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On Asymmetric Diophantine Approximations

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Aritmetica. — *On Asymmetric Diophantine Approximations* (*). Nota di L. C. EGGAN, presentata (**) dal Socio B. SEGRE.

1. In 1945 B. Segre [3] introduced the idea of asymmetric approximations by proving the following theorem in the case $m = 1$.

THEOREM 1. – *If m is a positive integer, t a non-negative real number and θ an irrational number with continued fraction expansion $[a_0, a_1, \cdots]$ such that $a_{2j+1} \geq m$ for infinitely many j, then there exist infinitely many rationals p/q with $q > 0$ such that*

$$\frac{-1}{\sqrt{m^2+4t}\; q^2} < \frac{p}{q} - \theta < \frac{t}{\sqrt{m^2+4t}\; q^2} .$$

Moreover, if $t = 0$, or $1/t$ is an integer, the constant $\sqrt{m^2+4t}$ cannot be increased.

If $a_{2j} \geq m$ for infinitely many j, the statement holds with $p/q - \theta$ replaced by $\theta - p/q$.

By taking $t = 1$, this theorem, recently proved by E. A. Maier, also yields a theorem of Müller [2].

In this Note, we consider $t = 0$ or $1/n$, n an integer, and announce a generalization of Maier's theorem, for these values of t, by finding the best constant c_k such that

$$-1/c_k q^2 \leq p/q - \theta \leq t/c_k q$$

has at least k solutions p/q, when θ is as in the statement of Theorem 1.

In particular, the theorem is as follows.

THEOREM 2. – *Let m and n be positive integers and let $\theta_n = [0, \overline{m, mn}]$ and $\varphi_n = [0, \overline{mn, m}]$ have convergents P_j/Q_j and H_j/K_j, respectively. If $\theta = [a_0, a_1, a_2, \cdots]$ is any irrational number such that $a_{2j+1} \geq m$ for infinitely many values of j, then for any positive integer k, there exists at least k rational number p/q with $q > 0$ satisfying*

$$\frac{-1}{c(m,n,k)\, q^2} \leq \frac{p}{q} - \theta \leq \frac{(1/n)}{c(m,n,k)\, q^2} ,$$

where $c(m,n,k) = \varphi_n + m + H_{2k-1}/K_{2k-1}$. Moreover, the constant $c(m,n,k)$ cannot be improved and equality is attained on the right hand side for $\theta = \theta_n$.

If $a_{2j} \geq m$ for infinitely many j, the statement holds with $p/q - \theta$ replaced by $\theta - p/q$ and the left hand side is best possible for $\theta = \varphi_n$.

The case for $n = 1$ is already known [1, Theorem 2.3], and Theorem 2 is proved in a similar fashion.

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2. We obtain Maier's theorem for $t = 0$ or $1/n$, n a positive integer by taking limits. Besides the other results already mentioned, we also obtain Theorems 11 and 12 of Segre [3]. Also, by the proof, we are able to obtain the following strengthened version of Theorem 13 of [3].

COROLLARY. – *Let m and n be positive integers. If $\theta = [a_0, a_1, a_2, \cdots]$ is an irrational number for which $a_{2j+1} \geq m$ for infinitely many j, but θ is not properly equivalent to $\theta_n = [0, \overline{m, mn}]$ then there exist infinitely many rational numbers p/q with $q > 0$ such that*

$$-1/cq^2 < p/q - \theta < 1/ncq^2,$$

where

$$c = (\sqrt{m^2 + 4/n} + m + 2/mn)/2.$$

REFERENCES.

[1] L. C. EGGAN, *On Diophantine approximations*, «Trans. Amer. Math. Soc.», *99*, 102–117 (1961).

[2] M. MÜLLER, *Über die Approximation reeler Zahlen durch die Näherungsbrüche ihres regelmässigen Kettenbruches*, «Arch. Math.», *6*, 253–258 (1955).

[3] B. SEGRE, *Lattice points in infinite domains and asymmetric Diophantine approximations*, «Duke Math. J.», *12*, 337–365 (1945).

Note added in proof: the details of the proofs of these results will appear in «The Journal of the London Math. Soc.».