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Almost-periodicity for some differential equations in Hilbert spaces

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Analisi matematica. — *Almost-periodicity for some differential equations in Hilbert spaces.* Nota di SAMUEL ZAIDMAN (*), presentata (**) dal Corrisp. L. AMERIO.

1. We present a new proof of an almost-periodicity theorem for the bounded solutions of the (elliptic) equation

$$\ddot{u} - Au = f \tag{1}$$

in a Hilbert space H, A being self-adjoint and $\geqq 0$, whereas $f(t)$ is a (Bochner) almost-periodic function from $-\infty < t < +\infty$ to H.

The line of this proof was suggested to us by L. Nirenberg, during a course CIME on «Abstract differential equations», Varenna (Lake of Como) 1963, and in Montréal, in March 1964.

2. Let V, H be two Hilbert spaces, $V \subset H$ with continuous immersion, V is dense in H.

Consider on $V \times V$ a sesqui-linear continuous symmetric form $a(u, v)$, such that for every $\beta > 0$, there exists $\alpha > 0$, with the property that

$$a(u, u) + \beta |u|_H^2 > \alpha |u|_V^2, \quad \text{for every} \quad u \in V. \tag{2}$$

The sesqui-linear form $a(u, v)$ defines a self adjoint operator A in H, with domain $D_A \subset V \subset H$, by means of the relation

$$(Au, v)_H = a(u, v), \quad \text{for any} \quad u \in D_A, v \in V \tag{3}$$

$$D_A = \{u \in V, |a(u, v)| < C_u |v|_H\}, \quad \text{for any} \quad v \in V. \tag{4}$$

We deduce from (2) that $(Au, u) \geqq 0$, for any $u \in D_A$.

It is also easily seen that

$$V = D_{(A+I)^{1/2}} \tag{5}$$

(with equivalent norms: $|u|_V^2 \approx |u|_H^2 + |(A + I)^{1/2} u|_H^2$).

3. As a corollary of a more general result (See [5]) we have the following:

THEOREM 1.—*Given an arbitrary function* $f(t) \in L^2_{loc}(-\infty, \infty; H)$, *there exists at least a function* $u(t) \in L^2_{loc}(-\infty, \infty; H)$ *such that the relation*

$$\int_{-\infty}^{+\infty} (u(t), \ddot{\varphi} - A\varphi)_H \, dt = \int_{-\infty}^{+\infty} (f, \varphi)_H \, dt, \tag{6}$$

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holds for every

$$\varphi \in K_A = \{ \varphi \in C_0^2 (-\infty , \infty ; H) , \varphi \in D_A , A\varphi \text{ continuous in } H \}.$$

In our particular case (A self-adjoint $\geqq 0$), we may regularize our solution. Precisely, we have the following:

THEOREM 2.—*Every solution $u(t)$ of 6) has the following supplementary properties*:

1° $u(t)$, $\dot{u}(t) \in L^2_{loc}(-\infty , \infty ; V)$, $\ddot{u}(t) \in L^2_{loc}(-\infty , \infty ; H)$;

2° $u(t) \in D_A$ *almost everywhere, and* $Au(t) \in L^2_{loc}(-\infty , \infty ; H)$;

3° *The relation.*

$$\ddot{u} - Au = f \tag{7}$$

holds, almost everywhere in $-\infty < t < +\infty$.

4° *An inequality of the form*

$$\int_a^b \{ |u(t)|_V^2 + |\dot{u}(t)|_V^2 + |\ddot{u}(t)|_H^2 + |Au(t)|_H^2 \} \, dt < C_\delta \int_{a-\delta}^{b+\delta} \{ |u(t)|_H^2 + |f(t)|_H^2 \} \, dt, \tag{8}$$

holds where C_δ *does not depend from* u, *nor from* (a , b).

One proves this result with an easy adaptation of the method given in Agmon-Nirenberg ([1] – Ch. IV). (See for details [6]).

We need also a little more precise inequality than (8), where C_δ will be precised better.

THEOREM 3.—*In the hypothesis of the preceeding theorem, the following estimate holds*

$$\int_a^b (|\ddot{u}|_H^2 + |Au|_H^2) \, dt < \frac{C}{\delta^4} \int_{a-\delta}^{b+\delta} |u|^2 \, dt + \int_{a-\delta}^{b+\delta} |f|^2 \, dt \tag{9}$$

where C *non depends from* u *or* (a , b).

We take $\zeta(t) = \varphi^4(t)$, *where* $\varphi \in C_0^\infty = \begin{cases} 1 , a \leqq t \leqq b \\ 0 , t < a - \delta , t > b + \delta , \\ < 1 \text{ in other points} \end{cases}$

and such that $|\dot{\varphi}| < \frac{C}{\delta}$, $|\ddot{\varphi}| < \frac{C}{\delta^2}$. After, by (7) we have

$$\int_{a-\delta}^{b+\delta} \zeta (\ddot{u} - Au , \ddot{u} - Au)_H \, dt = \int_{a-\delta}^{b+\delta} \zeta |f|^2 \, dt,$$

or

$$\int_{a-\delta}^{b+\delta} \zeta (|\ddot{u}|^2 + |Au|^2) \, dt - \int_{a-\delta}^{b+\delta} \zeta [(\ddot{u} , Au) + (Au , \ddot{u})] \, dt = \int_{a-\delta}^{b+\delta} \zeta |f|^2 \, dt.$$

Partial integration and self-adjointness gives us that

$$\int_{a-\delta}^{b+\delta} \zeta\,[(\ddot{u}\,,\,\mathrm{A}u) + (\mathrm{A}u\,,\,\ddot{u})]\,dt = -2\int_{a-\delta}^{b+\delta} \zeta(\mathrm{A}\dot{u}\,,\,\dot{u})\,dt + \int_{a-\delta}^{b+\delta} \ddot{\zeta}\,(u\,,\,\mathrm{A}u)\,dt\,.$$

Consequently

$$\int_{a-\delta}^{b+\delta} \zeta(|\ddot{u}|^2 + |\mathrm{A}u|^2)\,dt = \int_{a-\delta}^{b+\delta} \zeta\,|f|^2\,dt - 2\int_{a-\delta}^{b+\delta} \zeta\,(\mathrm{A}\dot{u}\,,\,\dot{u})\,dt + \int_{a-\delta}^{b+\delta} \ddot{\zeta}\,(\mathrm{A}u\,,\,u)\,dt$$

and from $(\mathrm{A}\dot{u}\,,\,\dot{u}) > 0$, it follows

$$\int_{a-\delta}^{b+\delta} \zeta(|\ddot{u}|^2 + |\mathrm{Au}|^2)\,dt < \int_{a-\delta}^{b+\delta} \zeta\,|f|^2\,dt + \int_{a-\delta}^{b+\delta} \ddot{\zeta}(\mathrm{A}u\,,\,u)\,dt\,.$$

Now

$$|\ddot{\zeta}\,(\mathrm{A}u\,,\,u)| < \sqrt{\zeta}\,|\mathrm{A}u|\,\frac{|\ddot{\zeta}|}{|\sqrt{\zeta}|}\,|u| < \frac{1}{2}\zeta\,|\mathrm{A}u|^2 + \frac{1}{2\zeta}\,|\ddot{\zeta}|^2\,|u|^2$$

and obtain

$$\int_{a-\delta}^{b+\delta} (\zeta\,|\ddot{u}|^2 + \zeta\,|\mathrm{A}u|^2)\,dt < \int_{a-\delta}^{b+\delta} \zeta\,|f|^2\,dt + \frac{1}{2}\int_{a-\delta}^{b+\delta} \zeta\,|\mathrm{A}u|^2\,dt + \frac{1}{2}\int_{a-\delta}^{b+\delta} \frac{|\ddot{\zeta}|^2}{\zeta}\,|u|^2\,dt$$

and from this

$$\int_{a}^{b} (|\ddot{u}|^2 + |\mathrm{A}u|^2)\,dt < 2\int_{a-\delta}^{b+\delta} |f|^2\,dt + \int_{a-\delta}^{b+\delta} \frac{|\ddot{\zeta}|^2}{\zeta}\,|u|^2\,dt.$$

But $\frac{|\ddot{\zeta}|^2}{\zeta} < \frac{\mathrm{A}}{\delta^4}$, as easily results from $\zeta = \varphi^4$. Hence the theorem is proved.

4. We pass now to the theorem of almost-periodicity for the solutions of equation (7), when the known term $f(t)$ is an almost-periodic function from $-\infty < t < +\infty$ to H.

We note firstly that due to the facts: $\dot{u}(t) \in \mathrm{L}^2_{\mathrm{loc}}(-\infty\,,\,\infty\,;\,\mathrm{V})$, and $\ddot{u}(t) \in \mathrm{L}^2_{\mathrm{loc}}(-\infty\,,\,\infty\,;\,\mathrm{H})$, it follows that $u(t)$ is continuous in V, and $\dot{u}(t)$ is continuous in H.

We have the following:

THEOREM 4 (1).—*If $f(t)$ is almost-periodic from $-\infty < t < +\infty$ to* H, *and if $u(t)$ satisfies 7), and the relation $|u(t)|_{\mathrm{H}} < \mathrm{L}\,,\, -\infty < t < +\infty$, then $u(t)$ is almost-periodic in* V, *and $\dot{u}(t)$ is almost-periodic in* H.

(1) It was proved already with a different method, in [6].

We prove firstly that $\dot{u}(t)$ is almost-periodic in H. Take an $\varepsilon > 0$. There exists a relatively dense set of ε-almost-periods τ for $f(t)$; that is

$$\sup_{-\infty < t < +\infty} |f(t+\tau) - f(t)|_H < \varepsilon .$$

If $\ddot{u} - Au = f$, then obviously $v(t) = u(t+\tau) - u(t)$ satisfies the equation

$$\ddot{v} - Av = f(t+\tau) - f(t) = g(t), \qquad \text{almost-everywhere.}$$

We use the following inequality (Gagliardo [3], Nirenberg [4]).

$$|\dot{\omega}(T)|_H < C\left(\int_{T-d}^{T+d} |\ddot{\omega}(t)|_H^2 \, dt\right)^{1/3} \sup_{T-d<t<T+d} |\omega(t)|_H^{1/3} + \frac{C}{d} \sup_{T-d<t<T+d} |\omega(t)|_H$$

where C non depends of ω and d, and which holds for scalar as well for vector-valued function ω, for which, $\omega, \dot{\omega}, \ddot{\omega} \in L^2_{loc}(-\infty, \infty; H)$.

We apply this inequality to $v(t)$, and obtain, with $d = L/\varepsilon$, the relation

$$|\dot{v}(T)|_H < C\left(\int_{T-L/\varepsilon}^{T+L/\varepsilon} |\ddot{v}(t)|_H^2 \, dt\right)^{1/3} (2L)^{1/3} + \frac{C\varepsilon}{L} \cdot 2L .$$

Now we apply 9) with $\delta = L/\varepsilon$. We have

$$\int_{T-L/\varepsilon}^{T+L/\varepsilon} |\ddot{v}(t)|_H^2 \, dt < \frac{C\varepsilon^4}{L^4} \int_{T-2L/\varepsilon}^{T+2L/\varepsilon} |v|_H^2 \, dt + \int_{T-2L/\varepsilon}^{T+2L/\varepsilon} |g|_H^2 \, dt < \frac{C\varepsilon^4}{L^4} 4L^2 \cdot \frac{4L}{\varepsilon} + \varepsilon^2 \frac{4L}{\varepsilon} = \frac{16C}{L} \varepsilon^3 + 4L\varepsilon .$$

Hence

$$|\dot{v}(T)|_H < C_1\left(\frac{16C\varepsilon^3}{L} + 4L\varepsilon\right)^{1/3} \cdot (2L)^{1/3} + 2C\varepsilon = D_1(\varepsilon), \ D_1(\varepsilon) \to 0$$

as

$$\varepsilon \to 0, \ -\infty < T < +\infty .$$

and this proves the almost-periodicity of $\dot{u}(t)$ in H. As $u(t)$ is bounded in H, by a theorem of Amerio [2], $u(t)$ is almost-periodic in H.

To prove the almost-periodicity of $u(t)$ in V, we use (8), applied again to $u(t+\tau) - u(t) = v(t), f(t+\tau) - f(t) = g(t)$

$$\int_{T-1}^{T+1} |v(t)|_V^2 \, dt < C \int_{T-2}^{T+2} \{|v(t)|_H^2 + |g(t)|_H^2\} \, dt .$$

Take $\varepsilon > 0$; and τ a common ε-almost-period for $f(t)$ and $u(t)$ in H. (This is possible for a relatively dense set of τ). Then $|v(t)|_H < \varepsilon, |g(t)|_H < \varepsilon$, $-\infty < t < \infty$; hence

$$\int_{T-1}^{T+1} |v(t)|_V^2 \, dt < C \cdot 8\varepsilon^2,$$

for every $T, -\infty < T < +\infty$.

Take t_T such that $|v(t_T)|_V = \inf_{T-1<t<T+1} |v(t)|_V$. Then

$$|v(t_T)|_V^2 < \frac{1}{2}\int_{T-1}^{T+1} |v(t)|_V^2 \, dt < K\varepsilon^2 .$$

For an arbitrary real T,

$$v(T) = v(t_T) + \int_{t_T}^{T} \dot{v}(t)\, dt \quad , \quad |v(T)|_V < \sqrt{K}\varepsilon + \sqrt{2}\left\{\int_{T-1}^{T+1} |\dot{v}(t)|_V^2 \, dt\right\}^{1/2} .$$

Apply again (8), and obtain $|v(T)|_V < D_2(\varepsilon)$, $D_2(\varepsilon) \to 0$ as $\varepsilon \to 0$. This proves the theorem.

BIBLIOGRAPHY.

[1] AGMON S., NIRENBERG L., *Properties of Solutions of Ordinary Differential Equations in Banach Space*, « Comm. Pure Appl. Math. », May 1963, 121–239.

[2] AMERIO L., *Sull'integrazione delle funzioni quasi-periodiche astratte*, « Ann. di Matem. » vol. LIII, 371–382 (1961).

[3] GAGLIARDO E., *Proprietà di alcune classi di funzioni di più variabili*, « Ricerche di Matem. » (1959).

[4] NIRENBERG L., *Lectures on Elliptic Equations*, « Ann. Sc. Normale Superiore », Pisa (1959).

[5] ZAIDMAN S., *Un teorema di esistenza globale per alcune equazioni differenziali astratte*, « Ricerche di Matem. », vol. 13 (1964).

[6] ZAIDMAN S., *Soluzioni quasi-periodiche per alcune equazioni differenziali in spazi hilbertiani*, « Ricerche di Matem. », vol. 13 (1964).