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On Asymmetric diophantine approximation

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NOTE PRESENTATE DA SOCI

Aritmetica. — *On Asymmetric diophantine approximation.* Nota di EUGENE ALFRED MAIER, presentata (*) dal Socio B. SEGRE.

1. It is the purpose of this paper to prove the following generalization of theorems of Segre [5] and Müller [2].

THEOREM.—*Let* θ *be irrational with continued-fraction expansion* $<a_0, a_1, a_2, \cdots>$, *let* t *be a non-negative real number and* m *a positive integer.*

If $a_{2j+1} \geq m$ *for infinitely many* j, *then there exist infinitely many rational numbers* h/k *with* $k > 0$ *such that*

$$-\frac{1}{\sqrt{m^2+4t}\;k^2} < \frac{h}{k} - \theta < \frac{t}{\sqrt{m^2+4t}\;k^2}. \tag{1}$$

If $t = 0$ *or if* t *is the reciprocal of an integer, this result becomes false if* $\sqrt{m^2+4t}$ *is replaced by a larger constant.*

If $a_{2j} \geq m$ *for infinitely many,* j, *the statement holds with* $h/k - \theta$ *replaced by* $\theta - h/k$.

2. The proof of the theorem is based upon elementary properties of continued fractions and the following lemma.

LEMMA.—*If* x *and* y *are integers and* α *and* β *are positive real numbers such that*

$$\frac{1}{\alpha x^2} + \frac{1}{\beta y^2} \leq \frac{1}{xy}, \tag{2}$$

then

$$\frac{1}{2}\beta - \gamma \leq x/y \leq \frac{1}{2}\beta + \gamma \tag{3}$$

where $\gamma = \sqrt{\beta^2/4 - \beta/\alpha}$. *Furthermore the inequality is strict if one of* β *and* γ *is rational and the other is irrational.*

Proof.—From (2) we have

$$\frac{\beta}{\alpha} + \left(\frac{x}{y}\right)^2 \leq \beta\frac{x}{y}.$$

Completing the square on x/y, we obtain

$$\left[\frac{x}{y} - \frac{\beta}{2}\right]^2 \leq \frac{\beta^2}{4} - \frac{\beta}{\alpha} = \gamma^2$$

and (3) follows. If one of β and γ is irrational and the other rational, the $\frac{1}{2}\beta - \gamma$ is irrational and equality cannot hold since x/y is rational.

(*) Nella seduta del 14 novembre 1964.

3. To prove the first part of the theorem, we shall show that for each j such that $a_{2j+1} \geq m$, at least one of the convergents h_{2j-1}/k_{2j-1}, h_{2j}/k_{2j}, h_{2j+1}/k_{2j+1} of θ satisfies (1). For suppose none of these three convergents satisfies (1). Then, since

$$\frac{h_{2j}}{k_{2j}} < \theta < \frac{h_{2j+1}}{k_{2j+1}} < \frac{h_{2j-1}}{k_{2j-1}},$$

we have

$$\frac{h_{2j-1}}{k_{2j-1}} - \theta \geq \frac{t}{\sqrt{m^2+4t}\,k_{2j-1}^2}, \qquad \frac{h_{2j+1}}{k_{2j+1}} - \theta \geq \frac{t}{\sqrt{m^2+4t}\,k_{2j+1}^2}, \tag{4}$$

$$\theta - \frac{h_{2j}}{k_{2j}} \geq \frac{1}{\sqrt{m^2+4t}\,k_{2j}^2}.$$

Using these inequalities along with elementary properties of the convergents, we obtain

$$\frac{1}{k_{2j}\,k_{2j+1}} = \left(\frac{h_{2j+1}}{k_{2j+1}} - \theta\right) + \left(\theta - \frac{h_{2j}}{k_{2j}}\right) \geq \frac{1}{\sqrt{m^2+4t}\,k_{2j}^2} + \frac{t}{\sqrt{m^2+4t}\,k_{2j+1}^2} \tag{5}$$

and

$$\frac{1}{k_{2j}\,k_{2j-1}} = \left(\frac{h_{2j-1}}{k_{2j-1}} - \theta\right) + \left(\theta - \frac{h_{2j}}{k_{2j}}\right) \geq \frac{t}{\sqrt{m^2+4t}\,k_{2j-1}^2} + \frac{1}{\sqrt{m^2+4t}\,k_{2j}^2}. \tag{6}$$

From (5) and the lemma with $\alpha = \sqrt{m^2+4t}$ and $\beta = \sqrt{m^2+4t}/t$, we have

$$\frac{k_{2j}}{k_{2j+1}} \geq \frac{\sqrt{m^2+4t}}{2t} - \sqrt{\frac{m^2+4t}{4t^2} - \frac{1}{t}} = \frac{\sqrt{m^2+4t} - m}{2t}. \tag{7}$$

From (6) and the lemma with $\alpha = \sqrt{m^2+4t}/t$ and $\beta = \sqrt{m^2+4t}$, we have

$$\frac{k_{2j-1}}{k_{2j}} \geq \frac{\sqrt{m^2+4t}}{2} - \sqrt{\frac{m^2+4t}{4} - t} = \frac{\sqrt{m^2+4t} - m}{2}. \tag{8}$$

Multiplying (5) by $k_{2j}k_{2j+1}$, using (7) and (8), and the equality $k_{2j+1} = a_{2j+1}k_{2j} + k_{2j-1}$, we obtain

$$1 \geq \frac{k_{2j+1}}{k_{2j}} \cdot \frac{1}{\sqrt{m^2+4t}} + \frac{k_{2j}}{k_{2j+1}} \cdot \frac{t}{\sqrt{m^2+4t}} =$$

$$\left(a_{2j+1} + \frac{k_{2j-1}}{k_{2j}}\right)\frac{1}{\sqrt{m^2+4t}} + \frac{k_{2j+1}}{k_{2j}} \cdot \frac{t}{\sqrt{m^2+4t}} \geq \tag{9}$$

$$\left(m + \frac{\sqrt{m^2+4t} - m}{2}\right) \cdot \frac{1}{\sqrt{m^2+4t}} + \frac{\sqrt{m^2+4t} - m}{2t}\left(\frac{t}{\sqrt{m^2+4t}}\right) = 1.$$

If $\sqrt{m^2+4t}$ is rational, then the last inequality in (4) is strict since θ is irrational. Hence (5) is strict and thus the first inequality in (9) is strict, which yields the contradiction $1 > 1$. If $\sqrt{m^2+4t}$ is irrational, then with

$\alpha = \sqrt{m^2 + 4t}$ and $\beta = \sqrt{m^2 + 4t}/t$, we have $\gamma = m/2t$ which is rational. Thus, by the lemma, (7) is strict and therefore the second inequality in (9) is strict. This again yields the contradiction $1 > 1$.

To show the constant $\sqrt{m^2 + 4t}$ cannot be increased if $t = 0$ or if t is the reciprocal of an integer, let u be a non-negative real number such that for every irrational θ with $a_{2j+1} \geq m$ for infinitely many j, there exist infinitely many rationals h/k with $k > 0$ for which

$$(10) \qquad -\frac{1}{uk^2} < \frac{h}{k} - \theta < \frac{t}{uk^2} .$$

We shall show that $u \leq \sqrt{m^2 + 4t}$.

Under the above assumption, for $\theta = < mn, m, mn, m, \ldots > = (mn + \sqrt{m^2 n^2 + 4n})/2$, there exist infinitely many h/k with $k > 0$ such that (10) holds. The denominators of these fractions increase without bound; for if $k < M$ for all k, we have

$$|h| \leq |h - k\theta| + |k\theta| < 1/u + M|\theta|$$

and the numerators are also bounded, contradicting the existence of infinitely many h/k.

If, for the given value of θ, there exist infinitely many $h/k > \theta$ satisfying (10), then letting $\bar{\theta}$ denote the conjugate of θ, we have $h/k - \bar{\theta} = h/k - \theta + \sqrt{m^2n^2 + 4n} > 0$. Thus $h^2 - mnhk - nk^2 = k^2(h/k - \theta)(h/k - \bar{\theta}) > 0$. However $h^2 - mnhk - nk^2$ is an integer and is therefore greater than or equal to one.
Hence

$$\frac{1}{k^2(h/k - \bar{\theta})} \leq \frac{h^2 - mnhk - nk^2}{k^2(h/k - \bar{\theta})} = \frac{h}{k} - \theta < \frac{t}{uk^2}$$

whence it follows that $u < t(h/k - \bar{\theta})$. Taking limits as $k \to \infty$, we have $u \leq t(\theta - \bar{\theta}) = tn\sqrt{m^2 + 4/n}$. Since $t = 0$ is impossible in the case under consideration, t is the reciprocal of an integer. Setting $n = 1/t$, we have $u \leq \sqrt{m^2 + 4t}$.

The other possibility is that, for the given value of θ, there exist infinitely many $h/k < \theta$ satisfying (10). If $u < 1$, then $u < \sqrt{m^2 + 4t}$ for all m and t and hence we need only consider values of $u \geq 1$. In this case h/k is either a convergent or secondary convergent to θ and since $h/k < \theta$, we have one of the following three possibilities for infinitely many j:

$$\text{(i)} \qquad \frac{h}{k} = \frac{h_j}{k_j}, \ j \text{ even},$$

$$\text{(ii)} \qquad \frac{h}{k} = \frac{h_j + h_{j-1}}{k_j + k_{j+1}}, \ j \text{ odd}, \qquad \text{(iii)} \qquad \frac{h}{k} = \frac{h_j - h_{j-1}}{k_j - k_{j-1}}, \ j \text{ even}.$$

If $\theta = < a_0, a_1, a_2, \ldots >$, let $\alpha_j = < a_{j+1}, a_{j+2}, a_{j+3}, \ldots >$ and let $\beta_j = < a_j, a_{j-1}, \ldots, a_1 >$. For the value of θ under consideration, if j is even, $\alpha_j = m + 1/\theta$ and $\beta_j \to \theta$ as $j \to \infty$ whereas, if j is odd, $\alpha_j = \theta$ and $\beta_j \to m + 1/\theta$ as $j \to \infty$. Now if (i),

$$\frac{1}{u} > k_j^2 \left| \theta - \frac{h_j}{k_j} \right| = \frac{1}{\alpha_j + 1/\beta_j}, \; j \text{ even.}$$

Taking limits as $j \to \infty$ we have

$$\frac{1}{u} \geq \frac{1}{m + 2/\theta} = \frac{1}{\sqrt{m^2 + 4/n}} .$$

If (ii),

$$\frac{1}{u} > (k_j + k_{j-1})^2 \left| \theta - \frac{h_j + h_{j-1}}{k_j + k_{j-1}} \right| = \frac{(\alpha_j - 1)(\beta_j + 1)}{\alpha_j \beta_j + 1}, \; j \text{ odd,}$$

and taking limits

$$\frac{1}{u} \geq \frac{(\theta - 1)\left(m + \frac{1}{\theta} + 1\right)}{\theta\left(m + \frac{1}{\theta}\right) + 1} = \frac{\left(1 - \frac{1}{\theta}\right)\left(m + \frac{1}{\theta} + 1\right)}{m + 2/\theta} =$$

$$= \frac{m + 1 - \frac{1}{n}}{m + 2/\theta} > \frac{1}{m + 2/\theta} = \frac{1}{\sqrt{m^2 + 4/n}} .$$

If (iii),

$$\frac{1}{u} > (k_j - k_{j-1})^2 \left| \theta - \frac{h_j - h_{j-1}}{k_j - k_{j-1}} \right| = \frac{(\alpha_j + 1)(\beta_j - 1)}{\alpha_j \beta_j - 1}, \; j \text{ even,}$$

and again it follows that $1/u \geq 1/\sqrt{m^2 + 4/n}$. Thus $u \leq \sqrt{m^2 + 4/n}$ for all n. If t is the reciprocal of an integer, setting $n = 1/t$ we have $u \leq \sqrt{m^2 + 4t}$. If $t = 0$, the result follows by taking limits as $n \to \infty$.

The theorem is proved in a similar fashion if $a_{2j} \geq m$ for infinitely many j. In this case, for each j for which $a_{2j} \geq m$, at least one of the three convergents h_{2j-2}/k_{2j-2}, h_{2j-1}/k_{2j-1}, h_{2j}/k_{2j} satisfies the desired inequality. The inequality can be shown to be the best possible of its form when $t = 0$ or $1/t$ is an integer is the same fashion as above by considering $\theta = \langle m, mn, m, mn, \ldots \rangle$.

4. By setting $m = 1$ in the theorem we obtain Segre's theorem on asymmetriç approximation:

C o r o l l a r y 1 (Segre [5]).—*Let* θ *be irrational and let* t *be a non-negative real number. Then there exist infinitely many rational numbers* h/k *with* k *with* $k > 0$ *such that*

$$-\frac{1}{\sqrt{1 + 4t}\, k^2} < \frac{h}{k} - \theta < \frac{1}{\sqrt{1 + 4t}\, k^2} .$$

If $t = 0$ *or if* t *is the reciprocal of an integer, this result becomes false if* $\sqrt{1 + 4t}$ *is replaced by a larger constant. The statement also holds with* $h/k - \theta$ *replaced by* $\theta - h/k$.

Setting $t = 1$, we obtain a theorem of Muller:

Corollary 2 (Muller [2]).—*Let* θ *be irrational with continued-fraction expansion* $< a_0, a_1, a_2, \ldots >$. *If* $a_j > m$ *for infinitely many* j, *then there exist infinitely many rational numbers* h/k *such that*

$$\left| \theta - \frac{h}{k} \right| < \frac{1}{\sqrt{m^2 + 4}\, k^2} .$$

This result becomes false if $\sqrt{m^2 + 4}$ *is replaced by a larger costant.*

Finally, setting $t = m = 1$, we obtain Hurwitz's Theorem:

Corollary 3 (Hurwitz [1]).—*If* θ *is irrational, then there exist infinitely many rational numbers* h/k *such that*

$$\left| \theta - \frac{k}{h} \right| < \frac{1}{\sqrt{5}\, k^2} . \tag{11}$$

This result becomes false if $\sqrt{5}$ *is replaced by a larger constant.*

REFERENCES.

[1] A. HURWITZ, *Über die angenäherte Darstallung der Irrationalzahlen durch rational Bruche*, «Math. Annalen», *39*, 279–284 (1891).

[2] M. MÜLLER, *Über die Approximation reeller Zahlen durch die Näherungsbruche ihres regelmässigen Kettenbruches*, «Arch. Math.», *6*, 253–258 (1955).

[3] I. NIVEN and H. S. ZUCKERMAN, *An Introduction to the Theory of Numbers*, John Wiley and Sons, New York 1960.

[4] R. ROBINSON, *The approximation of irrational numbers by fractions with odd or even terms*, «Duke M. J.», *7*, 354–359 (1940).

[5] B. SEGRE, *Lattice points in infinite domains and asymmetric Diophantine Approximations*, «Duke J. Math.», *12*, 337–365 (1945).