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Upon the paradoxal character of the solutions of Kelvin-Somigliana and Kolossoff-Muskhelishvili in the plane elasticity

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Meccanica. — *Upon the paradoxal character of the solutions of Kelvin–Somigliana and Kolossoff–Muskhelishvili in the plane elasticity.* Nota di LIVIU SOLOMON, presentata (*) dal Socio B. FINZI.

1. It is well-known (see A. I. Lurye [1], chap. 2, § 1) that the fundamental solution of Lamé's equations (for isotropic and homogeneous bodies) can be obtained, by introducing in the representation of Grodskii

$$\bar{u} = \bar{B} - \frac{1}{4(1-\nu)} \operatorname{grad} (\bar{x} \cdot \bar{B} + B_0) \tag{1}$$

$$\Delta\bar{B} = -\frac{\bar{F}}{\mu} \quad , \quad \Delta B_0 = \frac{\bar{x} \cdot \bar{F}}{\mu} \tag{2}$$

(where $\bar{x}$ is the vector-radius of an arbitrary point; $\bar{u}$ – the displacement vector; $\bar{F}$ – the body force; μ – the constant of Coulomb; ν – the constant of Poisson), the fundamental solution of Laplace's equation.

In the case of the plane strain state, one has to take

$$\bar{B} = A \ln \frac{a}{\rho} \quad , \quad B_0 = 0 \quad , \quad \rho^2 = x^2 + y^2, \tag{3}$$

where a is an arbitrary and *undeterminable* constant, introduced in order to render the solution independent upon the choice of the unit of lenght.

This solution corresponds to the action of an *unit* concentrated force, if

$$A = 1/2\,\pi\mu\,. \tag{4}$$

Considering for simplicity the case of an unit force applied in the origin along the axis Ox (which does not affect the generality) we get from (1) the components of the plane tensor of Kelvin-Somigliana

$$v_{11} = \frac{1}{8\pi\mu(1-\nu)} \left[(3-4\nu)\ln\frac{a}{\rho} + \frac{x^2}{\rho^2}\right] \quad , \quad v_{12} = \frac{1}{8\pi\mu(1-\nu)} \frac{xy}{\rho^2} \tag{5}$$

or still in polar coordinates $z = \rho e^{i\chi}$:

$$v_{11} = \frac{1}{8\pi\mu(1-\nu)} \left[\cos^2\chi + (3-4\nu)\ln\frac{a}{\rho}\right] \quad , \quad v_{12} = \frac{1}{8\mu\pi(1-\nu)} \sin\chi\cos\chi\,. \tag{6}$$

The same expressions are appearing if we utilize the solution of Kolossoff-Muskhelishvili for a concentrated unit force directed along Ox (see N. I. Muskhelishvili [2], § 57). Namely, one gets

$$\varphi(z) = -\frac{1}{2\pi(\varkappa+1)} \ln z \quad , \quad \psi(z) = \frac{\varkappa}{2\pi(\varkappa+1)} \ln z \tag{7}$$

(*) Nella seduta del 10 giugno 1964.

with $\varkappa = 3 - 4\nu$, so that the plane complex displacement will be

$$(8) \qquad U = -\frac{3-4\nu}{16\pi\mu(1-\nu)} \ln \rho^2 + \frac{1}{16\pi\mu(1-\nu)} e^{2i\chi}$$

which leads—after separation of the real and imaginary parts—exactly to the components from (5) or (6), but for a constant term.

2. The above solution obviously contredicts the mechanical facts. The component v_{11} of the displacement along the direction of the acting force tends to infinity both for $\rho \to 0$ (which is logical), and for $\rho \to \infty$ (which is absurd); and the component v_{12} does not depend upon the distance ρ (which is also inacceptable).

For this reason, one admits usually in the theory of the plane problem—if it is necessary to suppose the displacements as being bounded at the infinity—that the forces acting upon any bounded boundary component, have a vanishing resultant force (see [2], § 36).

In order to examine the behaviour of the displacements (6), let us consider the curve

$$(9) \qquad F(x, y; a) \equiv \cos^2\chi + (3-4\nu) \ln \frac{a}{\rho} = 0$$

along which one has $v_{11} = 0$. One sees easily that the curves $F(x, y; a) = 0$ and $F(x, y; a) = \text{Const.}$ are homothetic with the curve $F(x, y; 1) = 0$, so that it remains to examine only this last one. For it, one has

$$(10) \qquad \rho = \exp[\cos^2\chi/(3-4\nu)]$$

which is the equation of a closed curve, having Ox, Oy as axes of symmetry.

Thus, the solution of Kelvin-Somigliana renders evident a closed curve, on which $v_{11} = 0$; in its interior one has $v_{11} > 0$, and in its exterior, $v_{11} < 0$. The modification of the constant a changes the dimensions of this " tube " of equal displacements—but does not modify its character.

All this is absurd—and is obviously due to the fact that the function $\ln z$ possesses *two* singular points (0 and ∞), while the function $1/R$ in the three-dimensional case is regular at the infinity.

It can be also noted that in the plane case, we have at our disposal *two* functions which can lead to solutions of the type of Kelvin-Somigliana: $\operatorname{Re} \ln z = \ln \rho$, and $\operatorname{Im} \ln z = \chi = \operatorname{arc\,tg} \frac{y}{x}$. Setting in (1)

$$(11) \qquad B_1 = 0 \quad , \quad B_2 = A\chi \quad , \quad B_0 = 0$$

one obtains a second solution of Kelvin's-Somigliana's type

$$(12) \qquad v'_{11} = \frac{1}{4(1-\nu)} \sin^2\chi \quad , \quad v'_{12} = \frac{1}{4(1-\nu)} [(3-4\nu)\chi - \sin\chi \cos\chi] ,$$

where in the case of the unit force

$$(13) \qquad A = \frac{1-\nu}{\pi\mu(1-2\nu)} .$$

This is also inacceptable, since it leads to multi-valued components of the displacement and, more than, does not depend upon ρ.

No other possibility to construct plane solutions of this type is existing.

The paradoxal character of the above results shows that it is not allowed to use the solutions *in displacements* of the plane problem, if it has been made use of the components of the tensor of Kelvin-Somigliana in an entire (non-differential) form, as it is for ex. the case in the construction of the particular solutions corresponding to non-vanishing body-forces (see [2], the end of § 57). In the contrary, all the calculations referring to the stresses are valid, since the derivatives of $\ln z$ are regular at the infinity.

3. The paradoxal character of the solution of Kelvin-Somigliana in the plane can be understood, taking into account that the problem of plane strain is in fact a three-dimensional one, with complete monotony of the data along the axis Oz.

Let us consider the three-dimensional infinite space, loaded along Oz by a system of constant loadings directed along Ox. We note with $\overline{x}$ the point of observation, and with $\overline{\xi}$ the point of source. If the loadings $q(\zeta)\,\overline{i}_1$ have the resultant force $\overline{i}_1$ (unit vector of Ox) on any interval of unit length on Oz, it follows $q(\zeta) = \text{const.} = 1$.

The solution of Kelvin–Somigliana in the three-dimensional space for a concentrated force $\overline{i}_1$ is

$$v_{11}(\overline{x}\,;\overline{\xi}) = c\left(\varkappa\frac{1}{R} + \frac{x^2}{R^3}\right) \quad , \quad v_{12}(\overline{x}\,;\overline{\xi}) = c\,\frac{xy}{R^3}\,, \tag{14}$$
$$v_{13}(\overline{x}\,;\overline{\xi}) = c\,\frac{x\,(z-\zeta)}{R^3}$$

where

$$R = \sqrt{x^2 + y^2 + (z-\zeta)^2} \quad , \quad c = \frac{1}{16\,\pi\mu\,(1-\nu)}\,, \tag{15}$$

Multiplying in (14) with $d\zeta$ and integrating from ζ_1 to ζ_2, we obtain the displacements of the three-dimensional space submitted to a « line » of loadings distributed along Oz, and having the resultant force $(\zeta_2 - \zeta_1)\,\overline{i}_1$.

We consider the change of variable

$$Z = z - \zeta \quad , \quad dZ = -d\zeta \tag{16}$$

and note with V_{1i} the integrals of the components from (14).

After certain elementary calculations, we obtain

$$\left\{\begin{aligned} V_{11} &= 2c\left[\frac{x^2}{\rho^2} + (3-4\nu)\ln\frac{a}{\rho}\right] + c\,(3-4\nu)\ln\frac{(R_1+Z_1)(R_2-Z_2)}{a^2} - \\ &\qquad - cx^2\left[\frac{1}{R_1(R_1+Z_1)} + \frac{1}{R_2(R_2-Z_2)}\right] \\ V_{12} &= 2c\,\frac{xy}{\rho^2} - cxy\left[\frac{1}{R_1(R_1+Z_1)} + \frac{1}{R_2(R_2-Z_2)}\right] \\ V_{13} &= cx\left[\frac{1}{R_2} - \frac{1}{R_1}\right] \end{aligned}\right. \tag{17}$$

where it has been noted

$$Z_i = z - \zeta_i \quad , \quad R = \sqrt{x^2 + y^2 + Z^2} = \sqrt{\rho^2 + Z^2} \quad , \quad R_i = \sqrt{\rho^2 + Z_i^2} \qquad (i = 1, 2) . \tag{18}$$

It is visible now, that the first terms from V_{11} and V_{12} coïncide with v_{11}, respectively with v_{12} for the plane case from (5).

4. It would be desirable to obtain from (17) the solution of the plane strain problem, by taking $Z_1, -Z_2 \to \infty$. In fact, it is visible that the last terms from V_{11}, V_{12}, together with V_{13}, tend in this case to zero. On the contrary, the second term from V_{11} tends now to infinity. This shows the reason for which the function v_{11} from (5) tends to infinity with ρ, in contradiction with the mechanical sense of the problem; things are happening so, only because in the solution of the plane problem—which had to have been obtained from (17)—one has *in advance* neglected the terms depending upon z, and therefore the second term from V_{11}, which tends also to ∞, has been overlooked.

Let us group together in (17) the terms in $\cos^2 \chi$, $\cos \chi \sin \chi$, and the logarithmic ones. After elementary calculations, we obtain

$$\left\{\begin{aligned}
V_{11}(x, y, z) &= 2c \left\{ \left[1 - \frac{1}{2\sqrt{1 + \frac{Z_1^2}{\rho^2}} \left(\sqrt{1 + \frac{Z_1^2}{\rho^2}} + \frac{Z_1}{\rho} \right)} - \right.\right. \\
&\qquad \left. - \frac{1}{2\sqrt{1 + \frac{Z_2^2}{\rho^2}} \left(\sqrt{1 + \frac{Z_2^2}{\rho^2}} - \frac{Z_2}{\rho} \right)} \right] \cos^2 \chi + \\
&\qquad \left. + \frac{3 - 4\nu}{2} \left[\ln \left(\sqrt{1 + \frac{Z_1^2}{\rho^2}} + \frac{Z_1}{\rho} \right) + \ln \left(\sqrt{1 + \frac{Z_2^2}{\rho^2}} - \frac{Z_2}{\rho} \right) \right] \right\} \\
V_{12}(x, y, z) &= 2c \left[1 - \frac{1}{2\sqrt{1 + \frac{Z_1^2}{\rho^2}} \left(\sqrt{1 + \frac{Z_1^2}{\rho^2}} + \frac{Z_1}{\rho} \right)} - \right. \\
&\qquad \left. - \frac{1}{2\sqrt{1 + \frac{Z_2^2}{\rho^2}} \left(\sqrt{1 + \frac{Z_2^2}{\rho^2}} - \frac{Z_2}{\rho} \right)} \right] \cos \chi \sin \chi \\
V_{13}(x, y, z) &= c \left[\frac{1}{\sqrt{1 + \frac{Z_2^2}{\rho^2}}} - \frac{1}{\sqrt{1 + \frac{Z_1^2}{\rho^2}}} \right] \cos \chi
\end{aligned}\right. \tag{19}$$

(where the variable z appears through Z_1, Z_2).

The coefficient of $\cos^2 \chi$ and the logarithmic term from V_{11} are both positive for any values of ρ, χ, Z_1, Z_2—so that we have everywhere $V_{11} > 0$, as it was to be expected.

Further, the limits of the expressions (19) for any *finite* Z_1, Z_2 are *zero*. Therefore, this solution gives displacements which keep a constant sign along the direction of the loading, and tend to zero at the infinity.

On the contrary, for ρ finite, we get from (19)

$$\lim_{Z_1, -Z_2 \to \infty} V_{11} = (3 - 4\nu)\, c \ln \frac{Z_1 Z_2}{\rho^2} \to \infty \quad , \quad \lim_{Z_1, -Z_2 \to \infty} V_{12} = 2c \sin \chi \cos \chi \, , \tag{20}$$

$$\lim_{Z_1, -Z_2 \to \infty} V_{13} = 0$$

which is also easily understandable: if $Z_1, -Z_2 \to \infty$, the resultant force applied along Oz upon the elastic space tends also to ∞, so that it is to be expected, that the displacements cannot remain finite.

In the practice, one cannot meet neither the case $Z_1, -Z_2 \to \infty$, nor the case $\rho \to \infty$. But the values of the displacements can *never* be searched in the form (5) or (8)—which have no mechanical meaning—but only starting from (19) and evaluating in which measure these formulae, established for the infinite space loaded along Oz, remain admissible for the considered problem.

This remains valid also in connection with the above said at the end of § 2. Further, this must permit to clear up certain difficulties appearing in the plane contact problem, in the plane problem of stress-concentration etc.—all these connected in a certain measure with the behaviour of the displacements at the infinity.

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