-2} defined on C_1 by system of order $mj + (j-1)\,n_1$ having P_t as base-points of virtual multiplicities $jv_t + (j-1)\,a_t$.

The points $P_{s,u}$ $(u = 1, 2, \cdots, 9)$ was not supposed to be the base of a pencil of cubics. The points P_t do not then satisfy the conditions for the mentioned curve C of order m to be transformable into a curve belonging to such a pencil.

After the way R is constructed, it is seen that the case II does not occur by R. Further it is noted that $\Sigma\, a_t\, v_t = n_1\, m - 1$ and, as a consequence, the curves of order $m + n_1$ having P_t as base points of multiplicities $v_t + a_t$ form a system of virtual genus and dimension 1. The system does not satisfy I and II.

Vice versa: A system (φ) of order mr having the base-points P_t of virtual multiplicities $v_t\, r$ $(t = 1, 2, \cdots)$, not satisfying the conditions I and II and having the properties above, corresponds to a Halphen-pencil when r is > 1.

Let C be the curve of order m having P_t as multiple points of virtual multiplicities v_t $(t = 1, 2, \cdots)$ and R a curve of order n_1 completely determined by having P_t as base-points of virtual multiplicities, a_t, making the virtual genus O, not satisfying II and where $\Sigma\, a_t\, v_t = n_1\, m - 1$. The system (C'), of order $n' = n + n_1\,(r-1)$, not satisfying I and II and having P_t as base-points of multiplicities $R_t = r_t + (r-1)\,a_t$ cuts out on C a g_{r-1}^{r-2} if C is irreducible. If C is reducible, (C') is cutting out a g_{r-1}^{r-2} on a part C_1 of C. This may be shown like this. Since I and II are not the cases with (φ), so neither are the cases with C. Thus C degenerates in a part, C_1, and rational curves and "point-lines", having the complete number of virtual intersections with C at the base. Let us denote the order of one of these parts by m'_s and by $v_{t,s}$ the multiplicity of the part at P_t. As I and II are not occurring by (C'), we have $n'\, m'_s - \Sigma\, R_t\, v_t \geq 0$ and therefore $n_1\, m'_s - \Sigma\, a_t\, v_{t,s} \geq 0$. From $\Sigma\, a_t\, v_t = n_1\, m - 1$ it follows that only by one curve C_1 having the order m'_1 and base-points of multiplicities $v_{t,1}$ we have $\Sigma\, a_t\, v_{t,1} = n_1\, m'_1 - 1$, and hence (C') cuts out a g_{r-1}^{r-2} on C_1. We assume then that this g_{r-1}^{r-2} has an $(r-1)$–ple point at the point A of intersection between R and C, or a part C_1 of C. In order to show that in this case (φ) is transformable into a Halphen-pencil, provided that A is not also the mentioned j–ple point, consider the system (φ''), of order $m + n_1$, having P_t as base-points of multiplicities $v_t + a_t$ not satisfying I and II. As (φ'') get the virtual genus and virtual dimension 1, it is transformable into a system (φ_s'') of order 3 having 8 simple base-points. Because of invariance by such transformations, we have $3 = m_s + n_{s,1}$ where m_s and $n_{s,1}$ are the orders of the transformed system C_s of C and R. As $m_s \geq 3$ and $n_{s,1} \geq 0$, because R is not satisfying II, we obtain $m_s = 3$ and $n_{s,1} = 0$. C_s is then given by the points $P_{s,1}, P_{s,2}, \cdots, P_{s,9}$, (φ''_s) by $P_{s,1}, P_{s,2}, \cdots, P_{s,8}$ and the transformed curve of R by the "pointline" $P_{s,9}$. The transformed curve (φ_s) of (φ) is then given by the base-points $P_{s,q}$ $(q = 1, 2, \cdots, 9)$ of virtual multiplicities r and, still because of invariance by q–transformations, the order of the transformed system (C_s') of (C') is $n_s' = n_s + (r - 1)\, n_{s,1} = n_s = 3r$, where n_s is the order of (φ_s). The multiplicites of (C_s') at the points $P_{s,1}$, $P_{s,2}, \cdots, P_{s,8}$ are $r + (r - 1)\, 0 = r$ and, at $P_{s,9}$, it gets the multiplicity $r + (r - 1)(-1) = 1$.

Now, as supposed, there exists a curve in (C') not intersecting C_1 in points distinct from A. Since R is transformed into the "pointline" given by $P_{s,9}$, the transformed degenerated or not degenerated curves cannot then intersect C_s in a point distinct from $P_{s,9}$ and so $P_{s,9}$ is an r–ple point in the g_r^{r-1} cut out on C_s by the system having $P_{s,1}, P_{s,2}, \cdots, P_{s,8}$ as base-points of multiplicities r. (φ_s) is then a Halphen-pencil if $P_{s,9}$ is not also such a j–ple point: A is a $(j - 1)$–ple point, as mentioned before, We may therefore state the following theorem.

Let (φ) *be a system of order* n *having the base-points* $P_1, P_2, \cdots, P_i$, *of virtual multiplicities* $r_1, r_2, \cdots, r_i$, *making the virtual genus* 1, *the virtual dimension* 0 *and not satisfying I and II. If this system should be transformable into a Halphen-pencil, it is necessary that* r_t *and* n *have a common factor* $r > 1$. *There exists then a curve* C *of order* m *having* P_t $(t = 1, 2, \cdots, i)$ *as base-points of virtual multiplicities* v_t, *where* $n = rm$ *and* $r_t = rv_t$, *and a curve* R *of order* n_1 *completely determined by having* P_t *as base-points of virtual multiplicities,* a_t, *making the virtual genus* 0, *not satisfying II and where* $\Sigma\, a_t\, v_t = n_1\, m - 1$, *such that the system of order* $m + n_1$ *having* P_t *as base-points of virtual multiplicities* $v_t + a_t$ *does not satisfy I or II.* (φ) *is transformable into a Halphen-pencil if, and only if, the point* A *of intersection between* R *and* C *is an* $(r - 1)$*–ple point in the* g_{r-1}^{r-2} *cut out on* C, *or a part of* C, *by the system of order* $n + n_1 (r - 1)$ *having* P_t *as base-points of multiplicities* $r_t + a_t (r - 1)$. *A is not supposed to be a* $(j - 1)$*–ple point in a* g_{j-1}^{j-2} *cut out on* C *by a system of order* $mj + n_1 (j - 1)$ *having* P_t *as base-points of multiplicities* $jv_t + a_t (j - 1)$ *where* $j = \left(\frac{r}{h}\right) (h < r)$. *Neither do the* P_t*'s satisfy the conditions for* C *to be transformable into a pencil of cubics.*

If for istance $r_i = r$, the curve R may be a "pointline".

3. ON THE REDUCIBILITY OF ELLIPTIC SYSTEMS OF VIRTUAL DIMENSION ONE.—Let (φ) be a system of order n having P_t $(t = 1, 2, \cdots, i)$ as base-points, of virtual multiplicities r_t making the virtual genus and dimension

one. As pointed out in the Introduction, the system (φ_s) reached by the transformation T composed of the q–transformations $T_1, T_2, \cdots$ is of order 3 and has 8 base-points $P_{s,1}, P_{s,2}, \cdots, P_{s,8}$ of virtual multiplicities 1, provided the case I or II is not occurring. (φ) is then irreducible if the 9th base-point $P_{s,9}$ in the pencil of cubics defined by $P_{s,1}, P_{s,2}, \cdots, P_{s,8}$ is not an f–point of the inverse transformation T′ of T.

Consider this last case. The curves in (φ_s) together with a "pointline" at $P_{s,9}$ give then a curve having the assigned multiplicities at each of the f–points of T′. A possible f–point in the 1st neighbourhood of $P_{s,9}$ get now the effctive multiplicity 1. A "pointline" at that point gives the effective multiplicity equal to the virtual one again. Thus we may continue. It is seen that there exists at least one "pointline" not passing through another f–point of T′. As in the Section 2 in [3], it is then seen that this "pointline" corresponds to a curve of order greater than zero. (φ_s) corresponds then to a system of order lower than n and the points $P_1, P_2, \cdots, P_i$ are consequently given in such a special position that they are base-points of a system, of order lower than n, which is transformable into a pencil of cubics. The system (φ) is in this case reducible, or does not have the assigned multiplicities. In all the other cases, (φ) is irreducible and the effective multiplicities at the points P_t are the virtual ones.

We may therefore state the following theorem.

A system (φ) *of order n given by the base-points* $P_1, P_2, \cdots, P_i$, *of virtual multiplicities* $r_1, r_2, \cdots, r_i$, *making the virtual genus and dimension one is reducible, or does not have the assigned multiplicities at the base, if, and only if, there is the case I, II or* $P_1, P_2, \cdots, P_i$ *are given in such a special position that they are-base points of a system of order lower than n which is transformable into a pencil of cubics. By adding this system with certain rational curves (and "pointlines"), we obtain a system of curves having multiplicity* r_t *at each point* P_t.

4. ON THE REDUCIBILITY OF ELLIPTIC SYSTEMS OF VIRTUAL DIMENSION ZERO.—Let (φ) be a system of order n having the base-points $P_t (t=1, 2, \cdots, i)$, of virtual multiplicities r_t, making the virtual genus 1 and the virtual dimension O. As pointed out in the Introduction, the system (φ_s), reached by the transformation T composed of the q–transformations T_1, $T_2, \cdots, T_s$, is of order $3r$ and has 9 base-points $P_{s,1}, P_{s,2}, \cdots, P_{s,9}$ of virtual multiplicities r, provided that the case I or II is not occurring. Suppose $r = 1$. Since the case I is not occurring, the system (φ_s) degenerates if, and only if, three of the points $P_{s,u}$ $(u = 1, 2, \cdots, 9)$ are situated on a straight line l and 6 on a conic k such that $P_{s,u}$ do not form the base of a pencil of cubics. The conic may also degenerate. If l and k are not used as f–lines by the inverse transformation T′ of T, l and k are mapped into at least two rational curves which then form together a curve having the effective multiplicities at P_t equal to the virtual ones. As $P_{s,u}$ impose too many conditions on l and k, the points P_t have to be in such a special position that they form

a superabundant base making the virtual dimension negative, of at least two irreducible rational curves. If " pointlines " are regarded as special rational curves, the conclusions above are true even if l and k are f–lines by T′.

Consider the case when (φ_s) is irreducible. The system (φ) is then also irreducible, if (φ_s) does not pass through some of the f–points $P_{s,10}$, $P_{s,11}, \cdots$ of T′ having there virtual multiplicity zero. This may occur when $P_{s,1}$, $P_{s,2}, \cdots, P_{s,9}$ is not the base of a pencil of cubics. The cubic through $P_{s,1}$, $P_{s,2}, \cdots, P_{s,9}$, $P_{s,10}, \cdots$ and " pointlines " at $P_{s,10}, \cdots$ give then a curve, having the effective multiplicities equal to the virtual ones at all the f–points of T′. The system (φ) is then in this case composed of an elliptic curve C, of order lower than n, of rational curves and of some possible " pointlines ". As the points $P_{s,1}$, $P_{s,2}, \cdots, P_{s,10}, \cdots$ impose an aboundance of linear conditions on (φ_s), the points P_1, $P_2, \cdots, P_i$ are in such a special position that—taken with appropriate multiplicities—they form a superabundant base, making the virtual dimension negative, of an irreducible elliptic curve C of order lower than n.

Consider the case when $r > 1$. The sistem (φ_s) is then, and only then, irreducible, if $P_{s,1}$, $P_{s,2}, \cdots, P_{s,9}$ form a base of a Halhen-pencil. The system (φ) is then irreducible, only if (φ) is transformable into such a one.

In conclusion, we may state the following theorem.

Consider a system (φ) of order n having the base-points P_1, $P_2, \cdots, P_i$, of virtual multiplicities r_1, $r_2, \cdots, r_i$, making the virtual genus one and the virtual dimension zero. If n and r_t $(t = 1, 2, \cdots, i)$ are relatively prime, (φ) is reducible or does not exist with the assigned multiplicities if, and only if, there is case I or II and then (φ) satisfies the conditions for being transformable into a pencil of cubics. If this is not the case, we have only two possibilities:

(1) *The points P_t are given in such a special position that—taken with appropriate multiplicities—they form a superabundant base, making the virtual dimension negative, of at least two rational curves (or " pointlines ") which, together with other possible rational curves (and " pointlines "), give a curve having each point P_t as point of effective multiplicity r_t.*

(2) *The points P_t are in such a special position that – taken with appropriate multiplicities – they form a superabundant base, making the virtual dimension negative, of an irreducible elliptic curve of order lower than n which together wit rational curves (possible " pointlines "), gives a curve having each point P_t as a point of effective multiplicity r_t.*

When n and r_t $(t = 1, 2, \cdots, i)$ have a common factor $r > 1$, (φ) is reducible or does not have the assigned multiplicities if, and only if, there is the case I, II or the condition for (φ) to be transformable into a Halphen-pencil is not satisfied.

REFERENCES.

[1] K. L. SUNDET, *On the Existence of Systems of Plane Rational and Elliptic Algebraic Curves, given by a Group of Points. A numerical Criterion*, « Rendiconti di Matematica », *22*, 469–488 (1963).

[2] F. ENRIQUES, O. CHISINI, *Teoria geometrica delle equazioni e delle funzioni algebriche*, vol. III, Nicola Zanichelli, Bologna 1915.

[3] K. L. SUNDET, *On Cremonian reducibility of a linear system to a pencil of cubics*. « Rendic. Acc. Naz. Lincei », (8) *38*, 000 (1965)1.

[4] B. SEGRE, *Alcune questioni su insiemi finiti di punti in geometria algebrica*, in *Atti del Convegno Internazionale di Geometria Algebrica*, Torino 1961, 15–33.

RIASSUNTO. — Si consideri un sistema (φ) di curve piane algebriche d'ordine n, avente $P_1, P_2, \cdots, P_i$ come punti base di molteplicità virtuali $r_1, r_2, \cdots, r_i$ ed il genere virtuale 1. In questa Nota si esamina anzitutto la possibilità che (φ) sia trasformabile in un fascio di Halphen mediante una trasformazione cremoniana, sotto la condizione che (φ) abbia dimensione virtuale zero. All'uopo è necessario che n e $r_t (t = 1, 2, \cdots, i)$ ammettano un comune fattore $r > 1$. Supposta questa condizione verificata, denotiamo con C la curva d'ordine m che passa per i punti base $P_t (t = 1, 2, \cdots, i)$ con molteplicità virtuali v_t, dove $n = mr$ e $r_t = rv_t$. Viene allora dimostrato che il sistema (φ), è trasformabile in un fascio di Halphen se, e solamente se, il punto d'intercezione – distinto dai P_t – di una certa curva R d'ordine n_1 e della C è tra i punti $(r-1)$–pli della g_{r-1}^{r-2} segata su C o su una parte di C da un certo sistema di curve d'ordine $n + (r-1) n_1$.

Applicando questo risultato insieme con la condizione per la riducibilità ad un fascio di cubiche, ottenute in [3], è possibbile di caratterizzare tutti i casi in cui (φ) degenera, la dimensione virtuale essendo uno o zero.