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Notes on rings of ordinary differential operators

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Matematica. — *Notes on rings of ordinary differential operators.* Nota di HISASI MORIKAWA, presentata [(*)] dal Socio straniero O. ZARISKI.

1. Let $\mathfrak{M}$ be a closed Riemann surface and $K_{\mathfrak{M}}$ be the field of meromorphic functions on $\mathfrak{M}$. Let $\mathfrak{S}$ be the direct set consisting of all the finite point sets on $\mathfrak{M}$, where the order in $\mathfrak{S}$ is given by the set theoretical inclusion. If $S' \supset S$ $(S, S' \in \mathfrak{S})$, there exists the canonical homomorphism of the fundamental group $\pi_1(\mathfrak{M} - S')$ of $\mathfrak{M} - S'$. We denote by $G_{\mathfrak{M}}$ the inverse limit $\lim_{\overrightarrow{S}} \pi_1(\mathfrak{M} - S)$ with respect to the canonical homomorphisms. Let $\mathfrak{D}_{\mathfrak{M}}$ be the ring of differential operators with coefficients in $K_{\mathfrak{M}}$, i.e. the ring consisting of all the linear operators $\sum_{l=0}^{n} a_l D^l$ with a_l in $K_{\mathfrak{M}}$, where is a fixed non-trivial derivation of $K_{\mathfrak{M}}$ over the field $\mathfrak{C}$ of complex numbers. We fix once and for all in the following a non-trivial derivation D; other derivations are expressed by a D with a in $K_{\mathfrak{M}}$. We denote by V_f the finite vector space of all the solutions of the differential equation $f(D)(y) = 0$ and by $\Omega_{\mathfrak{M}}$ the union $\bigcup_{f \in \mathfrak{D}_{\mathfrak{M}}} V_f$. Then $\Omega_{\mathfrak{M}}$ is regarded as a ring by the usual sum and product and the group $G_{\mathfrak{M}}$ operates on $\Omega_{\mathfrak{M}}$ in the natural way. Thus we can consider $\Omega_{\mathfrak{M}}$ as a $\mathfrak{C}[G_{\mathfrak{M}}]$-module or a $K_{\mathfrak{M}}[G_{\mathfrak{M}}]$-module.

For a non-zero left ideal $\mathfrak{a}$ of $\mathfrak{D}_{\mathfrak{M}}$ we denote by $V_{\mathfrak{a}}$ the vector space $\bigcap_{f \in \mathfrak{a}} V_f$ and call the vector space of solutions of $\mathfrak{a}$. Let $G_{(\mathfrak{M})}$ be the Galois group in Galois theory. Choosing a $\mathfrak{C}$-base of $V_{\mathfrak{a}}$, we have a representation $\Gamma_{\mathfrak{a}}$ of $G_{(\mathfrak{a})}$; we call the matric group $\Gamma_{\mathfrak{a}}$ the matric Galois group of the left ideal $\mathfrak{a}$. We denote by comm $(\Gamma_{\mathfrak{a}})$ the commutor algebra of $\Gamma_{\mathfrak{a}}$ in the full matric ring of degree dim $V_{\mathfrak{a}}$.

2. We shall start from the notion "proper ring" due to Ore:

Definition 1. – We mean by the proper ring of a left ideal $\mathfrak{a}$ in a ring A the subring $B_{\mathfrak{a}} = \{b \in A \mid \mathfrak{a} b \in \mathfrak{a}\}$.

We shall define primary left ideals in the most natural way as follows:

Definition 2. – If the residue ring $B_{\mathfrak{a}}/\mathfrak{a}$ is a primary ring, the left ideal $\mathfrak{a}$ is called a primary left ideal.

Let $\mathfrak{a}$ be a non-zero left ideal of $\mathfrak{D}_{\mathfrak{M}}$. Then, since $\mathfrak{a} b \in \mathfrak{a}$ for every b in $B_{\mathfrak{a}}$ and $\mathfrak{a}(V_{\mathfrak{a}}) = 0$, the residue ring $B_{\mathfrak{a}}/\mathfrak{a}$ operates on $V_{\mathfrak{a}}$ canonically and faithfully.

THEOREM 1. – *Let $\mathfrak{a}$ be a non-zero left ideal of $\mathfrak{D}_{\mathfrak{M}}$. Then the residue ring $B_{\mathfrak{a}}/\mathfrak{a}$ is canonically isomorphic to the commutor algebra comm $(\Gamma_{\mathfrak{a}})$ of the matric Galois group* $\Gamma_{\mathfrak{a}}$ of $\mathfrak{a}$.

(*) Nella seduta del 14 marzo 1964.

Proof. – Let $(\xi_1, \cdots, \xi_n)$ be a $\mathfrak{C}$–base of $V_\mathfrak{a}$ and (α_{ij}) be any elements in comm $(\Gamma_\mathfrak{a})$. Then, since the Wronskian $W(\xi_1, \cdots, \xi_n)$ of $(\xi_1, \cdots, \xi_n)$ is not zero, we can put

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} \xi_1 & \xi'_1 & \cdots & \xi_n^{(n-1)} \\ \vdots & \vdots & & \vdots \\ \xi_n & \xi'_n & \cdots & \xi_n^{(n-1)} \end{pmatrix} \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix} \begin{pmatrix} \xi_1 & \xi'_1 & \cdots & \xi_n^{(n-1)} \\ \vdots & \vdots & & \vdots \\ \xi_n & \xi_n & \cdots & \xi_n^{(n-1)} \end{pmatrix}$$

and get a matrix (a_{ij}) with coefficients in $K_\mathfrak{M}$, where $\xi^{(l)}$ means $D^l(\xi)$. Put $f(D) = \sum_{l=1}^{n} a_{l1} D^l$. Then we have $f(D)(\xi_i) = \sum_{l=0}^{n} \alpha_{il}\, \xi_l \ (i = 1, 2, \cdots, n)$, because

$$\begin{pmatrix} f(D)(\xi_1) \\ \vdots \\ f(D)(\xi_n) \end{pmatrix} = \begin{pmatrix} \xi_1 & \xi'_1 & \cdots & \xi_n^{(n-1)} \\ \vdots & \vdots & & \vdots \\ \xi_n & \xi'_n & \cdots & \xi_n^{(n-1)} \end{pmatrix} \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix},$$

$$\begin{pmatrix} \xi_1 & \xi'_1 & \cdots & \xi_1^{(n-1)} \\ \vdots & \vdots & & \vdots \\ \xi_n & \xi'_n & \cdots & \xi_n^{(n-1)} \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \cdots & \alpha_{1n} \\ \vdots & & \vdots \\ \alpha_{n1} & \cdots & \alpha_{nn} \end{pmatrix} \begin{pmatrix} \xi_1 & \xi'_1 & \cdots & \xi_1^{(n-1)} \\ \vdots & \vdots & & \vdots \\ \xi_n & \xi'_n & \cdots & \xi_n^{(n-1)} \end{pmatrix}.$$

This proves Theorem 1.

As consequences from Theorem 1 we have the following algebraic criterion of the indecomposability and infiniteness of matric Galois groups.

THEOREM 2. – *Let* $\mathfrak{a}$ *be a non-zero left ideal in* $\mathfrak{D}_\mathfrak{M}$. *Then we have the diagram*:

(I) $\mathfrak{a}$ is primary left ideal. $\Longleftrightarrow$ (II) comm $(\Gamma_\mathfrak{a})$ is a primary ring.

$\Updownarrow \quad \Updownarrow$

(III) $V_\mathfrak{a}$ is an indecomposable $\Gamma_\mathfrak{a}$–module.

Proof. – By virtue of Theorem 1 (I) $\Longleftrightarrow$ (II) is already proved. (I) $\Rightarrow$ (III) is obvious, because of $V_\mathfrak{a}$ is a decomposable $\Gamma_\mathfrak{a}$–module the commutor algebra comm $(\Gamma_\mathfrak{a})$ of $\Gamma_\mathfrak{a}$ has a zero divisor which is not nilpotent. We shall prove (III) $\Rightarrow$ (II). Assume for a moment comm $(\Gamma_\mathfrak{a})$ is not primary. Then there exist non-zero elements x and y in comm $(\Gamma_\mathfrak{a})$ such that $xy = 0$ and y is not nilpotent. Put $V_1 = \bigcup_n$ kernel (y^n) and $V_2 = \bigcup_n$ image (y^n). The by virtue of Fitting's lemma $V_\mathfrak{a}$ is a direct sum $V_1 \otimes V_2$. Since $x(V_2) = 0$, if we assume $V_1 = \{0\}$ we have $x(V_\mathfrak{a}) = 0$. Therefore $V_1 \neq \{0\}$. This contradicts with the indecomposability of $\Gamma_\mathfrak{a}$–module $V_\mathfrak{a}$.

THEOREM 3. – *Let* $\mathfrak{a}$ *be a primary left ideal of* $\mathfrak{D}_\mathfrak{M}$. *Then if* $\mathfrak{a}$ *is not a maximal left ideal, the matric Galois group* $\Gamma_\mathfrak{a}$ *of* $\mathfrak{a}$ *is an infinite group*.

Proof. – Assume for a moment that the matric Galois group $\Gamma_\mathfrak{a}$ of $\mathfrak{a}$ not maximal primary left ideal $\mathfrak{a}$ is a finite group. Since every representation of a finite group in $\mathfrak{C}$ is semi-simple, the commutor algebra comm $(\Gamma_{\mathfrak{a}})$ of $\Gamma_{\mathfrak{a}}$ also semi-simple. Since $\mathfrak{a}$ is a primary left ideal, by virtue of Theorem 2 the commutor algebra must be isomorphic to C. This shows $\Gamma_{\mathfrak{a}}$ is an irreducible representation, i.e. $V_{\mathfrak{a}}$ has no proper submodule. Hence a must be a maximal left ideal. This is a contradiction with the assumption on $\mathfrak{a}$.