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A canonical decomposition of additive functors of modules

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Matematica. — *A canonical decomposition of additive functors of modules*. Nota di ALEXANDRU SOLIAN, presentata (*) dal Socio B. SEGRE.

Suppose T is a covariant functor from one category of modules to another and that for any A of the first category there is given a Z–homomorphism $A \to T(A)$, so that these homomorphisms satisfy a natural condition of making commutative certain diagrams; in this case the functor is provided with an additional structure which, by definition, makes it a functor atomized by the given class. The canonical decomposition mentioned in the title refers to the process of passing from arguments to values of T and is made in terms of successive passing to values of covariant functors atomized by a canonical class of homomorphisms, of functors Hom (L ,) or Hom (, L) and to inverse values of functors of the first type. The complete result is given by the Principal Theorem. In the construction made enters as an essential fact the possibility of associating to every ring-module its additive group structure and also the isomorphism $\mathrm{Hom}_\Lambda(\Lambda, A) \approx A$.

In what follows Λ, Γ, etc. will represent associative rings with an identity element; by Z we shall denote the ring of integers. $\mathfrak{M}_\Lambda$ will mean the category of (unitary) Λ–modules and Λ–homomorphisms in which it will be determined but unspecified whether question runs about right or left Λ–modules. In general, the employed terminology will be that of [1] but we shall make some references also to [3].

1. Let T be a covariant functor defined on $\mathfrak{M}_\Lambda$ with values in $\mathfrak{M}_\Gamma$; suppose that together with T there is given a class $\{\varphi_A\}$ of Z–homomorphisms $\varphi_A : A \to T(A)$, where A is any object of $\mathfrak{M}_\Lambda$, so that if $f \in \mathrm{Hom}_\Lambda(A, C)$, the diagram

(1)
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\varphi_A"'] & C \arrow[d, "\varphi_C"] \\ T(A) \arrow[r, "T(f)"] & T(C) \end{tikzcd}$$

is commutative; in this case we say T *is an atomized functor by the class* $\{\varphi_A\}$ of Z–homomorphisms.

Thus, the class $\{\varphi_A\}$ provides T with an additional structure. As an example of the manner in which the structure of T is affected by the class $\{\varphi_A\}$ we give the following proposition:

(*) Nella seduta dell'8 febbraio 1964.

PROPOSITION 1. – *Let* T *be a covariant functor from* $\mathfrak{M}_\Lambda$ *to* $\mathfrak{M}_\Gamma$ *which is atomized by the class* $\{\varphi_A\}$; *if all* φ_A *are epimorphisms, then* T *is an additive functor.*

Proof. – Let $f, g \colon A \to C$ be Λ–homomorphisms and let $x \in T(A)$; if $x = \varphi_A a$, $a \in A$, then we have

$$T(f+g)(x) = T(f+g)\,\varphi_A\, a = \varphi_C (f+g)\, a = \varphi_C fa + \varphi_C ga = T(f)\,\varphi_A\, a + \\ + T(g)\,\varphi_A\, a = (T(f) + T(g))\, x .$$

Remark. – The class $\{\varphi_A\}$ which occurs in the above definition determines in the general case, a natural transformation of two functors; indeed, let us denote by I_Λ the functor $\mathfrak{M}_\Lambda \to \mathfrak{M}_Z$ which associates to a Λ–module A the Z–module A; then $\{\varphi_A\}$ determines a natural transformation of I_Λ into $I_\Gamma T$.

2. Let T be an additive covariant functor defined on $\mathfrak{M}_\Lambda$ with values in $\mathfrak{M}_\Gamma$, which is atomized by the class $\{\varphi_A\}$; if $f \colon A \to C$, $A, C \in \mathfrak{M}_\Lambda$, then since the diagram (1) is commutative, there exist uniquely determined Z–homomorphisms

$$\begin{aligned} f_1 &\colon \operatorname{Ker} \varphi_A \longrightarrow \operatorname{Ker} \varphi_C, \\ f_2 &\colon \operatorname{Coim} \varphi_A \longrightarrow \operatorname{Coim} \varphi_C, \\ f_3 &\colon \operatorname{Im} \varphi_A \longrightarrow \operatorname{Im} \varphi_C, \\ f_4 &\colon \operatorname{Coker} \varphi_A \longrightarrow \operatorname{Coker} \varphi_C, \end{aligned}$$

which render commutative the diagram

$$\begin{tikzcd} \operatorname{Ker} \varphi_A \arrow[r] \arrow[d, "f_1"'] & A \arrow[r] \arrow[d, "f"'] & \operatorname{Coim} \varphi_A \approx \operatorname{Im} \varphi_A \arrow[r] \arrow[d, "f_2"'] \arrow[d, shift left=8, "f_3"] & T(A) \arrow[r] \arrow[d, "T(f)"'] & \operatorname{Coker} \varphi_A \arrow[d, "f_4"'] \\ \operatorname{Ker} \varphi_C \arrow[r] & C \arrow[r] & \operatorname{Coim} \varphi_C \approx \operatorname{Im} \varphi_C \arrow[r] & T(C) \arrow[r] & \operatorname{Coker} \varphi_C \end{tikzcd} \tag{2}$$

where horizontal arrows represent canonical maps.

PROPOSITION 2. – *By passing from A to* Ker φ_A, *Coim* φ_A, *Im* φ_A, *Coker* φ_A *respectively and from f to* f_1, f_2, f_3, f_4 *respectively, there are defined covariant additive functors from* $\mathfrak{M}_\Lambda$ *to* $\mathfrak{M}_Z$.

These functors will be called respectively the *kernel*, the *coimage*, the *image*, and the *cokernel of* T *with respect to* $\{\varphi_A\}$ and will be denoted respectively by Ker $(T, \{\varphi_A\})$, Coim $(T, \{\varphi_A\})$, Im $(T, \{\varphi_A\})$ and Coker $(T, \{\varphi_A\})$.

The proof of Proposition 2 may be accomplished by simple verification.

From the commutative character of (2) it may be seen that Coim $(T, \{\varphi_A\})$, Im $(T, \{\varphi_A\})$, Coker $(T, \{\varphi_A\})$ are atomized by the class of respective canonical maps.

3. EXAMPLES. – 1) Let A be an abelian group and let F (A) represent its maximal torsion subgroup. Let T (A) be the factorgroup A/F (A) and

let $T(f)$, where $f: A \to C$, be the map $A/F(A) \to C/F(C)$ induced by f. Then T is a covariant additive functor from $\mathfrak{M}_Z$ to $\mathfrak{M}_Z$. Denoting by φ_A the natural epimorphism $A \to A/F(A)$, we obtain that T is atomized by the class $\{\varphi_A\}$.

We note that $\operatorname{Ker}(T, \{\varphi_A\})$ is the functor which associates to A, the subgroup $F(A)$, $A \in \mathfrak{M}_Z$.

2) Let S be a multiplicatively closed subset of the commutative ring Λ, such that $1 \in S$, $0 \notin S$ (1). If $A \in \mathfrak{M}_\Lambda$ and if A_S is the module of fractions of A with respect to S, then by passing from A to A_S and from $f: A \to C$ to the Λ–homomorphism $A_S \to C_S$ for which $[a/s] \to [fa/s]$ (where $a \in A$, $s \in S$, and $[a/s]$ is the class containing the formal quotient a/s) we obtain a covariant additive functor $T: \mathfrak{M}_\Lambda \to \mathfrak{M}_\Lambda$. Let φ_A be the homomorphism $A \to A_S$ for which $\varphi_A a = [a/1]$. Then T is atomized by the class $\{\varphi_A\}$.

Moreover, T may be interpreted as a functor from $\mathfrak{M}_\Lambda$ to $\mathfrak{M}_{\Lambda_s}$, which is atomized by the same class $\{\varphi_A\}$.

3) The following example does not enter in the limits of our theory but is illustrative as a generalization:

Let $\mathfrak{G}$ be the category of all groups and group homomorphisms. Denote by $[G, G]$ the commutator subgroup of the group G. Then a covariant functor $T: \mathfrak{G} \to \mathfrak{M}_Z$ may be defined by letting $T(G)$ be the factor-group $G/[G, G]$ and $T(f)$, where $f: G \to H$, the homomorphism $G/[G, G] \to H/[H, H]$ induced by f. If φ_G is the natural epimorphism $G \to G/[G, G]$ then for any $f: G \to H$, the diagram

$$\begin{tikzcd} G \arrow[r] \arrow[d] & H \arrow[d] \\ G/[G, G] \arrow[r] & H/[H, H] \end{tikzcd}$$

is commutative.

4. Let T be an additive functor defined on $\mathfrak{M}_\Lambda$ with values in $\mathfrak{M}_\Gamma$ (the variance of T is not yet specified). It is known (2) that if $A \in \mathfrak{M}_\Lambda$, then there exists a well-determined Λ–isomorphism

$$\eta_A : A \approx \operatorname{Hom}_\Lambda(\Lambda, A)$$

which maps $a \in A$ into that $f_a \in \operatorname{Hom}_\Lambda(\Lambda, A)$, for which $f_a(1) = a$.

Let us suppose that T is a covariant functor. Then T associates to $f_a: \Lambda \to A$, $a \in A$, the Γ–homomorphism $T(f_a) \in \operatorname{Hom}_\Gamma(T(\Lambda), T(A))$. The functor U defined from $\mathfrak{M}_\Lambda$ to $\mathfrak{M}_Z$ by

$$U(A) = \operatorname{Hom}_\Gamma(T(\Lambda), T(A)) \quad , \quad U(\psi) = \operatorname{Hom}(i_{T(\Lambda)}, T(\psi))$$

(1) See [3], § 8. 6.

(2) See [1], II §§ 2, 3.

(where $\psi : A \to C$ and $i_{T(\Lambda)}$ is the identity mapping of $T(\Lambda)$) is covariant and additive. Moreover, the mapping $\varphi_A : A \to \mathrm{Hom}_\Gamma(T(\Lambda), T(A)$ defined by

$$(3) \qquad \varphi_A\, a = T(f_a) \quad , \quad a \in A \quad , \quad A \in \mathfrak{M}_\Lambda ,$$

is a Z–homomorphism; indeed, by the additive character of T, we have

$$\varphi_A(a_1 + a_2) = T(f_{a_1+a_2}) = T(f_{a_1} + f_{a_2}) = T(f_{a_1}) + T(f_{a_2}) = \varphi_A\, a_1 + \varphi_A\, a_2 ,$$

where $a_1, a_2 \in A$.

Let $\psi : A \to C$ be *a* Λ–homomorphism, then $U(\psi) : U(A) \to U(C)$ and we have

$$U(\psi)\,\varphi_A\, a = \mathrm{Hom}(i_{T(\Lambda)}, T(\psi))\; T(f_a) = T(\psi)\, T(f_a) = T(\psi f_a)$$

$$\varphi_C\, \psi a = T(f_{\psi a})$$

for any $a \in A$; but $\psi f_a(1) = \psi\, a$ and $f_{\psi a}(1) = \psi a$. Hence, U is an atomized functor by the class $\{\varphi_A\}$, defined in (3).

Suppose now that T is contravariant; the T associates to f_a the Γ–homomorphism $T(f_a) \in \mathrm{Hom}_\Gamma(T(A), T(\Lambda))$. The functor U defined from $\mathfrak{M}_\Lambda$ to $\mathfrak{M}_Z$ by

$$U(A) = \mathrm{Hom}_\Gamma(T(A), T(\Lambda)) \quad , \quad U(\psi) = \mathrm{Hom}(T(\psi), i_{T(\Lambda)})$$

is anew covariant and additive. Let us define the class of mappings $\varphi_A : A \to U(A)$, $A \in \mathfrak{M}_\Lambda$, by the same relations (3); the Z–homomorphism character of φ_A may be proved exactly as before.

If $\psi : A \to C$ is now a homomorphism of the category $\mathfrak{M}_\Lambda$, then we have

$$U(\psi)\,\varphi_A a = \mathrm{Hom}(T(\psi), i_{T(\Lambda)})\, T(f_a) = T(f_a)\, T(\psi) = T(\psi f_a)$$

$$\varphi_C\, \psi a = T(f_{\psi a})$$

and exactly as before in the covariant case, it results that U is atomized by the class $\{\varphi_A\}$.

5. At the previous point 4, we have associated to every additive functor $T : \mathfrak{M}_\Lambda \to \mathfrak{M}_\Gamma$, an additive covariant functor $U : \mathfrak{M}_\Lambda \to \mathfrak{M}_Z$ defined by

$$(4) \qquad U(A) = \begin{cases} \mathrm{Hom}_\Gamma(T(\Lambda), T(A)), & \text{if T is covariant} \\ \mathrm{Hom}_\Gamma(T(A), T(\Lambda)), & \text{if T is contravariant} \end{cases}$$

such that, additionally U is atomized by the class $\{\varphi_A\}$ of Z–homomorphisms, defined by (3) (3). We shall call U, the *atomistical part* of T and we shall denote it by P (T).

From relations (4) it may be seen, that, in order to pass from the values of P (T) to those of T, there must be taken some (well-determined)

(3) Such functors and similar homomorphisms were considered also by DOLD [2], but in another connexion.

inverse images of the former by the functor R_T defined from $\mathfrak{M}_\Gamma$ to $\mathfrak{M}_Z$ by the formulas

$$(5) \qquad R_T(C) = \begin{cases} \mathrm{Hom}_\Gamma(K, C) & \text{if T is covariant} \\ \mathrm{Hom}_\Gamma(C, K) & \text{if T is contravariant} \end{cases}$$

where $C \in \mathfrak{M}_\Gamma$ and $K = T(\Lambda)$. R_T is consequently a functor $\mathrm{Hom}_\Gamma(K, \ )$ or $\mathrm{Hom}_\Gamma(\ , K)$.

Hence we have

THEOREM 1. – *In order to obtain values (modules and homomorphisms) of an additive functor* $T : \mathfrak{M}_\Lambda \to \mathfrak{M}_\Gamma$, *one must obtain the values of its atomistical part* $P(T) : \mathfrak{M}_\Lambda \to \mathfrak{M}_Z$ *and then pass to well-determined inverse images of the latter under the functor* R_T *defined from* $\mathfrak{M}_\Gamma$ *to* $\mathfrak{M}_Z$ *by formulas* (5).

6. We remark that every covariant functor T defined from a category of modules to a category of modules is an atomized functor by the class $\{O_A\}$ of the null mappings $A \to T(A)$. This makes the problem raise whether the additional structure of atomizing class of homomorphisms constitutes effectively something new and whether it may not be the trivial class. To this problem it may be answered, in the first place, by recalling that the class $\{\varphi_A\}$ constructed at point 4. is canonical and not arbitrary; in the second place, to the question in what case this canonical class is the null class, by the following

PROPOSITION 3. – *Let* T *be an additive functor from* $\mathfrak{M}_\Lambda$ *to* $\mathfrak{M}_\Gamma$; *the class* $\{\varphi_A\}$ *defined in* (3) *is the class of null homomorphisms if and only if* $T(\Lambda) = O$.

Proof. – If φ_Λ is the null mapping $\Lambda \to (P(T))(\Lambda)$ then $\varphi_\Lambda(1) = T(i_\Lambda)$ (where i_Λ is the identity mapping of Λ) is as well the O of $\mathrm{Hom}_\Gamma(T(\Lambda), T(\Lambda))$ and the identity mapping of $T(\Lambda)$; so $T(\Lambda) = O$. The converse is obvious.

Proposition 3 shows that if the class $\{\varphi_A\}$ is trivial then P (T) is the null functor and consequently all the theory looses its significance; but this is a very particular case.

7. If the inverse way to that described in Theorem 1 is pursued, we obtain

THEOREM 2. – *Let* T *be an additive functor from* $\mathfrak{M}_\Lambda$ *to* $\mathfrak{M}_\Gamma$. *In order to pass from values of* T *to their inverse images (modules and homomorphisms) under* T *one must apply to the former the functor* R_T *defined at point* 5, *and then pass from the obtained values to well-determined inverse images of these under the atomistical part* P (T) *of* T.

In this connexion we can make following remarks: Suppose that $V : \mathfrak{M}_\Lambda \to \mathfrak{M}_\Gamma$ is atomized by the class $\{\varphi_A\}$ of Z–homomorphisms. The module A whose image under V is V (A) is an extension of Ker φ_A by Im φ_A and

$\psi : A \to C$ whose image under V is $V(\psi)$ must render commutative the diagram

$$\begin{array}{ccc} A & \xrightarrow{\psi} & C \\ \downarrow & & \downarrow \\ \operatorname{Im} \varphi_A & \xrightarrow{\psi^*} & \operatorname{Im} \varphi_C \end{array}$$

where the vertical maps are induced by φ_A and φ_C and where ψ^* is induced by $V(\psi)$. This remark could be useful for the construction of inverse images of a functor atomized by a class of homomorphisms when giving the kernels and images of these homomorphisms.

8. In particular, we can apply Theorem 2 for obtaining the way in which one must pass from values of the functor R_T defined at point 5 by formulas (5), to their inverse values.

Now, since R_T is defined from $\mathfrak{M}_\Gamma$ to $\mathfrak{M}_Z$, the functor R_{R_T} is defined from $\mathfrak{M}_Z$ to $\mathfrak{M}_Z$ and namely by formulas

$$(6) \qquad R_{R_T}(D) = \begin{cases} \operatorname{Hom}_Z(L, D) \text{ if } R_T \text{ is covariant} \\ \operatorname{Hom}_Z(D, L) \text{ if } R_T \text{ is contravariant} \end{cases}$$

where $D \in \mathfrak{M}_Z$ and $L = R_T(\Gamma)$. But from (5) we see that R_T is covariant-contravariant if T is, so that formulas (6) may be transcripted in the form

$$(7) \quad R_{R_T}(D) = \begin{cases} \operatorname{Hom}_Z(L, D) \text{ where } L = \operatorname{Hom}_\Gamma(K, \Gamma) \text{ if T is covariant} \\ \operatorname{Hom}_Z(D, L) \text{ where } L = \operatorname{Hom}_\Gamma(\Gamma, K) \text{ if T is contravariant} \end{cases}$$

where $D \in \mathfrak{M}_Z$ and $K = T(\Lambda)$.

Combining Theorems 1 and 2 we obtain

THEOREM 3. – (*Principal Theorem*). *Let* T *be an additive functor defined from the category* $\mathfrak{M}_\Lambda$ *to the category* $\mathfrak{M}_\Gamma$. *In order to obtain values (modules and homomorphisms) of* T *one must apply the atomistical part* P (T) *of* T, $P(T) : \mathfrak{M}_\Lambda \to \mathfrak{M}_Z$, *then the functor* $R_{R_T} : \mathfrak{M}_Z \to \mathfrak{M}_Z$ *defined by formulas* (7) *and then pass to certain (well-determined) inverse images of the obtained values under the atomistical part* $P(R_T)$ of R_T, $P(R_T) : \mathfrak{M}_\Gamma \to \mathfrak{M}_Z$, *where* $R_T : \mathfrak{M}_\Gamma \to \mathfrak{M}_Z$, *is defined by formulas* (5).

Remark. – It is seen that T is, in a certain sense, a « transformed functor » of the functor R_{R_T}, which is a functor $\operatorname{Hom}_Z(L, \ )$ or $\operatorname{Hom}_Z(\ , L)$, by covariant functors which are atomized by canonical classes of homomorphisms respectively. Consequently, the variance of T is given by the variance of R_{R_T}.

9. If renouncing to the additional structure of atomizing class of homomorphisms, the Principal Theorem may be extended, as easily seen, to arbitrary (eventually non-additive) functors of modules and, moreover, to

arbitrary functors defined from and to arbitrary categories. Indeed, let $T : \mathcal{C} \rightarrow \mathcal{C}'$ be such a functor; taking instead of the atomistical part of T the functor $\mathrm{Hom}_{\mathcal{C}'}(T(K), T(C))$ or $\mathrm{Hom}_{\mathcal{C}'}(T(C), T(K))$ according as T is covariant or contravariant, where K is an arbitrarily chosen but fixed object of $\mathcal{C}$ and $C \in \mathcal{C}$, the proof may be repeated, *mutatis mutandis*, in this general case.

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[3] D. G. NORTHCOTT, *An Introduction to Homological Algebra*, Cambridge, University Press (1960).