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The order of a rational function at a subvariety of an algebraic variety

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Matematica. — *The order of a rational function at a subvariety of an algebraic variety.* Nota di ALEXANDRU T. LASCU, presentata (*) dal Socio B. SEGRE.

In this note we give three interpretations of a valuation arising in Algebraic Geometry.

The results obtained here will be used in some future applications concerning theory of intersections for Segre's dilations on the one hand and structure of birational transformations on the other hand.

Let V be an abstract algebraic variety in the sense of Weil [1] over an universal domain Ω of arbitrary characteristic, U a subvariety of V and $\mathfrak{O} = \mathfrak{O}(U, V)$ the local ring of U on V. If U is a simple hypersurface (codim $U = 1$) $\mathfrak{O}$ is a valuation ring. Then for every rational function φ of V there are:

a) an integer $v_U(\varphi)$ (the order of φ at U), and

b) a rational function of U, induced by φ on U, provided that $v_U(\varphi) \geq 0$.

In this note we attempt to extend the properties *a*) and *b*) for codim $U > 1$. Under a suitable hypothesis (the hypothesis (*i*) bellow), which is satisfied for the simple subvarieties, we succeed in doing this for *a*). The integer $v_U(\varphi)$ may be determined algebraically or geometrically.

Algebraically one proceeds as follows. Let $\mathfrak{m}$ be the maximal ideal of $\mathfrak{O}$ and $F(\mathfrak{m}) = \mathfrak{O}/_{\mathfrak{m}} \oplus \mathfrak{m}/_{\mathfrak{m}^2} \oplus \cdots$ the associated graded ring of $\mathfrak{m}$. One supposes (the hypothesis (*i*)) that $F(\mathfrak{m})$ is an integral domain. Then $v_U(\varphi) = f$ if $\varphi = \alpha/\beta$ with $\alpha, \beta \in \mathfrak{O}$, $\alpha \in \mathfrak{m}^a$, $\alpha \notin \mathfrak{m}^{a+1}$, $\beta \in \mathfrak{m}^b$, $\beta \notin \mathfrak{m}^{b+1}$, and $f = a - b$. One knows that v_U is a valuation [3]. It is evident that, if codim $U = 1$, v_U is the valuation associated to U.

Geometrically we may get $v_U(\varphi)$ by means of the monoidal transformations of Zariski [2] of V. Namely, one shows (theorem 1) that if $\rho : V \to V'$ is such a transformation of center U then ρ blows up U to a simple hypersurface U' of V'. Thus one may define $v_U(\varphi)$ by putting $v_U(\varphi) = v_{U'}(\varphi \circ \rho^{-1})$. This definition does not depend on the choice of ρ, for the local ring $\mathfrak{O}(U', V')$ of U' on V' is independent of ρ. If $v_U(\varphi) \geq 0$, $\varphi' = \varphi \circ \rho^{-1}$ induces a rational function ψ on U' which is the residue class of φ' modulo the maximal ideal $\mathfrak{m}(U', V')$ in $\mathfrak{O}(U', V')$. One may extend the property *b*) by saying that φ *induces the function* ψ *on* U'. Let $\Omega(V)$, $\Omega(V')$ be respectively the functions fields of V, V' and $t : \Omega(V) \to \Omega(V')$ the isomorphism defined by ρ (for $\alpha \in \Omega(V)$, $t(\alpha) = \alpha \circ \rho^{-1}$). Then $t(\mathfrak{O}) \subseteq \mathfrak{O}' = \mathfrak{O}(U', V')$, $t(\mathfrak{m}) \subseteq \mathfrak{m}' = \mathfrak{m}(U', V')$, $\mathfrak{O}'/\mathfrak{m}' = \Omega(U')$. Hence $\Omega(U) \approx t(\mathfrak{O})/_{t(\mathfrak{m})} \subseteq \mathfrak{O}'/\mathfrak{m}$ and one sees

(*) Nella seduta del 20 aprile 1963.

easily that φ *induces a rational function on* U *if and only if* $\varphi' \in \mathfrak{O}'$ *and* φ' *is congruent modulo* $\mathfrak{m}'$ *with an element of* $t(\mathfrak{O})$. In particular, this is the case if $\varphi \in \mathfrak{O}$ or if φ' induces a constant function on U'. If U is a simple hypersurface of V, then $t(\mathfrak{O}) = \mathfrak{O}'$ and the above definition reduces to the classical one.

The summary of this note is the following. In § 1 we give some results about monoidal transformations. We show that the transform $\rho[U]$ of U by a monoidal transformation ρ of center U is a bunch of hypersurfaces; $\rho[U]$ is a simple hypersurface, provided the hypothesis (*i*) is satisfied. In § 2, one shows that the geometrical definition of v_U agrees with the algebraical one. § 3 contains another characterization of the integer $v_U(\varphi)$ by means of the divisor of φ on V, under the additional hypothesis that $\mathfrak{O}$ is and U.F.D. (unique factorization domain).

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Notations. – We have adopted, with a few modifications, the terminology of Weil's Foundations [1].

If A is an algebraic variety and $H \subseteq \Omega$ a field of definition of A, H(A) denotes the functions fields of A over H; $\mathfrak{O}_H = \mathfrak{O}_H(U, V) = \mathfrak{O}(U, V) \cap H(V)$, $\mathfrak{m}_H$ the maximal ideal of $\mathfrak{O}_H$ and $F_H(\mathfrak{m}) = \mathfrak{O}_{H/\mathfrak{m}_H} \oplus \mathfrak{m}_H/\mathfrak{m}_H^2 \oplus \cdots$ is the associated graded ring of $\mathfrak{m}_H$. P represents the s–dimensional projective space over Ω.

A rational map $\tau : A \to B$ is said to be *regular* (biregular) at a point $x \in A$, if there is a point $y \in B$ which corresponds by τ to x and the canonical isomorphism $t : \Omega(B) \to \Omega(A)$ defined by τ, maps $\mathfrak{O}(y, B)$ into (onto) $\mathfrak{O}(x, A)$; τ is *complete* over a subvariety B' of B, if every place of $\Omega(A)$ extending the canonical homomorphism $t(\mathfrak{O}(B', B)) \to t(\mathfrak{O}(B', B))/t(\mathfrak{m}(B', B))$ has a center on A ([1], p. 185, [5], p. 113). One says that τ is a *morphism* if τ is everywhere regular on A; a morphism $\tau : A \to B$ which is complete over every point of B is called a *proper morphism*. If $\tau \subset A \times B$ is the graph of τ and A' a subvariety of A, *the total transform of* A' by τ, $\tau\{A'\}$ is defined by $\tau\{A'\} = pr_B[(A' \times B) \cap T]$; *the transform of* A' *ny* τ, $\tau[A']$, is the bunch of those irreducible components of A' which correspond to A' by τ ([2]). If φ is a rational function of the variety A, (φ) shall represent the divisor of φ on A.

§ 1. Consider a system $\{\varphi_0, \cdots, \varphi_s\}$ of elements of $\mathfrak{O}_k$ which generates a primary ideal $\mathfrak{q}$ for $\mathfrak{m}$ in $\mathfrak{O}_k$ and let φ_i' be the residue class of φ_i in $\mathfrak{q}/\mathfrak{m}_k\mathfrak{q}$ $(i = 0, \cdots, s)$. Suppose that k is a field of definition of U, V, $\varphi_0, \cdots, \varphi_s$ and $x \in V$, $u^0 \in U$ are generic points over k of V, U respectively. Let $V' \subset V \times P^S$ be the locus of $x \times (\varphi_0(x), \cdots, \varphi_s(x))$ over k and $\tau : V' \to V$ the birational transformation defined over k for which $\tau(x \times (\varphi_0(x), \cdots, \varphi_s(x)) = x$. Consider the graded ring $G_k(\mathfrak{q}) = \mathfrak{O}/\mathfrak{m}_k \oplus \mathfrak{q}/\mathfrak{m}_k\mathfrak{q} \oplus \cdots \oplus \mathfrak{q}^r/\mathfrak{m}_k\mathfrak{q}^r \oplus \cdots$. Set $k_0 = k(u^0)$ and let $u : k_0[Y_0, \cdots, Y_s] \to G_k(\mathfrak{q})$ be the homomorphism which extends the canonical isomorphism $k_0 \approx k(U)$ to $k_0[Y_0, \cdots, Y_s]$ in such a way that $u(Y_i) = \varphi_i'(i = 0, \cdots, s)$; put $\mathfrak{a} = \ker u$ and let $W(u^0)$ be the algebraic set of zeros of $\mathfrak{a}$ in P^S.

Lemma 1. *Suppose that*: *(ii)* $W(u^\circ) = W(x^\circ)$ *for any* $x^\circ \in U$ *generic over* k *and put* $W = W(u^\circ)$. *Then* τ *is a proper morphism and* $U \times W = \tau^{-1}[U]$ (*the* total transform of U).

Proof. – Since τ is a projection, τ is everywhere defined on V'. Because P^S is a complete variety and $V' \subset V \times P^S$, τ is complete over every point of V. To conclude, it only remains to show that

(1) $$U' \subset V' \quad \text{and} \quad \tau(U') = U \quad \text{implies} \quad U' \subset U \times W$$

(2) $$U \times W \subset V'.$$

Let K be an extension of k_0 which is a definition field both of U' and $U \times W$; let $(x, y) \in V'$, $(x^\circ, y^\circ) \in U'$ be generic points over K of V', U'. Thus, by hypothesis, x° is a generic point of U over k. Because W is a closed set of P^S in the k_0 — topology of Zariski, the inclusion $U' \subset V \times W$ is equivalent with $y_0 \in W$. The homogeneous ideal $\mathfrak{a}$ of W in $k_0[Y_0, \cdots, Y_s]$ is generated by the homogeneous polynomials $P(Y_0, \cdots, Y_s) = \sum_i \alpha_i(x_0) M_i^r(Y_0, \cdots, Y_s)$ where $M_i^r(Y_0, \cdots, Y_s)$ are monomials of degree r in $Y_0, \cdots, Y_s$, $\alpha_i \in \mathfrak{O}_k$ and $\sum_i \alpha_i M_i^r(\varphi_0, \cdots, \varphi_s) \in \mathfrak{m}_k \mathfrak{q}^r$. Therefore $\sum_i \alpha_i M_i^r(\varphi_0, \cdots, \varphi_s) = \sum_i \beta_i M_i^r(\varphi_0, \cdots, \varphi_s)$ for suitable $\beta_i \in \mathfrak{m}_k$. If $y_j^\circ \neq 0$, by deshomogeneizing at φ_j, we get

$$\sum_i \alpha_i M_i^r\left(\frac{\varphi_0}{\varphi_j}, \cdots, \frac{\varphi_s}{\varphi_j}\right) = \sum_i \beta_i M_i^r\left(\frac{\varphi_0}{\varphi_j}, \cdots, \frac{\varphi_s}{\varphi_j}\right)$$

with $\beta_i \in \mathfrak{m}_k$. This gives

$$\sum_i \alpha_i(x) M_i^r\left(\frac{y_0}{y_j}, \cdots, \frac{y_s}{y_j}\right) = \sum_i \beta_i(x) M_i^r\left(\frac{y_0}{y_j}, \cdots, \frac{y_s}{y_j}\right),$$

i.e.,

$$\sum_i \alpha_i(x) M_i^r(y_0, \cdots, y_s) = \sum_i \beta_i(x) M_i^r(y_0, \cdots, y_s).$$

By specialization at (x°, y°) it results $\sum_i \alpha_i(x^\circ) M_i^r(y_0^\circ, \cdots, y_s^\circ) = 0$, because $\beta_i(x^\circ) = 0$. Thus $y^\circ \in W(x^\circ) = W$ and 1) is proved. To prove 2) consider a polinomial $P \in k[X, Y]$ which is homogeneous of degree r as a polynomial in indeterminates $Y_0, \cdots, Y_s$ and such that $P(x, y) = 0$. This implies $P(x, \varphi(x)) = 0$, which shows that $P(u^\circ, Y_0, \cdots, Y_s) \in \mathfrak{a}$ and so $P(u^\circ, z_0, \cdots, z_s) = 0$ for every $z = (z_0, \cdots, z_s) \in W$. Taking the point z generically over K for an irreducible component W_1 of W, (x°, z) is a generic point of $U \times W_1$ over K. Then $P(u^\circ, z) = 0$ implies $P(x^\circ, z) = 0$ which shows that P is null over $U \times W_1$ and thus $U \times W_1 \subset V'$.

Remarks:

1° If $\{\varphi_0, \cdots, \varphi_s\}$ is a system of parameters for $\mathfrak{O}$ then, in view of the "analitically independence" ([3], theorem 21, p. 292) $\mathfrak{a} = 0$. Thus $W = P^s$, $s = \operatorname{codim} U - 1$.

2° If $\mathfrak{m} = \mathfrak{q}$ then $G_k(\mathfrak{q}) = F_k(\mathfrak{m})$.

3° Evidently V is independent of k. If V″ is the variety obtained in the same way as V′ from the system of generators of $\mathfrak{q}^h$ formed by the monomials $\mu_0, \cdots, \mu_p$ of degree h in $\varphi_0, \cdots, \varphi_s$ then V′ is isomorphic to V″. Indeed, if $F \subset P^S$ is the locus of $(\varphi_0(x), \cdots, \varphi_s(x))$ over k then V′ is the graph of the rational transformation $V \xrightarrow{\psi} F$, defined over k, which maps (x) onto $(\varphi_0(x), \cdots, \varphi_s(x))$. Similarly if $F_h \subset P^p$ designates the locus of $(\mu_0(x), \cdots, \mu_p(x))$ over k, V″ is the graph of $\psi_h : V \rightarrow F_h$, where $\psi_h(x) = (\mu_0(x), \cdots, \mu_p(x))$. But F, F_h are isomorphic over k by the birational map l_h for which $l_h(\varphi_0(x), \cdots, \varphi_s(x)) = (\mu_0(x), \cdots, \mu_p(x))$. Thus $\psi_h = \psi_0 l_h^{-1}$ and V′, V″ are isomorphic over k.

4° If $\psi_0, \cdots, \psi_s$ is another system of generators of $\mathfrak{q}$ and $\tau_1 : V'_1 \rightarrow V$ $(V'_1 \subset V \times P)$ the birational map defined by them, then the irreducible components of $\tau^{-1}[U]$, $\tau_1^{-1}[U]$ correspond one-to-one biregular by $\tau^{-1} \circ \tau_1$. This is a consequence of the formulas $\psi_i \equiv \sum_j \lambda_{ij} \varphi_j$ modulo $\mathfrak{m}\mathfrak{q}$, $\varphi_j \equiv \sum_i \mu_{ji} \psi_j$ modulo $\mathfrak{m}\mathfrak{q}$ with suitable $\lambda_{ij}, \mu_{ji} \in \Omega(U) (= \mathfrak{O}/\mathfrak{m})$.

DEFINITION. – *The birational map* τ *above is called a monoidal transformation of* V *of center* $\mathfrak{q}$. *If* $\mathfrak{q} = \mathfrak{m}$ *then* τ *is called a monoidal transformation of* V *of center* U.

LEMMA 2. – *Let* τ *be a monoidal transformation of* V *of center* $\mathfrak{q}$. *Every irreductible component of* $\tau^{-1}[U]$ *is a hypersurface of* V′.

Proof ([5], p. 105). – In view of the local character of this lemma, we may suppose that V is a subvariety of an affine space A^r and replace P^s by an affine space A^s of suitable coordinates say $z_1 = y_1/y_0, \cdots, z_s = y/y_0$ if $y'_0 \neq 0$ on V′. Then $V' \subset A^r \times A^s = A^{r+s}$ is an affine variety: if K is an extension of K_0, $x = (x_1, \cdots, x_r) \in V$ a generic point over K and $z_i = \varphi_i(x)/\varphi_0(x)$ $(i = 1, \cdots, s)$, then (x, z) is a generic point of V′ over K. Let U′ be an irreducible component of $\tau^{-1}[U]$ defined over K and $(x', z') \in U'$ a generic point over K. We may suppose that $\varphi_i(x_1, \cdots, x_s)$ is a polynomial in $x_1, \cdots, x_r$ $(i = 1, \cdots, s)$; $x' \in U$ implies $\varphi_i(x') = 0$ $(i = 1, \cdots, s)$. Thus U′ is a subset of the intersection of V′ by the hypersurface H_i of A^{r+s} of equation $\varphi_i(X_1, \cdots, X_r) = 0$. Since $y'_0 \neq 0$ on V′, it follows that $V \not\subseteq H_0$ and therefore every irreducible component U″ of $V' \cap H_0$ is a hypersurface of V′ (codim U″ = 1). Let U″ be such that $U' \subseteq U''$; let U″ be defined over K and (x'', t'') be a generic point of U″ over K. Then $(x'', z'') \xrightarrow{K} (x', z')$ and a fortiori $x'' \xrightarrow{K} x'$. But $z_i \varphi_0(x) = \varphi_i(x)$ implies $\varphi_i(x'') = 0$ which shows that $x'' \in U$. Therefore x'' is a generic point of U *i.e.* $\tau[U''] = U$. Thus the inclusion $U' \subseteq U''$ implies that $U' = U''$ *i.e* codim $U' = 1$. *q.e.d.*

THEOREM 1. – *Let* U *be a subvariety of* V *which satisfies the hypothesis* (*i*) (*i.e* F($\mathfrak{m}$) *is an integrity domain*) *and the hypothesis* (*ii*) (*of the lemma* 2). *Then every monoidal transformation of* V *of center* U, $\tau : V' \rightarrow V$, *is aproper morphism and* $\tau^{-1}[U]$ is a simple hypersurface of V′.

Proof. – By lemmas 1 and 2, $\tau^{-1}[U]$ is a hypersurface U′ of V′. To show that U′ is simple on V′ we shall use the notations from the proof of the lemma 2.

Let $\mathfrak{O}_1$ be the transform of $\mathfrak{O}$ by the canonical isomorphism, $t : \Omega(V) \to \Omega(V')$ associated to t. Let $\mathfrak{m}_1$ be the maximal ideal of $\mathfrak{O}_1$ and $\varphi'_i = t(\varphi_i)$ $(i = 0, \cdots, s)$. If $\mathfrak{O}' = \mathfrak{O}(U', V')$ then $\mathfrak{O}_1$ is dominated by $\mathfrak{O}'$, *i.e* $\mathfrak{O}_1 \subset \mathfrak{O}'$, $\mathfrak{m}_1 \subset m' = \mathfrak{m}(U', V')$, and because $\varphi'_i/\varphi'_0 = \zeta_i$ are functions of coordinate of V', in an affine neighborhood of U' on V', we have $\zeta \in \mathfrak{O}'$. To prove that $\mathfrak{O}'$ is regular, it is sufficient to show only that $\mathfrak{m}_1 \mathfrak{O}'$ is the maximal ideal $\mathfrak{m}'$ of $\mathfrak{O}'$. Indeed, since $\mathfrak{m}_1 = \sum_{i=0}^{s} \varphi'_i \mathfrak{O}_1$, $\mathfrak{m}_1 \mathfrak{O}' = \mathfrak{m}$ implies $\mathfrak{m}' = \Sigma \varphi'_i \mathfrak{O}'$ and then $\varphi'_i = \varphi'_0 \zeta_i$ $(i = 1, \cdots, s)$ shows that $\mathfrak{m}' = \varphi'_0 \mathfrak{O}'$. Suppose that W is defined over k and $F \in k[X_1, \cdots, X_r, Z_1, \cdots, Z_s]$ with $F(x', z') = 0$. If n is the degree of $F(x, Z) \in k(x)[Z]$ in $Z_1, \cdots, Z_s$, by multiplying it by $\varphi_0^n(x)$ and putting $\varphi_0(x) Z_i = Y_i$ we get $H(x, Y_0, \cdots, Y_s) = \varphi_0^n(x) F(x, Z_1, \cdots, Z_s)$ where $H(x, Y_0, \cdots, Y_s)$ is a homogeneous polynomial of degree n in $Y_0, \cdots, Y_s$. We have

$$H(x', 1, z'_1, \cdots, z'_s) = 0 .$$

Because (x', z') is a generic point of U', this shows that $H(x^0, Y) \in \mathfrak{a}$. Therefore $H(x, \varphi(x)) = \sum_i \alpha_i(x) M_i^{n+1}(\varphi_0(x), \cdots, \varphi_s(x))$ with $\alpha_i \in \mathfrak{O}$.

Thus $\varphi_0^n(x) F(x, \zeta(x)) = \Sigma \alpha_i(x) M_i^{n+1}(\varphi(x))$ which implies

$$F(x, \zeta(x)) = \Sigma \alpha_i M_{h_i}^n(\zeta(x)) \varphi_{l_i}(x), \qquad i.e.,$$

$F(x, \zeta(x)) \in \mathfrak{m}'_1 \mathfrak{O}'$. It follows that $\tau(\mathfrak{m}_k) \mathfrak{O}'_k = \mathfrak{m}'_k$ and consequently $\mathfrak{m}_1 \mathfrak{O} = \mathfrak{m}'$.

Remark. – Let $\mathfrak{L}$ be the linear system without basic divisors defined by the vector space L generated by $\varphi_0, \cdots, \varphi_s$ over Ω *i.e.* the family of divisors of V obtained from the divisors $(\varphi) + M$ where $\varphi \in L$ and $-M = \min(\varphi_i)$ after deleting the basic divisors. Let B be the set of basic points of $\mathfrak{L}$. V' is the graph of the rational map $V \xrightarrow{\psi} P^S$ associated to $\mathfrak{L}$. Therefore if $x \in V$ is a normal point the map $V \xrightarrow{\psi} P$ is regular at x when and only when $x \in B$ ([5]). It follows that if V is a normal variety and $B = U$, then τ is biregular in $V - U$.

§ 2. Let U be a subvariety of V which satisfies the hypotheses (*i*), (*ii*) and consider a monoidal transformation $\tau : V' \to V$ of center U; let $\tau^{-1}[U] = U'$ be the total transform of U by τ^{-1}. In view of remark 4 of lemma 1, V' is uniquely determined up to a birational mapping biregular over U'. We may therefore introduce the following

Definition. – *If* $t : \Omega(V) \to \Omega(V')$ *is the canonical isomorphism defined by* τ *and* $v_{U'}$, *the valuation defined by* U' *in* $\Omega(V')$, $v_U = v_{U'} \circ t$ *is called the valuation* v_U *associated to* U *on* V. According to the above definition, if $\varphi \in \Omega(V)$ we have

$$v_U(\varphi) = v_{U'}(t(\varphi)) \qquad (t(\varphi) = \varphi_0 t) .$$

We shall call the integer $v_U(\varphi)$ *the order of at* U.

THEOREM 2. – *Let* U *be a subvariety of* V *which satisfies the hypothesis* (*i*). *If* $\varphi = \mu/\nu$ *with* $\mu, \nu \in \mathfrak{O} = \mathfrak{O}(U, V)$ *and* $\mu \in \mathfrak{m}^a$, $\mu \notin \mathfrak{m}^{a+1}$, $\nu \in \mathfrak{m}^b$, $\nu \notin \mathfrak{m}^{b+1}$ *then* $v_U(\varphi) = a - b$.

Proof. – Let $\tau : V' \to V$ be a monoidal transformation of V of center U, defined over k. We shall use the notations of the proof of the lemma 2 and theorem 1. If $\varphi' = t(\varphi)$, $v_U(\varphi) = u$ implies $\varphi' = \varphi_0'^{\,u} \chi'$ where χ' is a unit of $\mathfrak{O}' = \mathfrak{O}(U', V')$. If $(y'_0, \cdots, y'_s) = (1, z'_1, \cdots, z'_s)$, $\chi'(x, y) = \dfrac{A(x, y)}{B(x, y)}$ with $A(x, Y)$, $B(x, Y) \in k[x][Y]$ and homogeneous of degree g in $Y_0, \cdots, Y_s$. Moreover $A(x', y') \neq 0$, $B(x', y') \neq 0$; thus if $\alpha, \beta \in k(V)$ are the functions for which $\alpha(x) = A(x, \varphi_0(x), \cdots, \varphi_s(x))$, $\beta(x) = B(x, \varphi_0(x), \cdots)$, then $\alpha, \beta \in \mathfrak{m}^g$, $\alpha, \beta \notin \mathfrak{m}^{g+1}$. But $\varphi_0 \notin \mathfrak{m}^2$ because $\varphi'_0 \notin \mathfrak{m}'^{\,2}$ as a regular parameter of $\mathfrak{O}'$. It follows $\varphi = \varphi_0^u \dfrac{\alpha}{\beta}$ with $\alpha\varphi_0^u \in \mathfrak{m}^{g+u}$, $\alpha\varphi_0^u \notin \mathfrak{m}^{g+u+1}$, $\beta \in \mathfrak{m}^g$, $\beta \notin \mathfrak{m}^{g+1}$, in view of the hypothesis (*i*) which shows that $u = a - b$.

§ 3. If A is a subvariety of the variety B, we shall designate by $m(A, B)$ the multiplicity of A on B.

THEOREM 3. – *Let* U *be a subvariety of* V *which satisfies the hypotheses* (*i*), (*ii*) *and for which the local ring* $\mathfrak{O}(U, V)$ *is an U.F.D. Let* $\varphi \in \Omega(V)$ and $(\varphi) = \sum_{i=1}^{p} a_i H_i$. *Then we have*

$$v_U(\varphi) = \sum_{i=1}^{p} a_i b_i ,$$

where $b_i = m(U, H_i)/m(U, V)$ ($b_i = 0$ if $V \not\subseteq H_i$),

First, we shall prove a lemma of local algebra. Let $\mathfrak{O}$ be a local ring which is an U.F.D., let $\mathfrak{m}$ be the maximal ideal of $\mathfrak{O}$ and suppose that $F(\mathfrak{m})$ is an U.F.D. Thus if K is the field of fractions of $\mathfrak{O}$, the function which associates to every element $x \in \mathfrak{O}$ the integer $r = r(x)$ defined by $x \in \mathfrak{m}^r$ $x \notin \mathfrak{m}^{r+1}$ (the order of x) is the restriction at $\mathfrak{O}$ of an uniquely determined valuation v of K ([3]). If $\mathfrak{p}$ is a prime ideal of $\mathfrak{O}$, $\mathfrak{O}_\mathfrak{p}$ represents the ring of fractions of $\mathfrak{O}$ with respect to $\mathfrak{p}$. If R is a local ring $m(R)$ is the multiplicity of R *i.e.* the multiplicity of his maximal ideal.

LEMMA 3. – *Let* $z \in K$, $z = z_1^{r_1}, \cdots, z_h^{r_h}$ *where* $z_1, \cdots, z_h \in \mathfrak{O}$ *are prime elements of* $\mathfrak{O}$ *and* $r_1, \cdots, r_h$ *are integers. If for* $i \neq j$ z_i *is relatively prime to* z_j $(i, j = 1, \cdots, h)$ *then*

$$v(z) = \sum_{i=1}^{h} m(z\mathfrak{O}_{\mathfrak{p}i})\, m(\mathfrak{O}/\mathfrak{p}_i)$$

where $\mathfrak{p}_i = z_i \mathfrak{O}$ *and* $m(z\mathfrak{O}_{\mathfrak{p}i}) = -m(z^{-1}\mathfrak{O}_{\mathfrak{p}i})$ *if* $r_i < 0$.

Proof. – Since $\mathfrak{O}_{\mathfrak{p}_i}$ is a one-dimensional regular local ring having z_i as a regular parameter and z_i is relatively prime to z_j for $j \neq i$, we have $r_i = m(z_i \mathfrak{O})$. As $v(z) = \sum_{i=1}^{p} r_i v(z_i)$ it is enough to show that $m(\mathfrak{O})\, v(z_i) = m(\mathfrak{O}/z_i \mathfrak{O})$. But $v(z_i) = s$ implies $z_i \in \mathfrak{m}^s$, $z_i \notin \mathfrak{m}^{s+1}$. Because $F(\mathfrak{m})$ is an U.F.D., $\mathfrak{m}^{s+t} : (z_i \mathfrak{O}) = m$. Hence $m(\mathfrak{O}/z_i \mathfrak{O}) = sm(\mathfrak{O})$ ([4]).

We can prove now the theorem 3. If in the above lemma we take $\mathfrak{O} = \mathfrak{O}(\mathrm{U}, \mathrm{V})$ and $\varphi = z$, it is easy to see that $p = h$ and, by suitable notations, $(z)_i = \mathrm{H}_i + \mathrm{X}_i$, where no component of X_i contains U. Thus $a_i = r_i\,(i = 1, \cdots, p)$. In view of the theorem 2, $v_{\mathrm{U}} = v$. By lemma 3, $v_{\mathrm{U}}(\varphi) = \Sigma m(\varphi\mathfrak{O}_{\mathfrak{p}i})\, m(\mathfrak{O}/_{\mathfrak{p}i})/_{m(\mathfrak{O})}$ But $r_i = m(\varphi\mathfrak{O}_{\mathfrak{p}i})$ and $\mathfrak{O}/_{\mathfrak{p}i} = \mathfrak{O}(\mathrm{U}, \mathrm{H}_i)$, *i.e.* $a_i = m(\varphi\mathfrak{O}_{\mathfrak{p}i})$ and $b_i = m(\mathfrak{O}/_{\mathfrak{p}i})/_{m(\mathfrak{O})} = m(\mathrm{U}, \mathrm{H}_i)/m(\mathrm{U}, \mathrm{V})$.

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