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Secants and transversals

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Matematica. — *Secants and transversals.* Nota di MICHAEL EDELSTEIN, presentata (*) dal Socio B. SEGRE.

1. INTRODUCTION. – A hyperplane π in E^n, will be called a secant of a subset $A \subset E^n$ if $\pi \cap A \neq \emptyset$. If A is connected and both components of $E^n - \pi$ meet A then we shall say that π is a transversal of A. Every transversal is clearly a secant.

If $\mathfrak{A}$ is a family of subsets of E^n, then by a k–secant (k–transversal) of $\mathfrak{A}$ we mean a hyperplane which is a secant (transversal) of exactly k members of $\mathfrak{A}$. The fact that a hyperplane is a k–transversal implies, then, that it is also a l–secant with $l \geq k$.

If $\mathfrak{A}$ is a finite nonempty family of compact subsets of E^n then a 1–secant always exists. If, in addition, all members of $\mathfrak{A}$ are pairwise disjoint, $\cup \mathfrak{A}$ is infinite and does not lie on a single straight line then a 2–secant exists [1].

It is the main purpose of the present paper to prove the following theorems.

THEOREM I. – *Let $\mathfrak{A} = \{A_1, A_2, \cdots, A_m\}$, be a finite family of compact, connected and mutually disjoint subsets of E^n and suppose that there exists an l–transversal ($l \geq 1$), of $\mathfrak{A}$. Then for every positive integer k with $k \leq l$ there exists a hyperplane π which is a k–secant of $\mathfrak{A}$. Moreover k–secants exist which are ($k - 1$)–transversals.*

THEOREM II. – *Let $\mathfrak{A}$ be as in Theorem I and suppose that no ($k + 1$)–secants exist. Then every k–transversal is also a k–secant.*

2. PRELIMINARIES.

2.1. Let P^n denote the projective n–space obtained by closing the E^n with the "infinite hyperplane." Let S be a fixed ($n - 1$)–sphere in E^n. We shall denote by $p(\pi)$ the point $p \in P^n$ which corresponds to the hyperplane π by polarity with respect to S. As usual we call p the pole of π; π the polar of p.

2.2. Consider the set of poles of all transversals of a given connected set $A \subset E^n$; if a_1 and a_2 are any two points of A strictly separated by some hyperplane π then these points will also be separated by hyperplanes whose parameters are sufficiently close to those of π. This clearly implies that the above set of poles is open (in the topology of P^n). Similarly, it can be seen that the set of poles of all hyperplanes disjoint from a given compact subset of E^n is also open.

(*) Nella seduta del 12 maggio 1962.

2.3. We now define a set Q^k as follows:

$$Q_i = \{ p(\pi) \mid \pi \cap A_i \neq \emptyset \}, \qquad (i = 1, 2, \cdots, m);$$

$$Q_{i_1 i_2, \cdots, i_k} = Q_{i_1} \cap Q_{i_2} \cap \cdots \cap Q_{i_k}, \quad (i_1, i_2, \cdots, i_k = 1, 2, \cdots, m);$$

$$Q^k = \bigcup_{1 \leq i_1 < i_2 < \cdots < i_k} Q_{i_1 i_2, \cdots, i_k}.$$

Q^k is a finite union of intersections of closed sets (2.2); hence closed. Let B denote its boundary. As $B \subset Q^k$, any hyperplane π with $p(\pi) \in B$ meets k members of $\mathfrak{A}$ at least. If π be an h–transversal then clearly $h \leq k - 1$ (otherwise $p(\pi) \in \operatorname{int} Q^k$). We thus arrive at the conclusion that polars of points of B which do not satisfy the requirements of the theorem have the property of meeting two sets at least without being a transversal of any of them. Such hyperplanes are then supports of at least two members of $\mathfrak{A}$. This property and the following two lemmas will be used in the proofs of our theorems.

3. LEMMA 1. – *If* K_1 *and* K_2 *are disjoint compact and convex subsets of* E^n, *then the set* M *of poles of all common supporting hyperplanes of both* K_1 *and* K_2 *is of dimension (in the sense of Menger-Urysohn) not exceeding* $n - 2$.

Proof. – We remark that the dimension of the set of poles of all supporting hyperplanes of any subset of E^h, does not exceed $h - 1$. This follows from [3, p. 46, Corollary 1], as such a set of poles is nowhere dense in P^h (any small parallel displacement of a supporting hyperplane yields a nonsupporting one).

Let now x be an arbitrary point of M and $\pi = p^{-1}(x)$ its polar. Either (a) π contains at least one of the K_1 $(i = 1, 2)$; or (b) it does not. In case (b) either (i) one of the two components of $E^n - \pi$ is free of $K_1 \cup K_2$; or (ii) no one is.

We denote by M_1 and M_2 the subsets of M with polars satisfying (a) or (i) and (a) or (ii) respectively. Both M_1 and M_2 are closed, and therefore compact, subsets of P^n. [Indeed if $x \in P^n - M_i$, $(i = 1, 2)$, then $p^{-1}(x)$ is not a supporting hyperplane of at least one of the K_i; but then all points of P^n sufficiently close to x have the same property.] As $M = M_1 \cup M_2$ it suffices, in order to prove the Lemma, to show that $\dim M_i \leq n - 2$.

To prove that $\dim M_1 \leq n - 2$ we consider the intersection H_1 of all closed half spaces determined by $p^{-1}(x)$, $x \in M_1$, which contain $K_1 \cup K_2$. Let ρ be any hyperplane strictly separating K_1 from K_2. The set $D_1 = \rho \cap H_1$ is convex and nonempty (any segment joining $y \in K_1$ with $z \in K_2$ meets D_1).

Any supporting hyperplane of H_1 intersects ρ along a $(n - 2)$–dimensional linear variety which is itself a supporting hyperplane, relative to ρ, of D_1. The separation property of ρ implies that intersections are different for different supporting hyperplanes of H_1. [Indeed should a pair $\pi_1 \neq \pi_2$, $p^{-1}(\pi_1)$, $p^{-1}(\pi_2) \in M_1$ exist such that $\pi_1 \cap \rho = \pi_2 \cap \rho$, then obviously $\pi_1 \cap \pi_2 \subset \rho$, and no point of $K_1 \cup K_2$ can lie on $\pi_1 \cap \pi_2$. Let $w_i \in K_i \cap \pi_1$, $z_i \in K_i \cap \pi_2$,

$(i = 1, 2)$, then w_1, w_2, z_1, z_2 are all distinct. Now the segment $[y_1, y_2]$ joining the middle points of the segments $[w_1, z_1]$ and $[w_2, z_2]$ crosses π_1 (and π_2) which is incompatible with the definition of M_1]. Thus a biunique correspondence c is set up between M_1 and the supporting hyperplanes of D_1 (in ρ). As M_1 is compact and the intersection $p^{-1}(x) \cap \rho$ depends continuously on x, c is seen to be a topological mapping of M_1 on a subset of P^{n-1} which is of dimension $\leq n-2$, by the remark at the beginning of this proof. Hence $\dim M_1 \leq n-2$.

In an analogous manner (here ρ' will be chosen parallel to ρ and so that $K_1 \cup K_2$ shall lie in one component of $E^n - \rho'$) it can be verified that $\dim M_2 \leq n-2$. Hence $\dim M \leq n-2$ as asserted.

LEMMA 2. – *If* K_1 *and* K_2 *are disjoint compact subsets of* E^n *then the dimension of the set* M *of the poles of all common supporting hyperplanes of both* K_1 *and* K_2 *does not exceed* $n-2$.

Proof. – Consider a cover C of $K_1 \cup K_2$ by open balls (i.e. interiors of $(n-1)$–spheres), centred at points of this set and of radius $r < d/2$, being the (shortest) distance between K_1 and K_2. C contains a finite subcover $C_1 \cup C_2$, $C_1 = \{S_1^1, S_2^1, \cdots, S_{k_1}^1\}$, $C_2 = \{S_1^2, S_2^2, \cdots, S_{k_2}^2\}$, such that $S^1 = \bigcup_1^{k_1} S_i^1 \supset K_1$, $S_2 = \bigcup_1^{k_2} S_i^2 \supset k_2$ and $S^1 \cap S^2 = \emptyset$. Any common supporting hyperplane π of both K_1 and K_2 is also a common supporting hyperplane of at least one pair of sets consisting of $S_i^1 \cap K_1$ and $S_j^2 \cap K_2$ for suitable i and j. Clearly π is also supporting the closures of the convex hulls of these two sets. The previous lemma applies and M is contained in a finite union of closed sets of dimension $\leq n-2$ each. Hence [3, p. 30] $\dim M \leq n-2$ as asserted.

4. PROOFS OF THEOREMS I AND II.

4.1. *Proof of Theorem I.* – The existence of an l–transversal with $l \geq k$ implies that $\operatorname{int} Q^k \neq \emptyset$. On the other hand $P^n - Q^k \neq \emptyset$ as, obviously, hyperplanes exist which are disjoint from $\cup A_i$. Thus B is disconnecting P^n. By a known theorem $\dim B \geq n-1$. Suppose, however, that contrary to the statement of the theorem no k–secant exists. Hence, as remarked in 2.3, all hyperplanes of p^{-1} (B) are supporting hyperplanes of two members of $\mathcal{A}$ at least and B is contained in the finite union of closed sets (of poles corresponding to common supporting hyperplanes of pairs of members of $\mathcal{A}$) each of dimension $\leq n-2$. Hence $\dim B \leq n-2$. This contradiction shows that k–secants do exist. Moreover as the above argument shows that p^{-1} (B) contains polars which support exactly one member of $\mathcal{A}$, k–secants exist which are $(k-1)$–transversals.

This concludes the proof of Theorem I.

4.2. *Proof of Theorem II.* – Let π be a k–transversal and suppose it is a l–secant with $l \neq k$. By hypothesis $l > k+1$. We may assume $A_1, A_2, \cdots A_k$ are the sets transversed by π and that $A_{k+1}, \cdots, A_l$ are those intersected but not transversed. Let $a \in \bigcup_{k+1}^{l} A_i$ and consider all hyperplanes through a which are common transversals of $A_1, \cdots, A_k$. We need only consider the case in which none of these hyperplanes is a k'–transversal with $k' > k$ as otherwise the assertion follows directly from the previous theorem. Suppose now that among these hyperplanes there is no k–secant and that, therefore, they are all supporting hyperplanes of at least two sets each. Their poles form, by Lemma 2, a set of dimension $\leq n - 2$. This however is impossible. Indeed as an intersection of an open set (of poles of k–transversals) and a hyperplane (locus of poles of all hyperplanes through a) it is $(n - 1)$–dimensional. This contradiction shows that k–transversals are here k–secants as asserted.

4.3. REMARK. – Theorems I and II, specialized to $n = 2$, imply the following result due to Grünbaum [2].

PROPOSITION. – *Let $\mathcal{A}$ be a family of (at least two) compact connected and mutually disjoint subsets of E^2 at least one of which is nondegenerate. If $\mathcal{A}$ has the property that each straight line that intersects a pair of members of $\mathcal{A}$ also intersects some other member of $\mathcal{A}$, then $\bigcup \mathcal{A}$ lies on a single straight line.*

Proof. – By hypothesis there are no 2–secants. Hence no l–transversal with $l > 1$ can, by Theorem I, exist. However, should $\bigcup \mathcal{A}$, contrary to the statement of Grünbaum's result, not lie on a straight line then a transversal could be found which is not a l–secant. Indeed assume A_1 is nondegenerate and let $a \in \bigcup \mathcal{A} - A_1$. If there is no straight line containing $\{a\} \cup A_1$ there must exist a line through $a_1^1, a_2^1 \in A_1$, $a_1^1 \neq a_2^1$, that does not contain a. But then the straight line through a and the midpoint of the segment $[a_1^1, a_2^1]$ will clearly be a transversal of A_1, that intersects some $A_j \neq A_1$. Since a is arbitrary the whole of $\bigcup \mathcal{A}$ must lie on the same line.

5. In the present section we use a notion of maximality for transversals to give a necessary and sufficient condition for a k–transversal to be a k–secant.

5.1. DEFINITION. – A hyperplane π is a maximal k-transversal if all hyperplanes sufficiently close to π are k'–transversals with $k' \leq k$. [Equivalently: There exists a neighbourhood $U \subset P^n$ of $p(\pi)$ such that all $\pi' \in p^{-1}(U)$ are k–transversals with $k' \leq k$].

5.2. PROPOSITION I. – *Let $\mathcal{A}$ be as in Theorem I and suppose that no member of $\mathcal{A}$ is a degenerate point set, then a necessary and sufficient condition or a k–transversal to be a k–secant is that it be a maximal k–transversal.*

Proof. Necessity. – If a hyperplane π is both a k–transversal and a k–secant then it intersects exactly k members of $\mathcal{A}$ and is a transversal of each of them. The same is obviously true for all sufficiently close hyperplanes.

Sufficiency. – Suppose the contrary is true. Without restricting generality we may assume that π is a transversal of $A_1, A_2, \cdots, A_k$ and a secant of A_{k+1}. Now $\operatorname{int} Q_{12,\ldots,k} \neq \emptyset$ and, since no set reduces to a single point,

int $Q_{k+1} \neq \emptyset$. But $p(\pi) \in Q_{k+1} \cap$ int $Q_{12,\ldots,k} \neq \emptyset$. Hence int $Q_{k+1} \cap$ int $Q_{12,\ldots,k} \neq \emptyset$ and any point of this intersection has a polar which is a $(k+1)$–transversal contrary to the assumption on π. This contradiction proves sufficiency and thus concludes the proof of the Proposition.

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