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On Cremonian reducibility of a linear system to a pencil of cubics

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Geometria. — *On Cremonian reducibility of a linear system to a pencil of cubics*. Nota di KNUT LAGE SUNDET, presentata (*) dal Socio B. SEGRE.

1. INTRODUCTION. — In my paper [1], the conditions for reducibility of algebraic curves of virtual genus $p = 0$ and $p = 1$ are treated. The last case, only when the virtual dimension d is ≥ 2. The reason for the lack of the cases $d \leq 1$ was that the questions of reducibility of a linear system to a pencil of cubics or to a Halphen-pencil by a plane Cremona transformation were not answered. The first question is dealt with here and the second one will be treated in a following paper.

In the mentioned paper [1], the cases occurring by successively applying a certain series of quadratic transformations $T_1, T_2, \cdots, T_s$ on a system (φ) of order n, given by the base-points $P_1, P_2, \cdots, P_i$ of virtual multiplicities $r_1, r_2, \cdots, r_i$, are sorted out when $p = 0$ and $p = 1$. The q–transformations $T_1, T_2, \cdots, T_s$ map successively (φ) into the systems (φ_1), (φ_2), $\cdots$, (φ_s) where (φ_s) has the order n_s and is given by the assigned points $P_{s,1}, P_{s,2}, \cdots P_{s,k}$, of virtual multiplicities $r_{s,1}, r_{s,2}, \cdots, r_{s,k}$. When the virtual genus is 1 and the virtual dimension is 0, the only cases to be consider are the following ones:

A) Some of the base-points, $P_{s,a}, P_{s,b}, P_{s,c}, \cdots$ say, are situated on a straight line and such that

$$r_{s,a} + r_{s,b} + r_{s,c} + \cdots > n_s ;$$

B) (φ_s) has the order $3r$ and is determined by 9 base-points of virtual multiplicities r.

When the case A) is reached, we only have for (φ) the possibilities I, II specified below.

I) The base-points are in a special position, such that an irreducible rational curve exists, of order m say, containing the P_t's with certain multiplicities a_t, determined by some of the P_t's and going through some of the others. the virtual number of intersections between this curve and the curves (φ) being too large (i,e., greater than mn). The integers m and a_t are not all equal to the corresponding integers n and r_t respectively ($t = 1, 2, \cdots, i$),

II) Integers $k_{\sigma,\tau}$, $\sigma = 0, 1, 2, \cdots, s$, $\tau = 1, 2, \cdots, i$ exist, such that

$$k_{s,1} + k_{s,2} > n_s > 1 ,$$

where $k_{0,\tau} = r_\tau$, $n_0 = n$, and $k_{\sigma,1}, k_{\sigma,2}, \cdots, k_{\sigma,i}$ are the integers

$$n_{\sigma-1} - k_{\sigma-1,1} - k_{\sigma-1,2} \,,\; n_{\sigma-1} - k_{\sigma-1,2} - k_{\sigma-1,3} \,,\; n_{\sigma-1} - k_{\sigma-1,3} - k_{\sigma-1,1}$$

(*) Nella seduta del 13 febbraio 1965.

$k_{\sigma-1,4}, \cdots, k_{\sigma-1,i}$ arranged in non-increasing order, and

$$n = 2n_{\sigma-1} - k_{\sigma-1,1} - k_{\sigma-1,2} - k_{\sigma-1,3}.$$

The assumption in both cases is that, when some base-points are adjacent to others, the sum of their virtual orders is equal to or less than the virtual orders of the latter. If I or II holds, the system (φ) is reducible or does not have the assigned multiplicities as effective ones.

On p. 180 of their work [2], F. Enriques and O. Chisini have considered algebraic curves of order zero. In the present paper, the set consisting of a point P and the points in the 1st neighbourhood of P will be regarded as an algebraic curve of order zero. To the point P we shall assign the multiplicity -1 and to the points in the 1st neighourhood the multiplicities $+1$. Such a curve of zero order will be referred to as a "pointline". If the point P is an f–point of a q–transformation, the "pointline" given by P is mapped, according to the usual formulae for the order and multiplicities of a transformed curve by a q–transformation, into an f–line of the inverse transformation; and vice-versa.

The virtual genus p and the virtual dimension d of a system (φ) of order n, having base-points of virtual multiplicities $r_1, r_2, \cdots, r_i$, are given by the classical formulae:

$$p = \frac{1}{2}(n-1)(n-2) - \frac{1}{2}\Sigma r_t(r_t - 1),$$

$$d = \frac{1}{2}n(n+3) - \frac{1}{2}\Sigma r_t(r_t+1).$$

If we put the order and the multiplicities of the points in a "pointline" into the expression for p, we get $p = 0$. The virtual dimension d goes to zero if a curve of order zero is given by a point of virtual multiplicity -1.

It is to be noted that if n, n_1 and n_2 are the orders of three curves and r_t, a_t and b_t their multiplicities at the points P_t $(t = 1, 2, \cdots, i)$, the invariance of the relations:

$$(1) \qquad r_t = a_t + b_t \quad \text{and} \quad n = n_1 + n_2$$

and of the difference:

$$(2) \qquad r_1 a_1 + r_2 a_2 + \cdots + r_i a_i - n n_1$$

by q–transformations are true, even when "pointlines" are concerned.

Because of the properties of "pointlines" mentioned above, they will be counted among the rational curves. However, the special reservation concerning adjacent points made in the conditions quoted above is then not necessary.

2. Necessary conditions for a system to be transformable into a pencil of cubics by a plane Cremona transformation.—Let a system (φ) of order n be given (as in § 1) by the base-points $P_1, P_2, \cdots, P_i$

of virtual multiplicities $r_1, r_2, \cdots, r_i$, making the virtual genus 1 and the virtual dimension zero. As pointed out in the Introduction, by successively using the quadratic transformations $T_1, T_2, \cdots, T_s$, if the case I or II is not occurring, we reach a system (φ_s) of order $3r$ with 9 base-points of virtual multiplicities r. As a common factor of $n, r_1, r_2, \cdots, r_i$ is invariant by q–transformations, a system of cubics is reached only when $n, r_1, r_2, \cdots, r_i$ have no other factor but 1 in common. A system (φ_s) of order 3 given by 9 simple basepoints $P_{s,1}, P_{s,2}, \cdots, P_{s,9}$ is then to be considered.

If the given system is of order higher than 3, it exists at least one f–point $P_{s,10}$ of the inverse Cremona transformation composed of $T_1, T_2, \cdots, T_s$ which has virtual multiplicity zero and is not one among the new base-points of virtual multiplicity zero introduced by a possible applied series of transformations, of the type prescibed by O. Chisini in the case of cuspidal branches, or of other auxiliary q–transformations applied to avoid some one with adjacent f–points. As (φ_s) is a pencil of cubics, it contains a curve, C_s say, passing through that point. The f–points of (φ_s) may be such that C_s degenerates into a cubic with a node or into a straight line and a conic, where the conic also may degenerate. The curve C_s and a " pointline " at $P_{s,10}$ form together a curve having the assigned multiplicity at that point. The possible f–points of virtual multiplicities zero in the 1st neighbourhood of $P_{s,10}$ have now got the multiplicities 1. " Pointlines " at these points, in addition to the former curves, give the multiplicities zero at these points also. So we may continue. The point $P_{s,10}$ may be in the neighbourhood of two adjacent points $P_{s,8}$ and $P_{s,9}$ of virtual multiplicities 1. C_s get then a node at $P_{s,8}$. By adding a " pointline " at $P_{s,8}$, we get the multiplcity 1 at this point but the multiplicities 2 and 1 at $P_{s,9}$ and $P_{s,10}$. " Pointlines " at $P_{s,9}$ and $P_{s,8}$ give the assigned multiplicities at these point.

It is thus seen that, if (φ_s) is a pencil of cubics, the f–points of the inverse of the applied Cremona transformation have to be in such a special position that they impose a superaboundance of linear conditions on a curve C_s of order 3 determined by them; moreover, C_s and some " pointlines " form together a curve having the assigned multiplicities at the f–points. The points P_t are consequently also in such a special position that—taken with appropriate multiplicities—they form a superabundant base, making namely the virtual dimension negative, of a certain curve C. The curve obtained by adding C with possible suitable rational curves and " pointlines " satisfies the conditions for (φ).

The order of C is less than the order of (φ) because at least one of the " pointlines ", which together with C_s form a curve of (φ_s), does not contain an f–point. Let the inverse of the applied q–transformations $T_1, T_2, \cdots, T_s$, mapping (φ) into (φ_s), be denoted by $T'_1, T'_2, \cdots, T'_s$ respectively, and the one that maps the mentioned " pointline " into a straight line by T'_t. It is then seen that the line corresponding by T'_t to the mentioned " pointline " does not contain other f–points that those of T_t making the complete number of intersections with the corresponding system (φ_{t-1}). This line or one of its transformed lines cannot be used as f–line again, because that implies an f–point of virtual multiplicity zero by one of the T_q ($q = 1, 2, \cdots$). The impossibility of this follows from the reservation concerning C_s. As the order of (φ) is the sum of the orders of C and of the transformed curves of the mentioned "pointlines", it follows that the order of C is less than the order of (φ).

As C_s may degenerate, C may degenerate into rational curves, rational curves and "pointlines", or "pointlines". Hence:

A system (φ) *of order* n, *virtual dimension* 0, *virtual genus* 1, *given by the assigned points* P_t ($t = 1, 2, \cdots, i$) *of virtual multiplicities* r_t, *and not satisfying I and II, is an irreducible system transformable into a pencil of cubics only if the integers* n *and* r_t *are relatively prime and the points* P_t *are in such a special position that—taken with appropriate multiplicities—they form a superabundant base, making the virtual dimension negative, of a reducible or irreducible curve* C *of order lower than* n. *By adding* C *with certain rational curves (and "pointlines"), we obtain a curve having multiplicity* r_t *at each point* P_t.

3. NECESSARY AND SUFFICIENT CONDITIONS FOR A SYSTEM TO BE TRANSFORMABLE INTO A PENCIL OF CUBICS.—As in the Introduction, the given system is now denoted by (φ) and the system reached by the series of q–transformations $T_1, T_2, \cdots, T_s$ by (φ_s). If (φ) does not satisfy I or II, the base-points of (φ_s) are denoted by $P_{s,1}, P_{s,2}, \cdots, P_{s,9}$. C_s is the special curve in (φ_s) mentioned in the precedent Section.

From the assumption that (φ_s) is a pencil of cubics, it follows that e.g. $P_{s,1}$ and $P_{s,2}$ are corresponding points in the g_2^1 cut out on C_s by a system (Π_s) of order 3 passing through $P_{s,u}$ ($u = 3, 4, \cdots, 9$) and a point Q_s fixed in a generic manner. Let (φ_t) and (Π_t) be the transformed systems of (φ_s) and (Π_s) by the series $T'_s, T'_{s-1}, \cdots, T'_{t-1}$, where no member has used $P_{s,1}, P_{s,2}$ or any of their transformed points as f–points. $T'_s, T'_{s-1}, \cdots$ are the inverse of the transformations $T_s, T_{s-1}, \cdots$ mentioned in the Introduction. The base-points of (φ_t) are denoted by $P_{t,q}$ ($q = 1, 2, \cdots$) and Q_t, $P_{t,1}$ and $P_{t,2}$ are the transformed points of Q_s, $P_{s,1}$ and $P_{s,2}$ respecitively. Suppose that $P_{t,1}$ and $P_{t,2}$ are distinct and both f–points by T'_t. Let the order of the transformed system (φ_{t-1}) of (φ_t) be n_{t-1} and the f–points of T_t be $P_{t-1,1}$, $P_{t-1,2}$ and $P_{t-1,3}$. The lines $P_{t-1,1}P_{t-1,2}$ and $P_{t-1,2}P_{t-1,3}$, corresponding to $P_{t,1}$ and $P_{t,2}$, intersect the transformed curve C_{t-1} of C_s in the points $A_{t-1,1}$ and $A_{t-1,2}$. These two points have to be corresponding points in the g_2^1 cut out on C_{t-1} by the corresponding system (Π_{t-1}) of (Π_t). Because, let us suppose the contrary; a curve in (Π_{t-1}) will then pass through $A_{t-1,1}$ and not through $A_{t-1,2}$. That curve contains $P_{t-1,1}P_{t-1,2}$. The other part of the curve is then mapped into a curve in (Π_t) going through $P_{t,1}$, but not through $P_{t,2}$.

It is seen that the order of (Π_{t-1}) is $n_{t-1}+2$ and the multiplicity of the base-points $P_{t-1,1}$, $P_{t-1,2}$ and $P_{t-1,3}$ are $r_{t-1,1}+1$, $r_{t-1,2}+2$ and $r_{t-1,3}+1$ respectively, where $r_{t-1,1}$, $r_{t-1,2}$ and $r_{t-1,3}$ are the multiplicities of (φ_{t-1}) at those points. If now $A_{t-1,1}$, $A_{t-1,2}$ or any of their transformed points are not used as f–points, $P_{t-1,1}P_{t-1,2}$, $P_{t-1,2}P_{t-1,3}$ or any of their transformed lines are not f–lines or they do not pass through any f–points of T'_{t-1}, $T'_{t-2}, \cdots, T'_1$, we may conclude:

If the system (φ) given by the base-points P_t $(t = 1, 2, \cdots, i)$ of virtual multiplicities r_t is transformable into a pencil of cubics, there exist two irreducible rational curves R_1 and R_2, of orders n_1 and n_2, having P_t as base-points of multiplicities a_t and b_t respectively, intersecting C in two points A_1 and A_2 different from P_t which are corresponding points in the g^1_2 cut out on C by a system of order $n + n_1 + n_2$ having the P_t's as base-points of multiplicities $r_t + a_t + b_t$ and passing through an arbitrary point. C is here the curve mentioned in the precedent Section.

The same procedure may be applied—with slight modifications—when the points $P_{t,1}$ and $P_{t,2}$ are adjacent or are not f–points in the same q–transformation. The conclusion is the same also in this case, except for the irreducibility in the adjacent case.

Consider now the general case. The mentioned systems (φ_t) and (Π_t) differ from each other in that the simple points $P_{t,1}$ and $P_{t,2}$ do not belong to the base of (Π_t). As systems having a simple base-point are uninteresting, $P_{t,1}$ and $P_{t,2}$ or one of their transformed points are assumed to be f–points by one of the T'_u $(u = t, t-1, \cdots, 1)$. To examine the transformed system (Π) of (Π_t) by T'_u, consider the "pointline" at $P_{t,v}$ $(v = 1, 2)$. They correspond to two rational curves R_1 and R_2, of orders n_1 and n_2 say, completely determined by having P_t $(t = 1, 2, \cdots, i)$ as base-points of virtual multiplicities a_t and b_t, making the number of virtual intersections with (φ) $nn_v - 1$ $(v = 1,2)$ and $\Sigma a_t b_t = n_1 n_2$. From the way the integers n_1, a_t and n_2, b_t are obtained, the case II in the Introduction cannot occur. The sum of the orders of (φ_t) and the "pointlines" and the sum of their virtual multiplicities at all the f–points $P_{t,q}$ $(q=1, 2, \cdots)$ are those of (Π_t). This is then also the case with the transformed systems. The order of (Π) is then $n+n_1+n_2$ and the virtual multiplicities at P_t are $r_t + a_t + b_t$. In addition, (Π) has the transformed point Q of Q_s as simple base-point.

The "pointlines" $P_{t,v}$ $(v = 1, 2)$ may have the f–points $P_{t,v,1}$, $P_{t,v,2}, \cdots$ in their 1st neighbourhood. These points have now got the actual multiplicity 1. "Pointlines" at these points give the effective multiplicity equal to the virtual ones and the "pointlines" $P_{t,v}$ $(v = 1, 2)$ "degenerate" into the 'pointlines" $P_{t,v}$, $P_{t,v,1}$, $P_{t,v,2}, \cdots$. The corresponding (virtual rational), curves will then also degenerate. If there are f-points in the neighbourhood of $P_{t,v,q}$ $(q=1, 2, \cdots)$, we may continue to add "pontlines" $P_{t,v,i,s}(s=1, 2, \cdots)$. It is seen that the "pointlines" given by $P_{t,v}$ contains a point $A_{t,v}$ different from the f–points and situated on the transformed curve C_t of C_s.

Now, a curve in (Π_t) going through $P_{t,1}$ also goes through $P_{t,2}$. This curve and the mentioned "pointlines" $P_{t,v}$ $(v = 1, 2)$ form a curve having the effective multiplicities at all the f–points equal to those of the general curve in (Π_t) and passing through $A_{t,v}$. Hence, the corresponding curve, which then is a member of (Π), passes through the points A_1, A_2 of intersections of R_1, R_2 and C.

On the other hand, let R_1 and R_2 be two curves of orders n_1 and n_2, completely determined by the base-points P_t of virtual multiplicities a_t and b_t making the virtual genus zero, having $nn_v - 1$ $(v = 1, 2)$ as virtual number of intersections with (φ), not satisfying condition II and such that $\Sigma a_t b_t = n_1 n_2$. Let further A_1 and A_2 be two points on the curve C, mentioned in the precedent Section, different from P_t and situated on both R_1 and R_2. If the curve of the system (Π) of order $n + n_1 + n_2$, having the P_t's as base-points of virtual multiplicities $r_t + a_t + b_t$, not satisfying I and II, going through a point Q not situated on C and passing through A_1, also passes through A_2, the system (φ) is transformable into a pencil of cubics. This may be verified in the following manner.

As the system (Π) has virtual dimension 1 and virtual genus 1, so (Π) must be irreducible if I and II do not occur. Therefore (Π) may be transformed into a system (Π_s) of cubics by a Cremona transformation. If n_s, $n_{s,1}$, and $n_{s,2}$ are the orders of the transformed curves of (φ), R_1 and R_2, we have

$$3 = n_s + n_{s,1} + n_{s,2} .$$

From the impossibility of $n_{s,1}$ and $n_{s,2}$ to be negative when II is not occurring and $n_s \geq 3$, we get $n_s = 3$; hence $n_{s,1} = n_{s,2} = 0$. The system (φ_s) is then given by 9 simple base-points, $P_{s,1}, P_{s,2}, \cdots, P_{s,9}$, and the curves corresponding to R_1 and R_2 by two "pointlines". As the sum of the virtual multiplicties of the base-points of (φ), R_1 and R_2 are equal to the multiplicities of (Π), the "pointlines" have to be given, for instance, by $P_{s,1}$ and $P_{s,2}$. (Π_s) is then defined by $P_{s,3}, P_{s,4}, \cdots, P_{s,9}$ and Q_s, where the latter is the transformed point of Q. The points A_v $(v = 1, 2)$ are transformed into the points $A_{s,v}$ and, as A_v is situated on R_v, they have to be in the neighbourhoods of $P_{s,v}$. Now (Π_s) contains a curve which, together with "pointlines" passes through $A_{s,v}$. The part of order different from zero must therefore contain $P_{s,v}$. As it was pointed out in the former Section, the points P_t have to be in a special position on a certain curve C. By adding C with suitable rational curves and "pointlines" we obtain a curve having the effective multiplicities at the points P_t equal to the virtual ones prescribed to (φ). This composed curve is then transformed by the mentioned Cremona transformation into a curve of order 3, where the part of non-zero order satisfies the conditions imposed by the simple points $P_{s,u}$ $(u = 1, 2, \cdots, 9)$. As C does not pass through Q, there are then two different curves satisfying these conditions. Hence, the $P_{s,u}$'s are the base-points of a pencil of cubics.

In conclusion, we may state the following theorem.

A system (φ) *of order* n, *virtual dimension* zero, *virtual genus* 1, *given by the assigned points* P_t $(t = 1, 2, \cdots, i)$ *of virtual multiplicities* r_t *and not satisfying I and II is an irreducible system transformable into a pencil of cubics only if the integers* n *and* r_t *are relatively prime and the points* P_t *are in such a special position that—taken with appropriate multiplicities—they form a superabundant base, making the virtual dimension negative, of a reducible or* *irreducible curve* C *of order lower than* n. C *together with certain rational curves (and " pointlines ") gives a curve having each point* P_t *as a point of multiplicity* r_t.

There exist then two curves R_1 *and* R_2, *completely determined by having the points* P_t *as base.points of virtual multiplicities* a_t *and* b_t, *having the orders* n_1 *and* n_2, *the virtual genus* zero, *not satisfying II, possessing* $nn_v - 1$ $(v = 1, 2)$ *as their virtual number of intersections with* (φ) *and such that* $\Sigma a_t b_t = n_1 n_2$. *Also there exists then a system* (Π) *of order* $n + n_1 + n_2$, *having* P_t *and a point* Q *not situated on* C *as base-points of virtual multiplicities* $r_t + a_t + b_t$ *and* 1, *and not satisfying I and II.* (φ) *is then transformable into a pencil of cubics if, and only if, the pair of points of intersections of* R_1, R_2 *and* C *belongs to the* g_2^1 *cut out by* (Π) *on* C, *or on a component of* C.

REMARK.—In order to determine the curves R_1, R_2 and the system (Π), we may consider the systems (φ'), (Π'), R_1' and R_2', of orders n, $n+n_1+n_2$, n_1 and n_2, given by distinct an arbitrarily chosen base-points P_t' $(t = 1, 2, \cdots, i)$, of the same virtual multiplicities as P_t by (φ), (Π), R_1 and R_2 respectively. The system (Π') has the virtual dimension 2 and the virtual genus 1, and is thus transformable by a series of q–transformations, which always have the three base-points of highest orders as f–points, into a system of cubics having 7 simple base-points. If the orders of the transformed systems of (φ'), R_1', and R_2' are n_s, $n_{s,1}$ and $n_{s,2}$, we have:

$$3 = n_s + n_{s,1} + n_{s,2} .$$

As it was seen before that $n_s = 3$ and $n_{s,1} = n_{s,2} = 0$, R_1 and R_2 are then given by two distinct among the 9 simple base-points of the transformed system of (φ'). The rational curves corresponding to those points have the numeric properties required from R_1 and R_2. R_1 and R_2 are then the curves obtained by altering the base from the points P_t' to the points P_t by two of them, in such a way to make the system (Π) not satisfying I and II.

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