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Upon the geometrical punch-penetration rigidity

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Meccanica. — *Upon the geometrical punch–penetration rigidity.*
Nota di LIVIU SOLOMON, presentata (*) dal Socio B. FINZI.

1. It is well known that the problem of the three-dimensional elastic frictionless contact (see H. Hertz [1]) is equivalent with that of the penetration of a rigid punch, having the form of an elliptical parabolloid (let us call it a parabolic punch), into an elastic half-space, both having perfectly smooth boundaries.

The general equations of the problem are studied, for ex., by I. Y. Staerman [2], especially for the punch with a plane elliptical basis-let us note this case with (′)—and for that of the parabolic punch–which will be noted with (″).

We shall note: μ – the constant of Coulomb; ν – the constant of Poisson; $\theta = (1 - \nu)/2\,\pi\mu$; $p(x, y)$ – the pression under the basis of the punch; $a \geq b$, D , k – the half-axes, the area, and respectively the eccentricity of the contact ellipse $\mathfrak{D}$; P – the total loading:

$$\text{P} = \iint_{\mathfrak{D}} p(\xi, \eta)\, d\xi\, d\eta\,; \tag{1}$$

δ – the central penetration of the punch; $a_0 = \text{P}/2\,\text{D}$ – the minimum pression under the punch with a plane elliptical basis; $a_1 = 3\,\text{P}/2\,\text{D}$ – the maximum pression under the parabolic punch.

We shall keep the notations, and utilize certain results from the papers of L. Solomon [3] , [4] , [5], L. Solomon and I. Zamfirescu [6].

2. The integral equation of the problem of central penetration (without rotation) for a punch with a plane (even non-elliptical) basis is

$$\frac{2\,\pi\mu}{1 - \nu}\,\delta = \iint_{\mathfrak{D}} \frac{p(\xi, \eta)}{\sqrt{(x - \xi)^2 + (y - \eta)^2}}\, d\xi\, d\eta\,. \tag{2}$$

In the case (′), the penetration is

$$\delta' = \theta\,\frac{\text{P}}{a}\,\text{K}(k)\,. \tag{3}$$

In the case (″), the penetration can be written in the form

$$\delta'' = \left(\frac{3}{2}\right)^{2/3} \text{K}(k)\,[\text{E}(k)]^{-1/3}\,(1 - k^2)^{1/3}\,\theta^{2/3}\,\text{P}^{2/3}\,\text{H}^{1/3} \tag{4}$$

(see formulae (12), (13) from [3], or—more detailed—(4.1) – (4.3) from [4]. K (k), E (k)—complete elliptic integrals of the first, respectively of the second kind; H—average curvature of the punch).

(*) Nella seduta del 10 giugno 1964.

Let us consider now the formula from the torsion's theory

$$\mathfrak{M}_3 = \mu\tau C \tag{5}$$

which binds the torsional couple $\mathfrak{M}_3$ (total characteristic of the loading), the constant μ (characteristic of the material), the torsional twist τ (global effect), and the geometrical torsional rigidity C (global geometrical characteristic of the body).

If isolating in the formulae (3), (4) the terms depending uniquely upon D and k, one gets two formulae of the same type as (5):

$$P = \frac{\mu}{1-\nu}\,\delta' C_c' \quad , \quad P = \frac{\mu}{1-\nu}\,\delta'' C_c'' \tag{6}$$

where

$$C_r' = \frac{4}{\sqrt{\pi}}\, D^{1/2}\left[\frac{2}{\pi}\,K(k)\,(1-k^2)^{1/4}\right]^{-1} \tag{7}$$

$$C_c'' = \frac{2}{3}\, C_c' . \tag{8}$$

The quantities C_c', C_c'' will be named *geometrical punch-penetration rigidities* for punches with elliptical bases-plane, or not.

The fact that $C_c'' < C_c'$ was to be expected, the stresses in the case (′) having to vainquish the increased resistance of the half-space along the singular line which is defining the boundary of $\mathfrak{D}$.

Noting with " o " the case of the circular contact domain, one gets a single formula (valid both in the case (′) and in (″)):

$$\frac{\delta}{\delta_0} = \frac{C_c(0)}{C_c(k)} = \frac{2}{\pi}\,K(k)\,(1-k^2)^{1/4} = 1 - \frac{1}{64}\,k^4\,l(k) \tag{9}$$

where

$$l(k) = 1 + k^2 + 0{,}88\,k^4 + \cdots , \qquad k \in (0,1) . \tag{10}$$

Therefore, the geometrical punch-penetration rigidity depends upon the contact *area*, and only in a non-essential manner upon the eccentricity, that means, upon the *shape* of the contact domain. (For $k \cong 1$, the formulation of the problem itself is to be modified: in this case, one has practically to deal with an initial contact along a line).

This explains the character of the results obtained by L. A. Galin [7], chap. 2, § 10, for punches with plane bases, and also the conclusion from [3], § 4, for parabolic punches. It is interesting to underline, that a similar fact appears in the theory of sound-propagation through elliptical orifices: see Lord Rayleigh [8], vol. 2, § 306.

3. The above said suggests to examine also the expression of the geometrical torsional rigidity for the elliptical cross-section. After simple calculations, the correspondent well-known formula can be written in the form

$$C = \frac{D^4}{4\pi^2 I_0} \cong \frac{1}{2\pi}\, D^2\left[1 - \frac{1}{8}\,k^4\,l(k)\right] \tag{11}$$

(I_0 – central polar moment of inertia) where—if neglecting higher-order terms—the function $l(k)$ is the same as that from (10).

This permits to transcribe the expression from (7), as

$$C'_c = 2{,}84\, D^{1/4}\, I_0^{1/8} \tag{12}$$

and a similar one for the quantity from (8). In this form, the geometrical punch-penetration rigidity does not more depend upon the specific elliptical form of $\mathfrak{D}$.

Remember now that the first expression from (11) has been proposed by B. de Saint-Venant [9] as an *approximative* one, for twisted bars with a non-elliptical cross-section. Although the limits of validity of such an approximative formula could not be established, the comparison with the exact solution for numerous non-elliptical cross-sections proved its usefulness.

4. Let us rewrite the equation (2) in the form

$$\delta' = \frac{1-\nu}{\mu} \frac{V(x,y,0)}{2\pi}, \tag{13}$$

the suffix (′) being now utilized for any punch with *plane* bounded basis, and $V(x, y, 0)$ being the limit-value of the single-layer potential with the density $p(x, y)$.

The comparison of (13) with (6) suggests to take the quantity

$$C'_c = 2\pi \frac{P}{V(x,y,0)} \tag{14}$$

as the *definition* of the geometrical punch-penetration rigidity also for punches with a plane bounded *non-elliptical* basis. Of course, this will have a sense only if C'_c from (14) does not depend upon the total loading and upon the material.

The comparison of (7), (9) and (11)—exact solutions—shows that the rôle of the shape of the contact domain in C'_c for the ellipse is by far less, than the rôle of the shape of the elliptical cross-section in C. This leads to the idea to take (12) as an approximative formula for the geometrical punch-penetration rigidity for bounded non-elliptical plane bases, instead of the exact expression (14), which cannot be in general calculated, inasmuch as no exact solutions for such punches are known.

5. As also for Saint-Venant's formula (11), such an assertion can be for the moment only verified with the help of several examples.

Owing to the absence of exact solutions, we are obliged to utilize with this aim the approximative solutions from [5], for the case "t" of the equilateral triangle with the height 3, and for the case "s" of the square with the side 2.

The functions and the tables from [5] permit to write—with the help of (14)—the values

$$C'_{c,t} = 2\pi \frac{j_t}{v_t} \cong 5{,}49\ \text{cm} \quad , \quad C'_{c,s} = 2\pi \frac{j_s}{v_s} \cong 4{,}60\ \text{cm}\,. \tag{15}$$

On the other hand, from (12) one gets firstly

$$C'_{c,t} = 2{,}31\, D_t^{1/2} \quad , \quad C'_{c,s} = 2{,}27\, D_s^{1/2} \tag{16}$$

(remember that, for equal areas, the geometrical punch-penetration rigidity must attain its minimum for the circle: see (9)).

From (16) it follows now for the considered examples

$$C'_{c,t} = 5{,}27 \text{ cm} \quad , \quad C'_{c,s} = 4{,}54 \text{ cm} . \tag{17}$$

Thus, the results from (15) and (17), obtained by two wholly different approximative methods, appear to be practically identic.

Therefore, the elementary formula (12) permits to determine effectively, with the help of (13), (14)—leading again to a formula of the same type as (6)—the value of δ', which is a global parameter of the greatest importance.

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