Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Two Intersection Formulas in Algebraic Geometry

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **35** (1963), n.6, p. 435–442.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1963_8_35_6_435_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1963.

Geometria. — *Two Intersection Formulas in Algebraic Geometry.* Nota di ALEXANDRU T. LASCU, presentata (*) dal Socio B. SEGRE.

As this note is a direct continuation of a previous note [9], we shall preserve here all the notations of [9], without any change. Our purpose is to prove here two interesection formulas, related with the monoidal transformations (the theorems 1 and 2). The first one is due to B. Segre [5] and will be proved here for arbitrary characteristic. The second one, due to M. Noether for curves in the complex projective plane, has been extended by D. G. Northcott [2] in a general context, for one-dimensional local rings. His formula reduces, in algebraic geometry, to the intersection of two hypersurfaces on an arbitrary algebraic variety; here is given a similar formula for the intersection of an hypersurface with a subvariety of arbitrary dimension. I must here express my heartly thanks to Prof. Gh. Galbură who kindly drew my attention to Noether's formula and encouraged me during this work.

Throughout this note we deal with an (abstract) algebraic variety V, in the sense of A. Weil [7]. We shall adopt the terminology of [7]; Ω shall designate the "universal domain" and will be supposed of arbitrary characteristic. U is a subvariety of V such that codim $U > 1$ and $t : V' \to V$ a birational transformation of V; except in lemma 3, t is supposed to be a monoidal transformation of base U in the sense of [9].

$\mathfrak{O}(M, N)$ shall designate the (absolute) local ring of the subvariety M of N and $\mathfrak{m}(M, N)$ the maximal ideal of $\mathfrak{O}(M, N)$. We shall designate, as in [9], the generators of the $\mathfrak{m}(U, V)$ employed in the definition of t by $\varphi_0, \cdots, \varphi_r$. It results that $V' \subset V \times P^r$, where P^r is the projective r–dimensional space over Ω. If $p : X \to X$ is a birational transformation and U a subvariety of X, by $p[Z]$ and $p\{Z\}$ we shall designate respectively the transform and the total tranform of Z by p; if p is regular at any point Z, then $p[Z] = p\{Z\}$ and this variety will be designated by $p(Z)$.

§ 1. For the lemmas 1 and 2 below, we shall suppose that t satisfies the hypotheses (i) and (ii) of [9] and $\mathfrak{O}(U, V)$ is an U.F.D.

LEMMA 1. – *If the simple hypersurfaces of* V', V *correspond biregularly by* t, *except* U', *and* $\varphi \in \Omega(V)$, $(\varphi) = \sum_{i=1}^{h} a_i H_i$, *then* $(\varphi_0 t) = \sum_{i=1}^{h} a_i H'_i + \left(\sum_{i=1}^{h} a_i b_i\right) U'$, *where* $t(H'_i) = H_i$, $b_i = \dfrac{m(U, H_i)}{m(U, V)}$ $(i = 1, \cdots, h)$.

Proof. By hypothesis, $v_{H_i}(\varphi) = v_{H'_i}(\varphi_0 t)$. It remains only to show that $v_{U'}(\varphi_0 t) = \sum_{i=1}^{h} a_i b_i$, which is the content of theorem 3, [9].

(*) Nella seduta del 14 dicembre 1963.

LEMMA 2. – *Under the hypotheses of the above lemma, if* U *is simple on* V *and* $X = \sum_{i=1}^{h} a_i H_i$ *is a divisor of* V, *then the cycle* $(X \times P^r).\ V'$ *is defined on* $V \times P^r$ *and* $(X \times P^r).\ V' = X' + \left(\sum_{i=1}^{h} a_i\, m(U, H_i)\right) U'$, *where* $X' = \sum_{i=1}^{h} a_i H'_i$, $t(H'_i) = H_i$ $(i = 1, 2, \cdots, h)$.

Proof. By linearity, it suffices to consider the case $X = H_1 = H$. $H \times P^r$ is a simple hypersurface of $V \times P^r$, and V' is not included in $H \times P^r$ because $pr_v V' = V$ is not included in H. Hence the product $(H \times P^r).\ V'$ is defined on $V \times P^r$. By the projection formula ([7], VII, § 6, th. 16), $pr_v[(H \times P^r) \cdot V'] = H$. As H' is simple on $V \times P^r$ (because $pr_v H'$ is simple on V) and $H' \subseteq (H \times P^r) \cap V'$, it follows that $(H \times P^r).\ V' = H' + \alpha U'$. Since $U' \subseteq (H \times P^r) \cdot V'$ implies (projecting on V) $U \subset H$, it results $\alpha = 0$ if U is not included in H; as this implies $\mathfrak{m}(U, H) = 0$, the lemma is proved in this case. Suppose now $U \subset H$, and let $\varphi \in \mathfrak{O}(U, V)$ be a generator of the ideal of H in $\mathfrak{O}(U, V)$. Then H is the unique component of (φ) which includes U and $v_H(\varphi) = 1$. By theorem 3 ([9]), $v_{U'}(\varphi_0 t) = m(U, H)$. The cycle $(\varphi) \times P^r$ is the divisor of the function $\psi \in \Omega(V \times P^r)$ which induces on V' the function $\varphi_0 t$. By a known theorem ([7], VIII), every simple hypersurface H' of V' which is simple on $V \times P^r$ appears in $(\varphi_0 t)$ with the same coefficient as in $(\psi) \cdot V'$. But $U' = U \times W$, with a suitable subvariety W of P^r ([9]). Hence U' is simple on $V \times P^r$, because U is simple on V by hypothesis. Thus $\mathfrak{m}(U, V) = v_{U'}(\varphi_0 t)$ is the coefficient of U' in $(\psi) \cdot V'$; $(\varphi) = H + X$ with U not included in supp X implies U' not included in supp $(X \times P^r) \cdot V'$, which shows that $\mathfrak{m}(U, V)$ is the coefficient α of U' in $(H \times P^r) \cdot V'$.

Remark. – The two lemmas above show that, if the simple hypersurfaces of V', V correspond biregularly by t, except U', and if U is simple on V, then for every $\varphi \in \Omega(V)$ the cycle $((\varphi) \times P^r) \cdot V'$ is defined on $V \times P^r$ and

$$(\varphi_0 t) = ((\varphi) \times P^r) \cdot V' = \sum_{i=1}^{h} a_i H'_i + \left(\sum_{i=1}^{h} a_i\, m(U, H_i)\right) U',$$

where

$$(\varphi) = \sum_{i=1}^{h} a_i H_i \quad , \quad t(H'_i) = H_i \qquad (i = 1, 2, \cdots, h).$$

Suppose that V is an affine variety. Let $\xi_1, \cdots, \xi_s \in \mathfrak{O}(U, V)$ form a system of coordinate functions of V and let W be a subvariety of V such that $U \subset W$. Denote by $\mathfrak{k}$ a definition field of $\xi_1, \cdots, \xi_s$, U, W, V. Let $\mathfrak{q} \subset \mathfrak{O}(U, V)$ be a primary ideal for $\mathfrak{m}(U, V)$ and $\mathfrak{n} \subset \mathfrak{O}(U, V)$ the ideal of W. We shall suppose that $\mathfrak{n} \subset \mathfrak{q}$. Let us consider a system of generators $\varphi_0, \cdots, \varphi_r$ of $\mathfrak{q}$ defined over $\mathfrak{k}$ and $t: V' \to V$, $V' \subset V \times P^r$ be the monoidal transformation of center $\mathfrak{q}$ defined by means of $\varphi_0, \cdots, \varphi_r$ ([9]). By A we shall designate a set of polynomials $P \in k[X_1, \cdots, X_s, Y_0, \cdots, Y_r]$ which are homogenenous in the indeterminates $y_0, \cdots, y_r$ and such that $\{P(\xi, \varphi_0, \cdots$

$\cdots, \varphi_r)\}_{P\in A}$ should generate $\mathfrak{u}$ in $\mathfrak{O}(U, V)$. The zeros of $\{P\}_{P\in A}$ in $V \times P^r$ form an algebraic set $\overline{W}$ over $\mathfrak{k}$. In the above situation we have:

LEMMA 3. – 1) $t^{-1}[W] = W'$ *is a subvariety of* V' *and is the unique irreducible component of* $t^{-1}\{W\}$ *whose projection by* t *on* V *strictly includes* U; 2) W' *is also the unique irreducible component of* $\overline{W} \cap V'$ *projecting by* t *on a subvariety which strictly includes* U; 3) *If, for every* $Q \in \mathfrak{k}[X, Y]$ *homogeneous of degree* h *in* $Y_0, \cdots, Y_r$, $Q(\xi, \varphi_0, \cdots, \varphi_r) \in \mathfrak{u}$, $Q(\xi, \varphi_0, \cdots, \varphi_r) \notin \mathfrak{m}\mathfrak{q}^h$ *implies* $Q(\xi, \varphi) = \sum_{i=1}^{s} \alpha_i P_i(\xi, \varphi) Q_i(\xi, \varphi)$, *whith* $P_i \in A$ $(1 \leq i \leq s)$, $Q_i \in \mathfrak{k}[X, Y]$, $h =$ *degree of* $P_i Q_i$ *in* $Y_0, \cdots, Y_r$ *and* $\alpha_i \in O(U, V)$, *then* W' *is the unique component of* $\overline{W} \cap V'$ *projecting by* t *on a subvariety which includes* U. *The irreducible components of* $t^{-1}[U] \cap W'$, $t^{-1}[U] \cap \overline{W}$ *projecting on* U *by* t *are the same.*

Proof. The first part of the lemma is an obvious consequence of the fact that t is biregular at every subvariety of V strictly including U.

If $(x, y) \in V'$ is generic over k, $P \in \mathfrak{k}[X, Y]$, then

$$(*) \qquad \varphi_p^h(x)\, P(x, y_0, \cdots, y_r) = y_p^h\, P(x, \varphi_0(x), \cdots, \varphi_r(x)),$$

$h =$ degree of P in $Y_0, \cdots, Y_r$. Let $(x', y') \in W'$ be a generic point over $\mathfrak{k}$. As x' is a generic point of W over $\mathfrak{k}$, $\varphi_0, \cdots, \varphi_r$ are definite and finite at x' and $\varphi_p(x') \neq 0$, with suitable p; it follows $y'_p \neq 0$. If $P \in A$, then $P(x', \varphi(x')) = 0$ and (*) shows that $P(x', y') = 0$, hence $W' \subseteq \overline{W}$. Conversely, if Z is an irreducible component of $\overline{W} \cap V'$ with $U \subset t(Z)$ and $(x'', y'') \in Z$ generic over $\mathfrak{k}$, then $\varphi_0, \cdots, \varphi_r$ are definite and finite at x'' and there exists a p such that $\varphi_p(x'') \neq 0$ (since $U \neq t(Z)$). Hence $y''_p \neq 0$ and, similarly as above, one may see that $P(x'', \varphi(x'')) = 0$ for every $P \in A$, i.e., $x'' \in W$. Therefore $Z \subset t^{-1}\{W\}$, which shows that $Z = W'$ (because $U \subset t(z)$).

To conclude, it remains only to show that, under the hypotheses of 3), W' includes every component C of $V' \cap \overline{W}$ for which $t(C) = U$. Let $(x^0, y^0) \in C$ be a generic point over $\bar{\mathfrak{k}}$; t induces on W' a birational transformation $t_1 : W' \to W$, and it is easy to see that t_1 is a monoidal transformation of base $\mathfrak{q}' = \mathfrak{q}/\mathfrak{u}$. Then $t_1^{-1}(x^0)$ is the algebraic set (x^0, y), where y is a zero of $P(x^0, Y)$ for every $P \in \mathfrak{k}[x, y]$ homogeneous of degree h in $Y_0, \cdots, Y_r$ such that $P(\xi', \varphi') \in \mathfrak{m}' \mathfrak{q}'^h$, $\xi'_j = \xi_j$ mod $\mathfrak{u}$, $\varphi'_i = \varphi_i$ mod $\mathfrak{u}$ ([9], lemma 1). These conditions are equivalent with $P(\xi, \varphi) \in \mathfrak{m}\mathfrak{q}^h + \mathfrak{u}$; hence we may write $P(\xi, \varphi) = R(\xi, \varphi) + Q(\xi, \varphi)$, where $Q(X, Y)$, $R(X, Y) \in \mathfrak{k}[X, Y]$ are homogeneous polynomials of degree h in $Y_0, \cdots, Y_r$, and $R(\xi, Y) \in \mathfrak{m}\mathfrak{k} \cdot [\xi, Y]$, $Q[\xi, \varphi) \in \mathfrak{u}$. We may suppose that $Q(\xi, \varphi) \notin \mathfrak{m}\mathfrak{q}^h$; hence, by hypothesis, $Q(\xi, \varphi) = \sum_{i=1}^{s} \alpha_i P_i(\xi, \varphi) Q_i(\xi, \varphi)$, with suitable $P_i \in A$, such that $P_i Q_i$ is homogeneous of degree h in $Y_0, \cdots, Y_r$ and $\alpha_i \in O(U, V)$ $(1 \leq i \leq s)$. Because $R(\xi, Y) \in \mathfrak{m}\mathfrak{k}[\xi, Y]$ and $x^0 \in U$ it results $R(x^0, y^0) = 0$. $C \subset \overline{W}$ implies $P_i(x^0, y^0) = 0$ and $C \subset V'$ shows that $Q(x^0, y^0) = \sum_{i=1}^{s} \alpha_i(x^0)\, P_i(x^0, y^0)\, Q_i(x^0, y^0)$, because $Q(x, y)$, P_i, Q_i are homogeneous polynomials of the same degree in $Y_0, \cdots, Y_r$. Hence $Q(x^0, y^0) = 0$. From $P(\xi, \varphi) = R(\xi, \varphi) + Q(\xi, \varphi)$, we have $P(x, y) = R(x, y) + Q(x, y)$ for every generic point (x, y) of V' over $\mathfrak{k}$. By specialisation at (x^0, y^0), this shows $P(x^0, y^0) = 0$; hence $(x^0, y^0) \in t_1^{-1}(x^0)$. As $t_1^{-1}[U]$ are algebraic over $\mathfrak{k}$ and (x^0, y^0) is a generic point of C over $\mathfrak{k}$, it follows $C \subseteq t_1^{-1}[U]$, hence *a fortiori* $C \subseteq W'$.

PROPOSITION 1. – *Suppose that* U *is simple on* V, *t a monoidal transformation of base* U *satisfying the hypotheses* (*i*) *and* (*ii*) *of* [9], *by which the simple hypersurfaces of* V, V′ *correspond biregularly, and* $U' = U \times P^s$, $s =$ codim $U - 1$. *If* $\mathfrak{k}$ *is a definition field of* U, V U′, V′ *and* $x^0 \in U$ *generic over* $\mathfrak{k}$ *then,* $t^{-1}(x^0) = x^0 \times P^s$. *If* $\psi \in O_{\mathfrak{k}}(U, V)$ *and* $v_{U'}(\psi) = 1$, *then* $(\psi) = H + \sum_{i=1}^{h} a_i H_i$, *with* U not included in H_i $(i = 1, 2, \cdots, h)$, $(\psi_0 t) = U' + H' + \sum_{i=1}^{h} a_i H'_i$, *and*

$$(x^0 \times P^s) \cdot \left(H' + \sum_{i=1}^{h} a_i H'_i\right) = x^0 \times P^{s-1},$$

where P^{s-1} *is a suitable subspace of* P^s.

Proof. $t^{-1}(x^0) \subseteq U' = U \times P^s$, because x^0 is generic over $\mathfrak{k}$ for U; hence $t^{-1}(x^0) \subseteq x^0 \times P^s$. which, together with $x^0 \times P^s \subseteq U'$, shows that $t^{-1}(x^0) = x^0 \times P^s$.

By theorem 2 of [9], $v_{U'}(\psi) = 1$ implies $\psi \in \mathfrak{m} (= \mathfrak{m}(U, V))$, $\psi \notin \mathfrak{m}^2$, which shows that the ideal $\psi\mathfrak{O}$ is prime because $\mathfrak{O} = \mathfrak{O}(U, V)$ is an unique factorization domain. Hence there is an unique simple hypersurface H of V which appears in (ψ) and contains U. Applying the theorem 3 of [9] it follows that $v_H(\psi) = 1$ and $\mathfrak{m}(U, H) = 1$. Hence we may write $(\psi) = H + \sum_{i=1}^{h} a_i H'_i$, with U not icluded in H_i $(i = 1, \cdots, h)$. By lemma 1, this implies $(\psi \circ t) = U' + H' + \sum_{i=1}^{h} a_i H'_i$; $\psi \in \mathfrak{k}(V)$ shows that (ψ) is algebraic over $\mathfrak{k}$, which proves that $x^0 \notin H_i$, because U not included in H_i and x^0 is a generic point of U over $\mathfrak{k}$. Therefore $(x^0 \times P^s) \cap H'_i = \emptyset$, i.e., the intersection product $(x^0 \times P^s) \cdot \left(\sum_{i=1}^{h} a_i H'_i\right)$ is defined and null on V′; $x^0 \times P^s \subseteq H'$, because $x^0 \times P^s$ contains a generic point over $\mathfrak{k}(x^0)$ which is generic on U′ over $\mathfrak{k}$, H′ is algebraic over $\mathfrak{k}$ and $U' \subseteq H'$. It follows that the cycle $(x^0 \times P^s)$. H′ is defined on V′, and to conclude we have only to show that $(x^0 \times P^s)$. $H' = x \times P^{s-1}$ in view to explain this, we can apply lemma 3. Indeed, with the notations of this lemma, we may suppose that there exists a polynomial $P \in \mathfrak{k}[X, Y]$ homogeneous of degree one in $Y_0, \cdots, Y_r$ such that $\psi = P(\xi, \varphi) = \sum_{i=0}^{r} \alpha_i(\xi) \varphi_i$; as $\psi \notin \mathfrak{m}^2$, it follows that $\mathfrak{m}^h : \psi^l = \mathfrak{m}^{h-l}$ for every h, l with $h \geq l$, because $F(\mathfrak{m})$ is an integrity domain ($\mathfrak{O}$ being regular). From this it is easy to see that $A = \{P(X, Y)\}$ satisfies the hypotheses of lemma 3,3). Therefore the irreducible components of $U' \cap H'$ which correspond to U by t coincide with the irreducible components N_δ of U' which are zeros of $P(X, Y)$ corresponding to U by t. Every irreducible component of $(x^0 \times P^s) \cap H'$ is contained in a subvariety N_δ, because x^0 is generic on U over $\mathfrak{k}$, which shows that $(x^0 \times P^s) \cap H' \subseteq x^0 \times P^{s-1}$, where P^{s-1} is the subspace of zeros of $P(x^0, Y)$ in P^s. It follows $(x^0 \times P^s) \cdot H = \alpha (x^0 \times P^{s-1})$, provided that $x^0 \times P^{s-1}$ is simple on V'. Because $U' = U \times P^s$, $x^0 \times P^s$ is simple on U'; hence $\mathfrak{O}(x^0 \times P^{s-1}, U')$ is regular. To prove that $x^0 \times P^{s-1}$ is simple on V', it suffices to show that the ideal $\mathfrak{u}$ of U' in $\mathfrak{O}(x^0 \times P^{s-1}; V')$ is principal, because $\mathfrak{O}(x^0 \times P^{s-1}; U') = \mathfrak{O}(x^0 \times P^{s-1}; V')/\mathfrak{u}$. Let $\chi \in \mathfrak{O}_{\mathfrak{k}}(U, V)$ be another function with $v_{U'}(\chi \circ t) = 1$; hence $(\chi) = L + \sum_{j=1}^{l} b_j L_j$, $U \subseteq L$, U is not included in L_j $(j = 1, \cdots, l)$. If P'^{s-1} is the subspace defined for χ similarly as P^{s-1} is for ψ, $L \neq H$ and $P^{s-1} \neq P'^{s-1}$ (χ can be choosen from φ_i $(0 \leq i \leq r)$). Then $x^0 \times P^{s-1}$ not included in L' shows that U' is the unique component of $\chi \circ t$ containing $x^0 \times P^{s-1}$, because $v_{U'}(\chi \circ t) = 1$, $\chi \circ t$ generates $\mathfrak{u}$ in $\mathfrak{O}(x^0 \times P^{s-1}; V')$.

If $\rho = \psi \circ t / \chi \circ t$, then $(\rho) = H' + \Sigma a_i H'_i - L' - \Sigma b_j L'_j$, where $x^0 \times P^{s-1}$ is not included in H'_i, L', L'_j, $\rho \in \mathfrak{O}(x^0 \times P^{s-1}; V')$; hence ρ generates the ideal of H' in $\mathfrak{O}(x^0 \times P^s; V')$. By a known intersection formula ([4] chap. II, § 5, no 7, b, théorème de réduction), $i(x^0 \times P^{s-1}, (x^0 \times P^s) \cdot H') = \mathfrak{m}(\hat{\rho}\, \mathfrak{O}(x^0 \times P^{s-1}; x^0 \times P^s)$, where $\hat{\rho}$ is the image of ρ by the canonical homomorphism $\mathfrak{O}(x^0 \times P^{s-1}, V') \to \mathfrak{O}(x^0 \times P^{s-1}; x^0 \times P^s)$. But $\hat{\rho}$ is induced by the quotient of two linear forms of $\mathfrak{k}(x^0)[Y_0, \cdots, Y_r]$ having non-zero denominator on $x^0 \times P^{s-1}$. Hence $\mathfrak{m}(\hat{\rho}\, \mathfrak{O}(x^0 \times P^{s-1}; x^0 \times P^s)) = 1$.

THEOREM 1 (Segre's formula, [5]). – *Let* $t: V' \to V$ *a birational transformation which coincides in a neighbourhood of* U *with a monoidal transformation of base* U. *If* U *is simple on* V $\mathfrak{k}$ *a definition field of* U, V, V', t, *and* $x^0 \in U$ *a generic point over* $\mathfrak{k}$, *then* $t^{s-1}(x^0) = P^s$, $s = \operatorname{codim} U - 1$. *For any divisor* X *of* V' *linearly equivalent with* U' *on* V' *such that,* U' is not included in supp X, *the cycle* $P^s \cdot X$ *is defined on* V' *and*

$$P^s \cdot X = - P^{s-1} + (\psi),$$

where P^{s-1} *is a suitable subspace of* P^s *and* ψ *is a rational function on* P^s.

Proof. In view of the local character of the theorem, we may replace t by a monoidal transformation of base U.

If $t_1: V'_1 \to V$ is another monoidal transformation of base U defined over $\mathfrak{k}$, then, by the remark 4 following lemma 1 of [9], V'_1, V' correspond biregularly at any subvariety projecting on V over a subvariety which includes U. Therefore $\psi \in \Omega(V')$, $(\psi) = U' - X$, and $\psi_1 = \psi \circ t^{-1} \circ t_1$ implies $(\psi_1) = U'_1 - X_1$, where $U'_1 = t_1^{-1}[U]$, and the components of X, X_1 projecting on V over a subvariety which includes U correspond biregularly. It follows that $P^s \cdot X$ and $P^s_1 \cdot X_1$, where $P^s_1 = t_1^{-1}(x_0)$, correspond biregularly by $t^{-1} \circ t_1$. By the remark just quoted, the birational transformation $P^s_1 \to P^s$ induced by $t^{-1} \circ t_1$ is a projective transformation. Hence we may prove the theorem for t_1, i.e., one may change the generators $\varphi_0, \cdots, \varphi_r$ of $\mathfrak{m}_k(U, V)$ employed in the definition of t, when necessary. Taking $\varphi_0, \cdots, \varphi_r$ such that they form a regular system of parameters in $\mathfrak{O}(U, V)$, we may suppose, owing to their analytical independence ([8]; theorem 21, p. 292), that t satisfies the hypotheses (i), (ii). It is easy to see that the hypersurfaces H of V', such that $U \subset t(H)$, correspond biregularly by t to the hypersurfaces of V including U. Therefore one may additionally suppose that the hypersurfaces of V', V correspond biregularly by t, except U'. By the proposition 1, $t^{-1}(x^0) = x^0 \times P^s$ with $s = \operatorname{codim} U - 1$.

Every point $(x^0, y^0) \in x^0 \times P^s$ is simple on V'. Indeed, one may suppose $V' \subset V \times P^s$; thus, if $y_p^0 \neq 0$, it is easy to see that the ideal i of V' in $\mathfrak{O}((x^0, y^0), V \times P^s)$ is generated by $\varphi_p \eta_{jp} - \varphi_j$ $(0 \leq j \leq s, j \neq p)$, where $\eta_{jp} \in \Omega(P^s)$ are the functions induced by y_j/y_p. As (x^0, y^0) is evidently simple on $V \times P^s$, and $\mathfrak{O}((x^0, y^0), V') = \mathfrak{O}((x^0, y^0), V \times P^0)/i$, this shows that (x^0, y^0) is simple on V'. Since $x^0 \times P^s$ is non-singular, two linearly equivalent divisors of V' cut on $x^0 \times P^s$ linearly equivalent divisors ([7], VIII § 2, th. 4, cor. 1). Then, to prove the formula above, it suffices to find a divisor Y linearly equivalent with U' on V', such that $(x^0 \times P^s). Y = -x^0 \times P^{s-1}$. Indeed, if so, then $(x^0 \times P^s)$. X is linearly equivalent with $-x^0 \times P^{s-1}$ on $x^0 \times P^s$. To conclude, it remains only to prove the existence of such a divisor Y. But this is an obvious consequence of prop. 1.

COROLLARY. – *Under the hypotheses of theorem 1, if every point of* U *is simple both on* U *and* V *and the coefficients of the components* C_i *of* X *such that* $t(C_i) \supset U$ *are all positive or negative, then the cycle* U'. X *is defined on* V' *and* $U' \cdot X = -U'' + \Sigma a_i U'_i + (\rho)$, *where* U'' *is birationally equivalent with* $U \times P^{s-1}$ ρ *is a rational function on* U' *and, for every* i, U'_i *is birationally equivalent with* $U_i \times P^s$, U_i *being a hypersurface of* U.

Proof. As above, we may replace V by an affine variety and suppose that $U' = U \times P^s$ and U', V' are non-singular. By the theorem, $(x^0 \times P^s) \cdot X = -x^0 \times P^{s-1} + (\psi)$. Because U' is not included in supp X, the cycle U'. X is defined on V'. The hypothesis that U', V' are non-singular, allows us to apply a theorem of [7] (VII, § 6, th. 18) to the diagram of inclusions

$$\begin{array}{ccc} & U' \rightarrow & V' \\ & \uparrow & \uparrow \\ x^0 \times & P^s & X \end{array}$$

It follows that the intersection $(x^0 \times P^s)(U' \cdot X))_{U'}$ is defined and equal to $((x^0 \times P^s) X)_{V'} = -x^0 \times P^{s-1} + (\psi)$. Hence, in view of the hypothesis above concerning $t(X)$, $U' \cdot X = -U'' + \Sigma a_i U'_i + (\rho)$, where U'' is as above and $U = t(U'_i \subset U)$. Then $U'_i \subseteq U_i \times P^s$ shows that U_i is a hypersurface of U, and $U'_i = U_i \times P^s$.

LEMMA 4. – *Let* $f: A' \rightarrow A$ *be a birational transformation which is complete over a subvariety* U *of* A. *Let* U_α $(1 \leq \alpha \leq l)$ *be the maximal subvarieties of* A' *corresponding to* U *by* f *and suppose that* $\dim U_\alpha = \dim U$. *If* f *is regular at* U_α *for* $\alpha = 1, \cdots, l$, *then*

$$m(\mathrm{U}, \mathrm{A}) = \sum_{\alpha=1}^{l} [\mathrm{U}_\alpha : \mathrm{U}]\, e(\mathfrak{m}\,\mathfrak{O}_\alpha),$$

where

$$\mathfrak{O}_\alpha = \mathfrak{O}(\mathrm{U}_\alpha, \mathrm{A}') \quad , \quad \mathfrak{m} = \mathfrak{m}(\mathrm{U}, \mathrm{V}).$$

Proof. As f is regular at U_α and $f(\mathrm{U}_\alpha) = \mathrm{U}$, we have $\mathfrak{O} = \mathfrak{O}(\mathrm{U}, \mathrm{V}) \subseteq \mathfrak{O}_\alpha$ and $\mathfrak{O}_\alpha$ dominates $\mathfrak{O}$. By using the normalization $s: \bar{\mathrm{A}} \to \mathrm{A}$, one may easily see (applying Zariski's main theorem to the birational transformation $f^{-1} \circ s$) that $\mathfrak{O}_\alpha$ is an integral extension of $\mathfrak{O}$. Therefore, if $\{\zeta_{\alpha\beta}\}_{1 \leq \beta \leq i_\alpha}$ is a system of coordinate functions of an affine neighbourhood of U_α on A' then $\zeta_{\alpha\beta} \in \mathfrak{O}_\alpha$ shows that $\zeta_{\alpha\beta}$ is an integral element over $\mathfrak{O}$. Therefore $\mathrm{R} = \mathfrak{O}[\zeta_{11}, \cdots, \zeta_{1 i_1}, \cdots, \zeta_{l1}, \cdots, \zeta_{l i_l}]$ is a finite extension of $\mathfrak{O}$ in $\Omega(\mathrm{A})$; this shows that R is a semi-local ring and there exists a (non divisor of zero) $\rho \in \mathfrak{O}$ such that $\rho \mathrm{R} \subseteq \mathfrak{O}$. Then the hypothesis about f to be complete over U proves that the maximal ideals $\mathfrak{p}_i$ of R such that $\dim \mathrm{R}_{\mathfrak{p}_i} = \dim \mathrm{U}$ correspond one-to-one to the subvarieties U_α ($\alpha = 1, \cdots, l$). If $\mathfrak{p}_\alpha$ is the maximal ideal of R corresponding to U_α, then it is easy to see that $\mathfrak{O}_\alpha = \mathrm{R}_{\mathfrak{p}_\alpha}$ and $\mathfrak{p}_\alpha \mathrm{R}_{\mathfrak{p}_\alpha}$ is the maximal ideal $\mathfrak{m}_\alpha$ of $\mathfrak{O}_\alpha$. By the extension formula ([1] or [3]), we get

$$e(\mathfrak{m}) = \Sigma [\mathfrak{O}_\alpha/\mathfrak{m}_\alpha : \mathfrak{O}/\mathfrak{m}]\, e(\mathfrak{m}\,\mathfrak{O}_\alpha)$$

But $e(\mathfrak{m}) = m(\mathrm{U}, \mathrm{V})$, $\mathfrak{O}_\alpha/\mathfrak{m}_\alpha = \Omega(\mathrm{U}_\alpha)$, $\mathfrak{O}/\mathfrak{m} = \Omega(\mathrm{U})$.

THEOREM 2 (Noether's formula). – *Let* A,B *be simple subvarieties of* V, codim$_\mathrm{V}$ B$=1$, *and suppose that the cycle* $\mathrm{A} \cdot \mathrm{B}$ *is defined on* V *and* U *is a proper component of* $\mathrm{A} \cdot \mathrm{B}$. *Consider a monoidal transformation t of* V ($t: \mathrm{V}' \to \mathrm{V}$) *of base* U.

If A', B' *are the subvarieties of* V' *corresponding to* A , B *by t respectively* (*i.e.* $\mathrm{A}' = t^{-1}[\mathrm{A}]$, $\mathrm{B}' = t^{-1}[\mathrm{B}]$), *then the cycle* $\mathrm{A}' \cdot \mathrm{B}'$ *is defined on* V' *and*

$$i(\mathrm{A} \cdot \mathrm{B}, \mathrm{U}; \mathrm{V}) = ab + c$$

where $a = \mathfrak{m}(\mathrm{U}, \mathrm{A})$, $b = \mathfrak{m}(\mathrm{U}, \mathrm{B})$ *and c is the coefficient of* U *in* $t(\mathrm{A}' \cdot \mathrm{B}')$.

Proof. Because A' is not included in U' the cycle $\mathrm{A}' \cdot \mathrm{U}'$ is defined on V', and we may write $\mathrm{A}' \cdot \mathrm{U}' = \sum_{\alpha=1}^{h} c_\alpha \mathrm{U}'_\alpha + \mathrm{Y}$ with $t(\mathrm{U}'_\alpha) = \mathrm{U}$ ($1 \leq \alpha \leq h$), $t(\mathrm{Y}) = 0$. If φ is an irreducible element of the ideal $\mathfrak{u}$ of B in $\mathfrak{O} = \mathfrak{O}(\mathrm{U}, \mathrm{V})$ then $\varphi\mathfrak{O} = \mathfrak{u}$ which shows that B is the unique component of (φ) containing U. Owing to the local character of the theorem, one may suppose, as in the proof of theorem 1, that t satisfies (i), (ii), $\mathrm{V}' \subseteq \mathrm{V} \times \mathrm{P}^r$, $(\varphi) = \mathrm{B}$, and that the hypersurfaces of V , V' (except U') correspond biregularly by t. Then, as U is simple in V, by the remark following lemma 2, $(((\varphi) \times \mathrm{P}^r) \cdot \mathrm{V}')_{\mathrm{V} \times \mathrm{P}^r} = (\varphi \circ t) = \mathrm{B}' + b\,\mathrm{U}'$. Because the subvarieties of U' are simple both on V' and $\mathrm{V} \times \mathrm{P}^r$, we may apply theorem 18 of [7] (ch. VII, § 6) to the following diagram of inclusions:

$$\begin{array}{ccc} \mathrm{V}' & \longrightarrow & \mathrm{V} \times \mathrm{P}^r \\ \uparrow & & \uparrow \\ \mathrm{A}' & & (\varphi) \times \mathrm{P}^r \end{array}$$

and conclude that $i((\varphi \circ t) \cdot \mathrm{A}', \mathrm{U}'_\alpha; \mathrm{V}') = i(((\varphi) \times \mathrm{P}^r) \cdot \mathrm{A}', \mathrm{U}'_\alpha; \mathrm{V} \times \mathrm{P}^r)$ (the cycles $((\varphi) \times \mathrm{P}^r) \cdot \mathrm{A}')_{\mathrm{V} \times \mathrm{P}^r}$, $((\varphi \circ t) \cdot \mathrm{A}')_{\mathrm{V}'}$ being defined, because A' is not included in U′, B′, supp $((\varphi) \times P^r)$). By the projection formula ([7], th. 18, ch. VII, § 6):

$$\sum_{\alpha=1}^{h} [U'_\alpha : U]\, i\,(A' \cdot B', U'_\alpha ; V') + b \sum_{\alpha=1}^{h} [U'_\alpha : U]\, i\,(A' \cdot U', U'_\alpha ; V') =$$

$$= i\,(A \cdot B , U ; V).$$

As $c = \sum_{\alpha=1}^{h} [U'_\alpha : U]\, i\,(A' \cdot B', U'_\alpha ; V')$, it remains only to show that $a = \Sigma\, [U'_\alpha : U]\, i\,(A' \cdot U', U'_\alpha ; V')$. To do this, we shall apply lemma 4 to the birational transformation $f : A' \to A$ induced by t. Indeed, it is easy to see that the hypotheses of the lemma are satisfied. Hence, $a = \Sigma [U'_\alpha : U] \cdot e\,(\mathfrak{m}\, \mathfrak{O}_\alpha)$. Let φ_i be such that $\varphi_j \circ t / \varphi_i \circ t \in \mathfrak{O}\,(U'_\alpha , V')$; $\varphi_i \circ t \in \mathfrak{O}\,(U', V')$ and generates the maximal ideal of $\mathfrak{O}\,(U', V')$ by [9]. One may see that U′ is the unique component of $(\varphi_i \circ t)$ including U'_α. Indeed, because $\varphi_j \circ t / \varphi_i \circ t \in \mathfrak{O}\,(U'_\alpha , V')$ by lemma 2, this is equivalent to show that (φ_i) has no common components including U with (φ_j) for a suitable $j \neq i$. As t is independent of the generators $\varphi_0 , \cdots, \varphi_r$ by [9] (remark 4 of lemma 1), it suffices to take $\varphi_0 , \cdots, \varphi_r$ irreducible in $\mathfrak{O}\,(U, V)$ and such that φ_i is not divisible by φ_j for $i \neq j$ (for instance, one may assume $\varphi_0 , \cdots, \varphi_r$ to be a regular system of parameters).

Then $\varphi_i \circ t$ generates the prime ideal of U′ in $\mathfrak{O}\,(U'_\alpha , V')$. By a general theorem ([4], chap. II, § 5, no 7, b), $i\,(A' \cdot U', U'_\alpha ; V') = e\,(\varphi'_i\, \mathfrak{O}_\alpha)$, where φ'_i is the image of $\varphi_i \circ t$ by the canonical homomorphism $\mathfrak{O}\,(U'_\alpha , V') \xrightarrow{g} \mathfrak{O}_\alpha = \mathfrak{O}\,(U'_\alpha , A')$. If $g\,(\varphi_j \circ t / \varphi_i \circ t) = \zeta'_j$, $g\,(\varphi_j \circ t) = \varphi'_j$, then $\varphi_j = \varphi_i\,(\varphi_j / \varphi_i)$ implies $\varphi'_j = \varphi'_i\, \zeta'_j$, which shows that $\varphi'_i\, \mathfrak{O}_\alpha = \Sigma\, \varphi'_j\, \mathfrak{O}_\alpha = \mathfrak{m}\, \mathfrak{O}_\alpha$, because $\mathfrak{m} = \Sigma\, \varphi'_i\, \mathfrak{O}'$ where $\mathfrak{O}' = \mathfrak{O}\,(U , A)$. Therefore $i\,(A' \cdot U', U'_\alpha ; V') = e\,(\mathfrak{m}\, \mathfrak{O}_\alpha)$.

Remarks. 1) Because $f : A' \to A$ is a monoidal transformation of base U, we have $[U'_\alpha : U] = 1$; 2) One may apply the above theorem repeatedly, to get $i\,(A \cdot B , U ; V)$ as a sum of products of multiplicities occuring by successive monoidal transformations ([6]).

BIBLIOGRAPHY.

[1] M. NAGATA, *The theory of multiplicity in general local rings*, Proc. Int. Symposium on Algebraic-Number Theory, Tokyo (1955).

[2] D. G. NORTHCOTT, *Theory of one dimensional local rings*, « Proc. London Math. Soc. » (3) *8*, 388–415 (1958).

[3] M. SAKUMA, *On the Theory of Multiplicites in Finite Modules over Semi-Local Rings*, « Journal Sc. Hiroshima Univ. », Ser. A, vol. 23, No. 1, April 1959, pp. 1–17.

[4] P. SAMUEL, *Méthodes d'Algèbre Abstraite en Géométrie Algébrique* (Springer, 1955).

[5] B. SEGRE, *Nuovi metodi e risultati nella geometria sulle varietà algebriche*, « Annali di Mat. pura, ed appl. » (1953).

[6] F. SEVERI, *Trattato di Geometria Algebrica*, vol. I (N. Zanichelli, 1926).

[7] A. WEIL, *Foundations of Algebraic Geometry*, « Am. Math. Soc. », Coll. Pubblications, vol. XXIX, New York (1946).

[8] O. ZARISKI and P. SAMUEL, *Commutative Algebra*, vol. II, van Nostrand, Princeton N.J. (1960).

[9] A. LASCU, *The order of a rational function at a subvariety of an algebraic variety*, « Atti Acc. Naz. Lincei », fasc. 4, vol. XXXIV (1963).