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Hideyuki Matsumura

On Algebraic Groups of Birational Transformations

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Geometria algebrica. — *On Algebraic Groups of Birational Transformations* (*). Nota di HIDEYUKI MATSUMURA, presentata (**) dal Socio B. SEGRE.

§ 1. We shall use the terminology of [3], [5], [7] freely. The characteristic of the universal domain is arbitrary.

Let V be a variety and let G be a connected algebraic group which operates on V. This implies that there exists a rational mapping $\lambda : G \times V \rightarrow V$ which, though not necessarily everywhere defined, satisfies for generic points the usual conditions of an action of a group on a set. Precisely: if K is a common field of definition for V, G and λ, and if x, y and P are independent generic points of G, G and V respectively, then we have $\lambda(x^{-1}, \lambda(x, P)) = P = \lambda(x, \lambda(x^{-1}, P)), \lambda(x, \lambda(y, P)) = \lambda(xy, P)$. Then it follows that $\lambda(a, P)$ is defined for any point a of G and for any generic point P of V over K (a), and the locus of $(P, \lambda(a, P))$ over K (a) is the graph of a birational transformation $\sigma(a) : V \rightarrow V$. The mapping $a \rightarrow \sigma(a)$ is a homomorphism of G into the group Bir (V) of all birational transformations of V onto itself. Bir (V) is the group of the automorphisms of the function field of V. When G operates on V faithfully (in the sense of [7]), then σ is injective. Moreover, if two connected algebraic groups G and G′ operates on V faithfully and have the same image in Bir (V), then they are necessarily isomorphic, so that we can speak of algebraic subgroups of Bir (V) without ambiguity.

If the operation of G on V is not faithful, there exist an algebraic group G′ operating faithfully on V and a rational homomorphism $\psi : G \rightarrow G'$ such that the operation of G on V is the composite of ψ and the operation of G′. In short, the image of σ is an algebraic subgroup of Bir (V). All these concepts, as well as the results which we are going to state in this paper, are birationally nvariant with respect to V.

According to the structure theorem of algebraic groups, there exists maximal connected linear subgroup H which contains all connected near subgroups of G, and the factor group G/H is an abelian variety ([2],]). We call H and G/H the linear part and the abelian part of G respectively.

§ 2. First we shall consider the linear part. Assume H operates on V non-trivially. Take a Borel subgroup (= maximal connected solvable sugroup) B of H. Since H is generated by its Borel subgroups and since all

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Borel subgroups are conjugate to each other, B operates on V non-trivially. B has a chain of connected subgroups

$$B = B_0 \supset B_1 \supset B_2 \supset \cdots \supset B_r = \{e\}$$

such that all B_i are normal in B and such that $\dim B_i = \dim B - i$. If α is the largest index i such that the action of B_i on V is not trivial, then let V_α be the variety of orbits of V with respect to B_α. By the cross-section theorem ([5]) we find that V is birationally equivalent to $L^1 \times V_\alpha$, where L^1 is the 1-dimensional projective space. The factor group B/B_α operates on V_α and we can repeat the same argument. Therefore:

TH. 1. – *Assume that a connected linear group* H *operates on* V *non-trivially, and let* B *be a Borel subgroup of* H. *Then* V *is birationally equivalent to* $L^s \times V_B$ *where* V_B *is the variety of* B*-orbits of* V *and* $0 < s \leq \dim B$.

COROLLARI 1. – *A necessary and sufficient condition that* Bir (V) *contains a linear algebraic group of positive dimension is that* V *is birationally equivalent to the product of* L^1 *and another variety. In particular, if a complete non-singular variety* V *has a multicanonical system* $|nK|$ *for some* $n > 0$, *then* Bir (V) *cannot contain any linear algebraic group of positive dimension.*

Proof. – The first half of the Cor. 1 is obvious. As for the second half, assume that V is birationally equivalent to $L^1 \times W$, and let W_0 be the open set of the simple points of W. Let us use the usual notation Ω^p to denote the sheaf of germs of regular differential forms of degree p, and let us denote by $(\Omega^p)^{\otimes n}$ the tensor product of n copies of Ω^p. Put $r = \dim V$. Then, by the " birational invariance of regular differential forms of the 1st kind ", we may consider $H^0(V, (\Omega^r)^{\otimes n})$ as a subspace of $H^0(L^1 \times W, (\Omega^r)^{\otimes n})$. But the latter is the tensor product

$$H^0(L^1, (\Omega^1)^{\otimes n}) \otimes H^0(W, (\Omega^{r-1})^{\otimes n})$$

by the Künneth formula (cf. [1]), and the first factor is zero since a projective space has no multicanonical systems. This contradicts to the hypothesis $l(nK) = \dim H^0(V \cdot (\Omega^r)^{\otimes n}) > 0$. Q.E.D.

The next corollary is rather classical: Enriques knew it at least for surfaces, and Washnitzer proved it in the general form about ten years ago

COROLLARI 2. – *If* V *is complete and non-singular, and if the rational mapping of* V *determined by* $|nK|$ *is birational for some* $n > 0$, *then* Bir (V) *is finite group.*

Proof. – Put $E = H^0(V, (\Omega^r)^{\otimes n})$, and denote by V′ the image of th rational mapping defined by $|nK|$. If $\dim E = q + 1$, V′ is imbedded i a projective space L^q and is not contained in any hyperplane of L^q. Th coordinate functions on V′ are the ratios of the elements of a base of . Every element σ of Bir (V) induces a linear transformation on E, hence a projective transformation σ' on V′. Since V and V′ are birationally relatd, σ is determined by σ'. Thus Bir (V) is isomorphic to the subgroup G of the projective transformation group of L^q consisting of those elements wh map V′ onto itself. G is clearly an algebraic linear group. By Cor. 1, the connected component G_0 must reduce to $\{e\}$. Thus G is a finite group. Q.E.D.

Remark. – The condition of birationality in Cor. 2 is satisfied if and only if

$$\lim_{n\to\infty}\cdot\sup l(nK)/n^r > 0.$$

COROLLARI 3 (Painlevé). – *Let* V *be a complete non-singular variety and let* G *be a connected algebraic group operating on* V. *Let* Ω *be any sheaf on* V *of the form* $\Omega^{p_1}\otimes\cdots\otimes\Omega^{p_m}$. *Then the elements of* $H^0(V, \Omega)$ *are* G*–invariant.*

(The original theorem of Painlevé [4] is the case of $\dim V = 2$, $\Omega = \Omega^1$. His proof was not quite rigorous).

Proof. – Let H be the linear part of G, let B be any Borel subgroup of H and let V_B be the variety of B–orbits of V. Since V_B is determined up to birational transformations, we may assume that V_B is non-singular (but not necessarily complete). Then, by what we have seen and by the Künneth formula, we obtain

$$H^0(V, \Omega) \subset H^0(L\times V_B, \Omega) = H^0(L, \mathfrak{O})\otimes H^0(V_B, \Omega) = H^0(V_B, \Omega),$$

hence the elements of $H^0(V, \Omega)$ are B–invariant. This being true for all B's, the linear part H induces identity on $H^0(V, \Omega)$. Now, it is easy to see that natural representation of G by $H^0(V, \Omega)$ is algebraic. H is in the kernel of this representation, and G/H, being an abelian variety, has no non-trivial linear representation. Therefore the whole group G acts trivially on $H^0(V, \Omega)$. Q.E.D.

Here we introduce a notion for developments in the future. Consider a function field $k(V)$, denoting by k the field of constants (= universal domain). Consider all connected linear algebraic groups H acting on V.

The subfield of $k(V)$ consisting of the functions which are invariant with respect to all H will be called the *linear kernel* of the function field $k(V)$ (or of V). Since each H is generated by its Borel subgroups, we may limit ourselves to the solvable H's to obtain the same definition. For a solvable H, we know by Th. 1 that $k(V)$ is a purely transcendental extension of $k(V_H)$, hence $k(V_H)$ is algebraically closed in $k(V)$. Then the linear kernel of $k(V)$ is also algebraically closed in $k(V)$, for it is the intersection of these $k(V_H)$'s.

The linear kernel is a characteristic subfield of $k(V)$ in the sense that it is mapped onto itself by any automorphism of $k(V)$ over k, and it contains another characteristic subfield of $k(V)$, the Albanese subfield (i.e. the function field of the image of the Albanese mapping $f: V\to A = \mathrm{Alb}(V)$).

§ 3. Now we turn to the abelian part. We consider a connected algebraic group G operating faithfully on a variety V. Then the following estimate immediately occurs in our mind:

Dim. – Of the abelian part of G $\leq$ irregularity of V (= dim Alb (V)).

Actually, the following Th. 2, which was recently found by Mieo Nishi, gives us a much stronger estimate. Nishi considered it for the case when V is complete and G operates on V regularly. Here we offer a simple proof which does not assume regularity.

Let $f: V \to A$ be the Albanese mapping, let V' be the closure of $f(V)$ in A and put

$$A' = \{a \in A : V'_a = V'\}.$$

Obviously A' is a closed subgroup of A. Since $A'_x \subset V'$ for $x \in V'$ (subscript implies translation), we have $\dim A' \leq \dim V'$, and the equality $\dim A' = \dim V'$ holds if and only if f is generically surjective.

Let $\lambda : G \times V \to V$ be the rational mapping which prescribes the action of G on V, and consider the mapping $f \circ \lambda : G \times V \to A$. By well-known properties of abelian varieties, there exist an endomorphism α of A, a homomorphism $\varphi : G \to A$ and a point $c \in A$ such that

$$f(\lambda(g, v)) = \varphi(g) + \alpha f(v) + c \qquad (g \in G \ , \ v \in V)$$

whenever both sides are defined. Putting $g = e$ we get $f(v) = \alpha f(v) + c$. Since V' generates A, we see easily that $\alpha = 1$, $c = 0$. Thus $f(\lambda(g, v)) = \varphi(g) + f(v)$. It follows that $\varphi(G) \subset A'$. Since the linear part H of G is in the kernel of φ, φ induces a homomorphism $\bar{\varphi} : \bar{G} \to A$, where $\bar{G} = G/H$.

TH. 2 (Nishi). – *Let* G *be a connected algebraic group operating faithfully on a variety* V. *Then, using the same notations as above,* $\bar{\varphi}$ *is an isogeny of the abelian part* $\bar{G}$ *of* G *with a subgroup of* A'.

First we prove a lemma.

LEMMA. – *Let* V *be a variety and let* G *be an algebraic group operating faithfully and regularly on* V. *Let* P *be a point of* V *and let* K_P *denote the stabiliser subgroup of* P *in* G:

$$K_P = \{g \in G : gP = P\}.$$

Then K_P *is linear.*

Proof. – Let $\mathfrak{O}$ be the local ring of V at P and let $\mathfrak{m}$ be its maximal ideal. The algebraic group K_P has a linear representation ρ_ν on the finite-dimensional vector space $\mathfrak{O}/\mathfrak{m}^\nu$, for each $\nu > 0$. It is easy to see that the ρ's are algebraic homomorphisms. Now the same argument as [5. Th. 13] shows that some ρ_ν is isomorphism (up to pure inseparability). Therefore K_P itself is linear ([5. Th. 12. Cor. 2]).

Remark. – It follows from this lemma that, if an abelian variety B operates regularly on a variety V, then all B–orbits have the same dimension.

Proof of Th. 2. – Assume the contrary. Then, replacing G by $(\operatorname{Ker} \varphi)_0$, we can assume that G has non-trivial abelian part and that $\varphi = 0$. First we consider the special case where G is an abelian variety. Replacing V by a suitable birational model we may assume that G operates on V regularly, though V may then be non-complete ([7]). Let k_0 be a common field of definition of V, G and λ, and let P be a generic point of V over k_0.

Let $\pi : V \to W$ be the natural rational mapping of V to the variety W of G–orbits of V, and put $Q = \pi(P)$. Then the subvariety GP of V is defined over $k_0(Q)$, and P is a generic point of GP over $k_0(Q)$. Take a point P_1 of GP which is algebraic over $k_0(Q)$, and let $P_1, P_2, \cdots, P_n$ be the different conjugates of P_1 over $k_0(Q)$. It easy to see that the morphism $G \rightarrow GP$ which sends $g \in G$ to gP is bijective. In fact, the stabiliser K_P of P in G is a finite group by the lemma. By specialization we see that the elements of K_P leave all points of V fixed. Since G operates on V faithfully, K_P must be equal to $\{e\}$. Therefore there exists one and only one element $x_i \in G$ such that $P = x_i P_i$, for each $i = 1, 2, \cdots, n$. If σ is an automorphism of the algebraic closure of $k_0(P)$ over $k_0(P)$, then we have $P = x_i^\sigma P_i^\sigma$. But $\{P_1^\sigma \cdots P_m^\sigma\}$ is a permutation of $\{P_1, \cdots, P_n\}$, so that the o–cycle $\Sigma(x_i)$ on G is σ–invariant. Therefore the o–cycle $p^e \Sigma(x_i)$ is rational over $k_0(P)$ for some positive integer e (p = the field characteristic). The sum $p^e \Sigma x_i$ is, then, a rational point of G over $k_0(P)$. Thus there is a rational mapping F from V into G, defined over k_0, such that

$$F(P) = p^e \sum_1^n x_i .$$

If x is a generic point of G over $k_0(P)$, then xP is a generic point of GP over $k_0(Q)$, hence over $k_0(Q, P_1, \cdots, P_n)$. It follows immediately that

$$F(xP) = F(P) + p^e nx.$$

Therefore $F(xP) \neq F(P)$. Since F goes through the Albanese variety A, we must have $f(xP) \neq f(P)$, contradiction.

Now we consider the general case. Let V_H be the variety of orbits on V with respect to the linear part H of G. The abelian part $\bar{G} = G/H$ operates on V_H, inducing an algebraic subgroup G' of Bir (V_H). Since G' operates on Alb (V_H) trivially, the abelian variety G' must reduce to $\{e\}$ by what we have just seen. In other words, G operates on V_H trivially.

Let P denote again a generic point of V over k_0, and let S denote the closure of the orbit HP in V. Since G operates trivially on V_H we see that G leaves S fixed. Thus G operates on S. But then, for a generic point g of G over $k_0(P)$, gP is a generic point of S. Hence $gP \in HP$. This implies $g \in HK_P$, where K_P is the stabiliser of P in G. Thus we have $G = HK_P$. Since G is connected, G is equal to $H(K_P)_0$, which is equal to H by the Lemma. Thus G is linear, contradicting to the hypothesis. Q.E.D.

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