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Christoph Lhotka

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Sitnikov's Planet

CHRISTOPH LHOTKA

"I do not know what I may appear to the world, but to myself I seem to have been only like a boy playing on the seashore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me." (Sir Isaac Newton)

Prolog

If evolution of mankind would have taken place on another Earth that was oscillating along a straight line through the barycenter of two massive stars (see Figure 1), would such a brilliant mathematician like Sir Isaac Newton (1643 - 1727) have discovered the same law of gravity? Would a person like Johannes Kepler (1571-1630) have formulated his three fundamental laws of planetary motion? Imagine, some religious beliefs would have supported the quite self-involved fact that

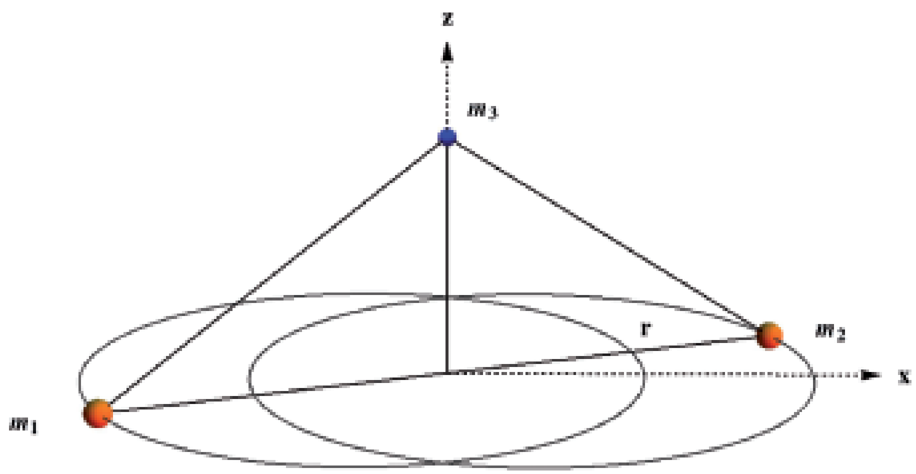

Figure. 1: Gravitational attraction of three masses (m_1, m_2, m_3) in special configuration.

the Earth is truly the center of the world. Which kind of observations would some brilliant man or woman need to make to be able to disprove this wrong statement? Would it have been easier for him, or her, than it was for Tycho Brahe (1546-1601)? Would a person like Albert Einstein have formulated the general theory of relativity in a different way? Let us follow this *gedankenexperiment* ([1]) for some moment, and let us assume that the natural environment of this planet supported the evolution of life, and that descent with modification allowed to create specimen that are capable of doing mathematics like we do. Let us imagine to be one of them:

We stand, quite similar to the observer Tycho Brahe, on the hemisphere of this planet that is found in daylight (daylight astronomy has shown to be much more interesting). Having performed enough observations we are able to draw the following conclusions: i) the stars appear in the morning (at the end of dawn) on the horizon in opposite directions, and disappear in the evening in not necessarily the same, but again opposite directions (at the beginning of dusk). Let us denote by T one complete cycle from dawn to dawn (or dusk to dusk). We further observe: ii) in addition to the vertical motion of period T the stars perform periodic motion of period P and parallel to the horizon. Let us assume $T/P >> 1$ so that we are able to observe that the horizontal motion projects paths into the sky that resemble ellipses of the same shape but mirrored to each other. We parametrize their shape by eccentricity e and semi-major axes a. Very probably, but not necessarily, we are able to formulate a variant of Kepler's first law in terms of our observations:

I The orbit of every *star* is an ellipse with the projection of the position of *our planet* to the orbital plane of the stars at the common focus.

We remark that the formulation of law I already required to exchange the role of the stars with that of the planets. It may be even more sophisticated to formulate the variant of Kepler's second law:

([1]) i.e., thought experiment.

II A line segment joining a *star* and *the projection of our planet on the plane of stellar orbits* sweeps out equal areas during equal intervals of time.

The statement is true if we already understood that we move relative to the orbital plane of the stars. Otherwise these areas would also change because their projection to the sky is coupled with our own movement. The formulation of Kepler law III becomes even more difficult. We remind ourselves of its standard, usual formulation:

III The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.

Historically, a careful work comparing the orbital motions of many planets was necessary to claim the third Kepler law in 1619. Most crucial, the orbital periods and the semi-major axes of different planets could be observed at that epoch. If we would just increase the number of stars in the vicinity of our planet the dynamics of the system would very probably become too unstable to justify our existence. Let us assume ([2]) that the people living on a Sitnikov's planet, starting from laws I & II, are able to deduce the following natural formulation of the gravity law (for instance, by using an equivalent of the Binet's formula): the attraction between any two bodies is equal to $\mathit{\Gamma}/\rho^2$, being ρ the distance between those bodies and $\mathit{\Gamma}$ some parameter. Moreover, it is also possible to determine the explicit value of $\mathit{\Gamma} \propto c^2/(a(1-e^2))$, where c is the constant angular momentum (see Appendix). Then, Kepler's law III directly follows from the fact that the constant c can also be related to the orbital period P by means of $c^2 = n^2 a^4 (1-e^2)$, where n is the mean motion related to the orbital period by $n = 2\pi/P$. In this context, the formulation of law III on the basis of pure observations and without a good theory of gravity seems impossible.

([2]) The author acknowledges the suggestion of an anonymous reviewer.

1. – Introduction

It turns out that the planet moves according to the following equation of motion:

$$\ddot{z} = -\frac{z}{\left(r(t;e)^2 + z^2\right)^{3/2}} \ . \tag{1}$$

Here, $z = z(t)$ is the distance of the planet (hereafter third body) from the common barycenter of the two stars (hereafter primaries), and $r(t;e)$ is the distance of one primary from the common center of gravity of the two primaries, that depends on orbital eccentricity e. A derivation of equation (1) from Newton's law of gravity is presented in the Appendix. The second order, non autonomous differential equation is valid as long as the scientists on our virtual planet choose their unit of mass, length and time to coincide with the sum of masses of the stars, revolution periods of the two stars, and semi-major axis of the orbital ellipse of one of the stars, respectively. The problem to describe possible solutions of equation (1), with given initial conditions $z(0), \dot{z}(0)$, is named after the Russian mathematician Kirill Aleksandrovich Sitnikov, who proved the existence of oscillatory motions in the three-body problem (Sitnikov, 1960) on the basis of this differential equation. The statement of the theorem requires some basic definitions related to symbolic dynamics that is based on $\mathbb{Z}$.

Let $s = [\ldots, s_{k-1}, s_k, s_{k+1}, \ldots]$ be an infinite sequence of integers, where the s_k are defined by the relation:

$$s_k = \left[\frac{t_{k+1} - t_k}{2\pi}\right] . \tag{2}$$

Here, the square bracket operator, $[x]$, denotes the integer part of x, and the $t_k \in \mathbb{R}$, with the property $t_k < t_{k+1}$, are discrete times with index $k \in \mathbb{Z}$, at which the solution of (1), say $z(t_k)$, becomes zero. With our choice of units, the s_k are therefore a measure of the number of complete revolutions of the primaries between two consecutive zeroes of the solution $z(t)$ of (1). In this setting, Sitnikov's theorem on oscillatory kind of motions in the three-body problem is as follows (version after Moser, 1973):

THEOREM 1. – *Given a sufficiently small eccentricity* $e > 0$ *there exists an integer* $m = m(e)$ *such that any sequence* s *with* $s_k \geq m$ *corresponds to a solution of the differential equation of the Sitnikov problem, namely* (1).

We remark that the s_k can be chosen completely independent from the initial conditions, with the only requirement that for a given eccentricity e, of the orbital ellipses of the primaries, m must be a lower fence to the s_k. It is therefore possible to choose an infinite, unbounded sequence of integers s_k such that the corresponding solution of (1) will be unbounded too, but will contain an infinite number of zeroes.

The proof of Theorem 1 can be found in Sitnikov (1960), Moser (1973), see also the discussion of the original proof in Wodnar (1993). The existence of oscillatory motions in the three-body problem has originally been shown for zero mass of the third body and small eccentricity e of the primaries. In Alekseev (1968a,b, 1969) the author generalizes the results to non-zero mass of the third body, and for the whole eccentricity interval $0 \leq e < 1$. A proof based on showing the topological equivalence of a discrete mapping, that describes the motion of the Sitnikov problem close to the critical velocity boundary ([3]) with the Bernoulli shift is given in Moser (1973).

We remark that Theorem 1 just states the existence of oscillatory kind of motions in the Sitnikov problem, the question remains open to find the relation $m(e)$, and to determine the initial conditions $(z(0), \dot{z}(0))$ for a given integer sequence s. No general recipe is known to the author. It would be a challenge to enlighten the scientific community with a proof of Theorem 1 that is also constructive in that sense that it provides a method to find $m(e)$, and $(z(0), \dot{z}(0))$ for given s too. So far, only numerical methods, based on trial and error or genetic algorithms, may be used to construct oscillatory solutions that are unbounded with a possible infinite number of zeros. It turns out, that finding oscillatory solutions of this kind, for given integer sequence s,

([3]) The critical velocity boundary is the curve in phase space that separates bounded and unbounded motions.

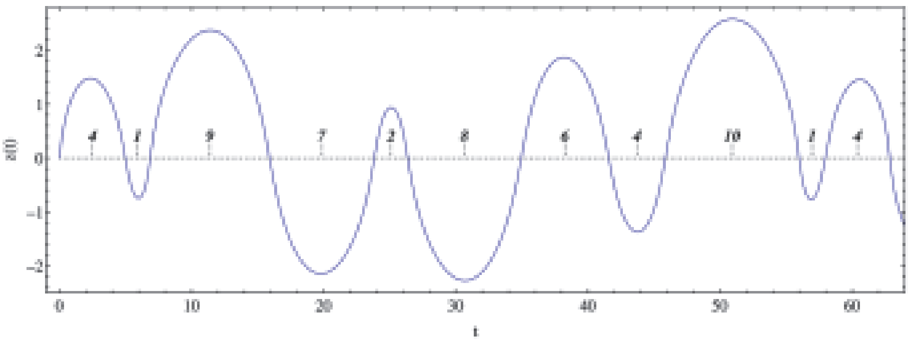

Figure. 2: A typical example of complex kind of oscillatory motion in the Sitnikov problem ($z(0) = 0.0$, $\dot{z}(0) = 2.5$, $e = 0.5$).

seems to be a numerical challenge too, since most of the numerically determined solutions turn out to be very sensitive in their initial conditions. Moreover, care is needed in the choice of numerical integration methods to propagate the orbit on long times. The reason is found in the presence of chaos in equations of the form (1), see also Section 3.

We demonstrate a possible solution of unbounded oscillating behaviour in Figure 2: using the initial conditions $z(0) = 0.0$, $\dot{z}(0) = 2.5$, and eccentricity parameter $e = 0.5$, the integer sequence around $t = 0$ turns out to be $s = (\dots, 4, 1, 9, 7, 2, 8, 6, 4, 10, 1, 4, \dots)$. We notice that unbounded integer sequences with an infinite number of zeroes may result in solutions of (1) with unusual structure: regular integer sequences with periodic sub-sequences (with contiguous elements) may be followed by sequences of irregular structure. This phenomenon is strongly related to the concept of intermittency in chaotic (but conservative) dynamical systems, where irregular alternation of phases of periodic and chaotic motions may take place.

As quoted in Moser (1973), Theorem 1 allows also to finding infinitely many periodic orbits ([4]) by choosing periodic integer sequences. As an example one could take the solution of (1) for $z(0) = 2.0$, $\dot{z}(0) = 0$, $e = 0.2$. The corresponding integer sequence around $t = 0$ turns out to

([4]) An orbit with the property $z(t + T) = z(t)$ for some real number $T > 0$.

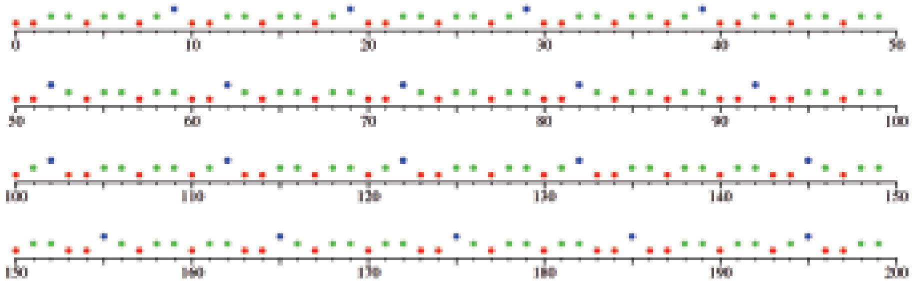

Figure. 3: 'Symphony' of integers s_k (k on abscissa) for $z(0) = 2.0, \dot{z}(0) = 0, e = 0.2$: symbol blue (top) for 7, green (middle) for 6, red (bottom) for 5.

be periodic in $s = (\dots, 5, 5, 6, 6, 5, 6, 6, 5, 6, 7, \dots)$. However, a small modulation of the underlying frequencies leads to a more complex overall integer sequence, as shown in Figure 3: we graphically represent the associated integer sequence $s_0, \dots, s_{200}$ (from upper left to lower right) on the number line, where the dots represent the numbers 5 (bottom, red), 6 (middle, green), and 7 (top, blue). This kind of representation of one orbit of (1) is closely related to symbolic dynamics. It would be interesting to investigate the kind of replacement rules acting on 3 different symbols that would reproduce this specific orbit for infinite times, without the need of numerical integrations of the equations of motion.

Let the initial position of the primaries in their orbit be parametrized by an angle ϕ (called true anomaly). The relation between the choice of the system parameters (e and ϕ) and the initial conditions ($z(0)$ and $\dot{z}(0)$) of the third body on the resulting integer sequence s can be visualized following an original idea of F. Vrabec (Dvorak et al., 1993): for a fixed parameter e and true anomaly $0° < \phi < 360°$ one numerically integrates (1) for different orbits of the third body for a given number of crossings through the barycenter, while counting the number of revolution periods of the primaries. If we choose the initial conditions on a grid $\dot{z}(0) \geq 0$ with $z(0) = 0$ we obtain a plot as shown in Figure 4. Here, the color code gives the number of revolution periods of the primaries between the 5-th and 6-th oscillation period of the third body in the parameter space $(\phi, \dot{z}(0))$. Black (bottom of color legend) indicates the region in

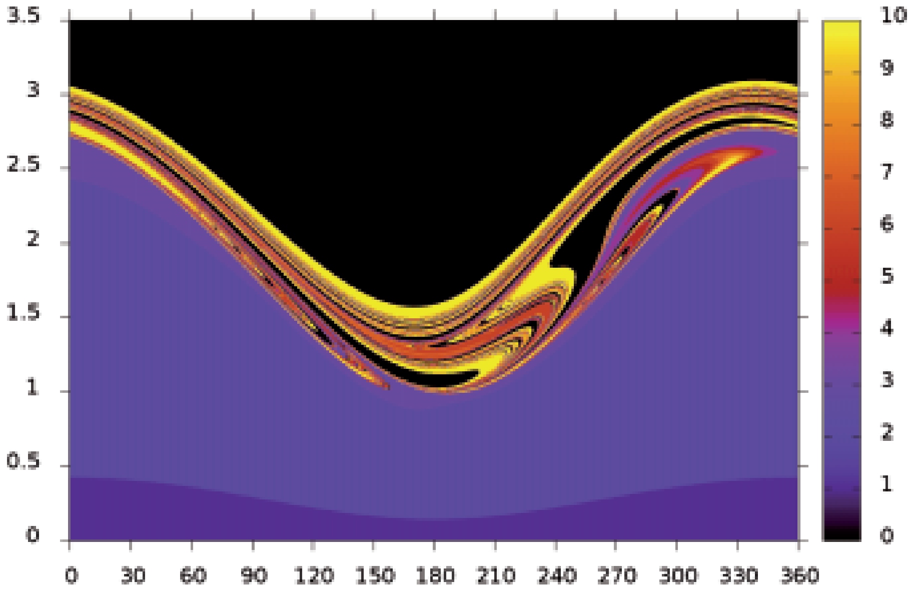

Figure. 4: Parameter study in the space $(\phi, \dot{z}(0))$ (initial true anomaly vs. initial velocity with $z(0) = 0$). The color code gives the crossings per revolution period of the primaries from one crossing to the other (after Lang (2011)). See also Dvorak and Lhotka (2013).

parameter space that leads to unbounded motions, while yellow (top of color legend) indicates the regime where the primaries take 10 (or more revolution periods) for one full oscillation cycle. We notice that we find complex patterns that also reveal regions of fractal structure close to the critical velocity boundary (see e.g., Dvorak and Lhotka, 2013).

In addition to the mathematical proof of oscillatory kind of motions in the three-body problem (Sitnikov, 1960; Alekseev, 1968a; Moser, 1973; Wodnar, 1993) the existence of different special kinds of orbits has been shown in Corbera and Llibre (2000); Chesley (1999); Martinez Alfaro and Chiralt (1993). We recommend that literature for further reading.

We conclude this section by returning to our gedankenexperiment, and try to understand how difficult it would be to define the length of a day for a population living on a Sitnikov planet. We first assume that the planet rotates with the spin axis normal to the orbital plane of the two stars (and therefore aligned with the orbit of the planet). Then, dusk ([5]) starts on one hemisphere exactly at the moment when $z(t_k) = 0$, where t_k is defined in (2), while dawn starts on the other hemisphere at the same moment. Since the number of revolution periods is given by s_k of Theorem 1 the length of day and night is also given by s_k. Since it is possible to construct arbitrary integer sequences s we may also construct arbitrary regular or irregular patterns of day and night changes together with arbitrary lengths of daylight durations. What a challenge for the people, living on a Sitnikov planet, to define their calendar!

2. – Historical remarks & the MacMillan problem

The dynamical system behind the Sitnikov problem was already known decades before Sitnikov provided his proof of oscillatory kind of motions. The case $e = 0$ was investigated by Pavanini (1907), and it was MacMillan (1911) who derived the oscillation period of the third body in terms of elliptic functions. The problem to describe solutions for zero eccentricity of the primaries is usually called MacMillan problem. The dynamical system behind is integrable. The integrability is easily understood, since for $e = 0$ equation (1) reduces to

$$\ddot{z} = -\frac{z}{(a^2 + z^2)^{3/2}} \ . \tag{3}$$

Here, $r(t; e)$ from (1) reduces to a constant radius a of one of the primaries from the common center of gravity, and the primaries stay in the same circular orbit, always separated from each other by 180°. The equation of motion (3) admits the energy integral

$$H = T + V = \frac{\dot{z}^2}{2} - \frac{1}{\sqrt{a^2 + z^2}} \ , \tag{4}$$

where H is the Hamiltonian, and T and V are the kinetic and po-

([5]) We use dusk / dawn to define the moment when the stars disappear / appear on the horizon (see also Prolog). This is a strong simplification since lighting conditions depend on many more parameters.

tential energies of the system. Since the Hamiltonian (4) is a conserved quantity and the dynamical system is only of dimension 1 it follows that (3) must be integrable. We notice, that $z = 0$ is an equilibrium point of (3) with $\ddot{z} = 0$. Let $z(0), \dot{z}(0) << 1$. If we expand the right hand side of (3) around $z = 0$ up to first order in z we get:

$$\ddot{z} = -Az \ , \tag{5}$$

with $A = 1/a^3$. The solution of this linear, second order ordinary differential equation becomes

$$z(t) = z(0) \cos(\omega_0 t) + \frac{\dot{z}(0)}{\omega_0} \sin(\omega_0 t) \ , \tag{6}$$

where the fundamental oscillation frequency is

$$\omega_0 = \sqrt{A} = \sqrt{\frac{1}{a^3}} \ . \tag{7}$$

We remark that the choice $a = 1$ leads to $\omega_0 = 1$, while choosing the unit of length to coincide with the distance of the primary bodies from each other gives $a = 1/2$ and thus $\omega_0 = \sqrt{8}$ (both definitions of the basic length unit exists in literature). For large $z(0), \dot{z}(0)$ the period of oscillation can be calculated in terms of elliptic functions. If we directly integrate (3) with respect to time t we get:

$$\left(\frac{dz}{dt}\right)^2 = \frac{2}{\sqrt{a^2 + z^2}} - 2C \ , \tag{8}$$

where the constant $C = -E$ (being E the total energy) depends on the choice of the initial conditions $z(0), \dot{z}(0)$. In MacMillan (1911) the author derives on the basis of the above equation the integral

$$\int_0^t dt = \int_0^v \frac{dv}{(1 - 2k^2 v^2)^2 \sqrt{(1 - v^2)(1 - k^2 v^2)}} \ , \tag{9}$$

where

$$k^2 = \frac{1}{2}(1 - C) \ , \tag{10}$$

and the new dependent variable v is related to z by means of $v^2 = \dfrac{1-u}{1-C}$ and $1 + z^2 = \dfrac{1}{u^2}$. The integration of (9) and inversion into the form $v(t)$ therefore shows also the integrability of (3). If we instead Taylor expand (9) with respect to k^2, the integral up to $v = 1$ gives quarter of a period. One full period in the MacMillan problem is then given by:

$$T = 2\pi\left(1 + \frac{9}{4}k^2 + \frac{345}{64}k^4 + \frac{3185}{256}k^6 + \dots\right), \tag{11}$$

with

$$k^2 = \frac{1}{2}\left(1 - \frac{1}{\sqrt{z(0)^2 + 1}}\right) \tag{12}$$

(valid for $\dot{z}(0) = 0$). We demonstrate the usage of (11) in Figure 5, where we calculate the period of oscillation in the space $(z, \dot{z})$ and superimposed with specific orbits obtained by numerical integration of (3). We notice that, due to the integrability of the MacMillan problem, all possible orbits in Figure 5 are regular closed curves or degenerate to the central equilibrium that is located at $z(0) = \dot{z}(0) = 0$.

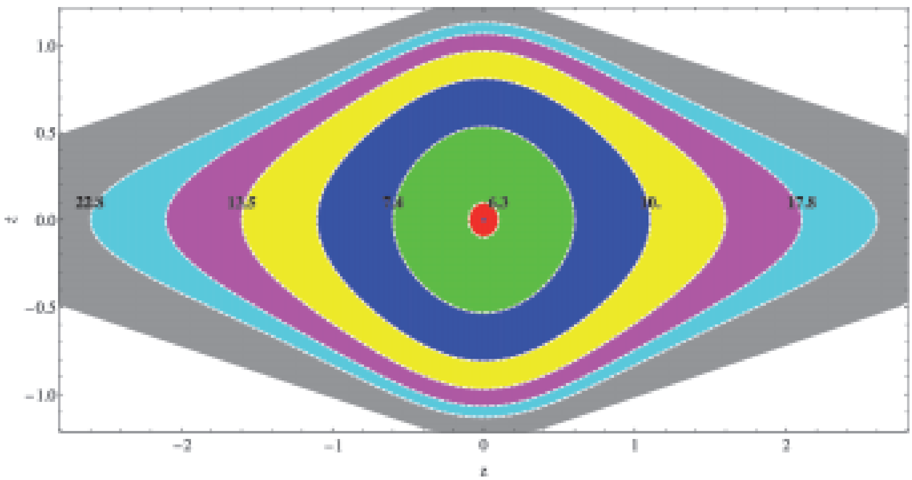

Figure. 5: Contours, calculated from (11), between different colors mark same oscillation periods in the space $(z, \dot{z})$. Dashed lines correspond to orbits directly obtained from (3) for $\dot{z}(0) = 0$ and $z(0) = 0., 0.5, \dots, 3.5$ (corresponding oscillation periods given in boldface).

The ‘circular Sitnikov problem’, and its generalizations, are still subject of scientific studies nowadays: in Pandey and Ahamad (2013) the authors study the different kinds of motions of the third body in the case where the primaries are three oblate spheroids – instead of point masses – and are situated at the edges of an equilateral triangle, in the circular problem. They establish a relation between the oblateness parameter of the primaries and the length of the sides of the equilateral triangle, and also provide the stability region as well as periodic orbits in the planar and 3D problem. The authors of Bountis and Papadakis (2009) investigate the generalized Sitnikov problem where the third body moves perpendicular on a line through the center of mass of $N-1$ equally massed primaries that themselves move on a circle. The study includes a detailed investigation of the intervals of stability and instability, along the z-axis, the investigation of periodic solutions that are found off the z-axis. The authors finally provide the solution of the problem for N tending to infinity. In Sidorenko (2011) the author studies the alternation of stability and instability within the family of periodic vertical motions, whenever their amplitude is varied in a continuous monotone manner. In Soulis et al. (2008), the authors study the existence and stability of straight line periodic orbits in a generalized Sitnikov problem, where three equally massed primaries move on a circular orbit. They provide a detailed study of the stability interval along the z-axis, and also investigate the bifurcation of 3D families of symmetric periodic orbits by means of the SALI method (Skokos et al., 2004) and suitably chosen Poincaré maps. In Perdios (2007) the author investigates the stability and extension of the family of straight line periodic orbits into 3D, also in the circular Sitnikov problem. Several new critical orbits are found at which families of three dimensional periodic orbits of the same or double period bifurcate. The study also investigates the influence of nearly equal primaries. A modern treatment to find the solution of the circular Sitnikov problem, and its period, in terms of elliptic functions can also be found in Belbruno et al. (1994). The authors study the linear stability of periodic orbits, and of the families of periodic orbits of the 3D circular restricted three body problem that bifurcate from them.

The MacMillan problem is also related to Euler's three-body problem [6] (Leonhard Euler, 1707-1783), where a particle moves in the gravitational field of two other point masses that are fixed in space. Since the primaries, in the MacMillan problem, move with constant angular speed around their common barycenter it is possible to introduce a rotating coordinate system, that is moving uniformly with the primaries, and to treat the MacMillan problem in a synodic frame, where the primaries are fixed. It remains to investigate the Euler problem with equal point masses and initial conditions of the third body restricted to the symmetry line normal to the line connecting the primaries, and to relate it to the inertial (non-rotating) frame. It turns out that Euler's problem is integrable in terms of Jacobi elliptic functions (e.g., see Whittaker, 1937). Contributions to the Euler problem have been made by the great mathematicians over centuries: Lagrange, Liouville, Laplace, Jacobi, Darboux, Le Verrier, Velde, Hamilton, Poincaré, and Birkhoff (to name a few).

3. – The phase space of the Sitnikov problem

It is natural to investigate low dynamical systems of the form (1) in the space $(z, \dot{z})$ like we already did with (3) in Figure 5, but with some minor modification: we first notice the presence of time t in (1) through $r(t) = r(t; e)$ that is absent in (3). We therefore can expect that phase portraits of the form of Figure 5 will change with time. Fortunately, the time dependency enters into equation (1) with a given period (2π in our choice of units). We can make use of this fact and draw phase portraits $(z(t), \dot{z}(t))$ modulo time 2π too.

3.1 – *Surfaces of section*

We start with 100 initial conditions within the region $z \times \dot{z} \in (-2.5, 2.5) \times (-2, 2)$ and integrate equation (1) for 1000 revolution periods of the primaries. We only keep points of the times series mo-

[6] Also named Euler-Jacobi problem and Two-Center Kepler problem.

dulo 2π and plot them in the space $(z, \dot{z})$. We sketch the relationship between the time series of specific orbits and the phase portrait modulo 2π – called the surface of section – for $e = 0.1$ in Figure 6: the equilibrium ($z(0) = \dot{z}(0) = 0.0$) evolves on a straight line (red) going through the central point of the phase portrait. A quasi-periodic orbit (in green), starting on an invariant curve, at $(z(0), \dot{z}(0)) = (0.3, 0.0)$, winds around the central line until it crosses the same curve on the phase portrait again after time 2π. We also demonstrate the structure of an orbit starting close to the 2:1 resonance in blue: during one revolution period of the primaries (in our units 2π) the third body performs only one half of its oscillation period (a full period would mean to return close to the region where the orbit started, see more explanations in Section 3.2).

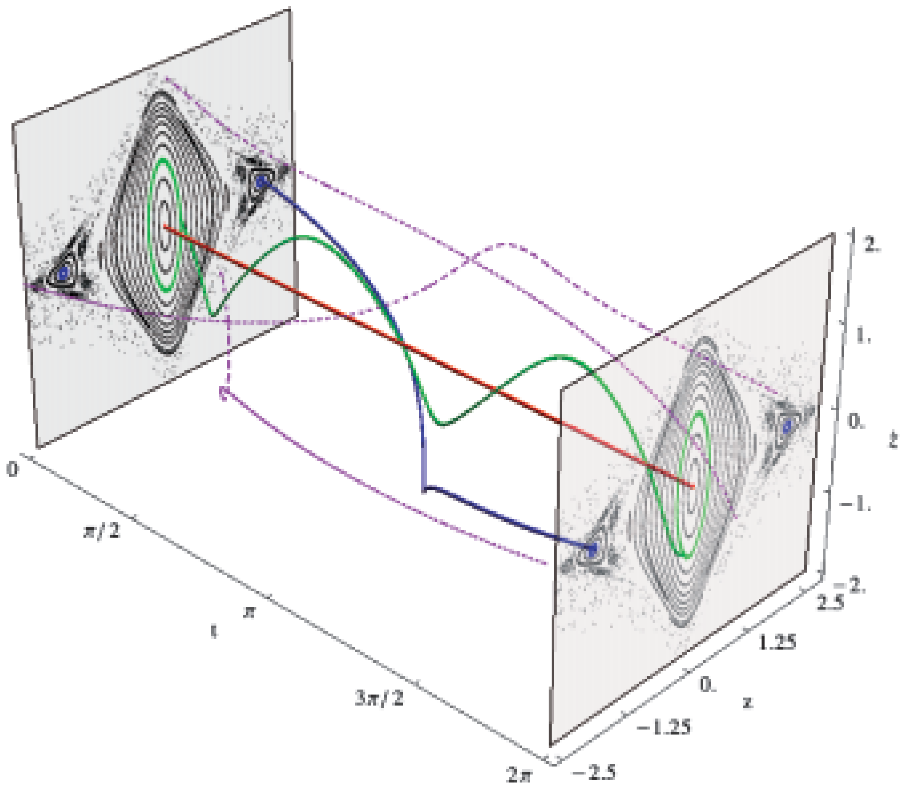

Figure. 6: Phase portrait modulo 2π for $e = 0.1$ and time series of specific orbits $(z(0), \dot{z}(0)) = (0, 0)$ (red), $(z(0), \dot{z}(0)) = (0.3, 0)$ (green), $(z(0), \dot{z}(0)) = (1.836..., 0.0482...)$ (blue), and $(z(0), \dot{z}(0)) = (0.1487..., 1.797...)$ (dotted magenta).

While all these previous kinds of orbits intersect the phase portrait in such a way to form regular smooth invariant manifolds on the surfaces of sections, we also demonstrate the structure of a chaotic orbit in Figure 6 (magenta, dotted): starting with initial conditions inside the chaotic (dotted) domain, the orbit evolves in time on a smooth curve, but lacking of any visible regular pattern projected to the phase space. Moreover, the orbit does not return close to its initial conditions, and eventually (not shown here for the clarity of visualization) fills up the whole chaotic regime.

We remark that from the manifolds on the phase portraits modulo 2π we have sufficient information to judge (by visual inspection) if the corresponding orbit is an equilibrium, (quasi-) periodic, or of chaotic nature. In the following, we will therefore only provide surfaces of section to qualitatively describe the dynamics. We also notice, that the space $(z, \dot{z})$ modulo 2π is also named in literature stroboscopic map, or (more often) Poincaré surface of section [7].

3.2 – *Qualitative description of the phase space*

In this section we provide surfaces of section of the Sitnikov problem for different values of the parameter e. We focus on the regime of phase space close to the central equilibrium and close to the 2:1 resonance. We start with the phase portrait modulo 2π for $e = 0$, shown in Figure 7 (left), that should be compared with Figure 5: the phase space is dominated by closed invariant curves that resemble ellipses close to the central equilibrium (located at $z(0) = \dot{z}(0) = 0$), while they become reshaped into diamond like curves further away from the origin. Each curve in these kinds of plots represents a manifold of

[7] To be precise, the construction of a classical Poincaré surface of section requires to choose initial conditions on a given energy level of the Hamiltonian. In case of the Sitnikov problem the Hamiltonian is not a constant of motion - strictly speaking we do not present Poincaré surface of sections in the classical sense, but rather surfaces of section in a much broader context.

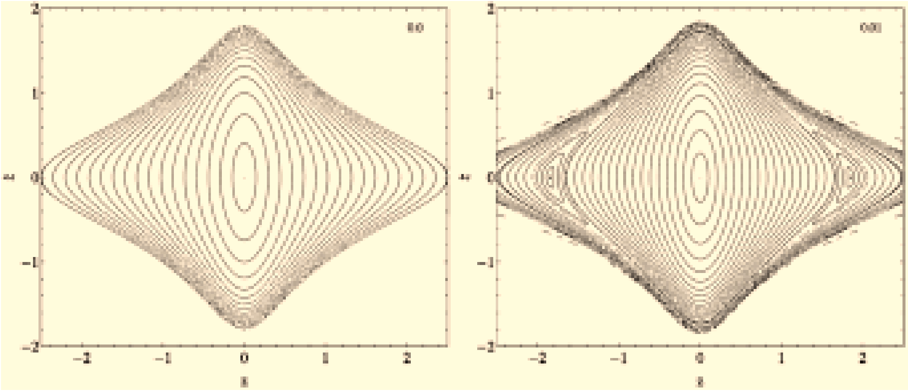

Figure. 7: Left: phase portrait modulo 2π for the Sitnikov problem with $e = 0$ – the MacMillan problem. Right: the case $e = 0.01$ – the 2:1 resonant islands are 'born'.

initial conditions that result in the same orbit – with same irrational frequency of oscillation – as it was already indicated by the color code in Figure 5.

If the oscillation period of the third body and the revolution period of the primaries become rationally dependent the curves are replaced by so-called resonant islands. For $e = 0.01$ two islands, with their centers located at roughly $z \simeq \pm 1.85$ and $\dot{z} = 0$ show up in phase space as shown in Figure 7 (right). The two points correspond to the 2:1 resonant orbit of the third body, and mark the locations where the third body is situated after one revolution period of the primaries. Starting close to the point with positive $z(0)$ the third body is located close to the point with negative z after the first revolution period of the primaries. The orbit returns back to the point with positive z after the second revolution period of the primaries. One full oscillation cycle therefore takes two revolution periods of the primary bodies to complete.

In Figure 7 (right) the resonant islands are still surrounded by invariant curves that circumscribe the central main equilibrium point. Eventually, chaos will destroy some of these invariant curves as already has been shown on the sections in Figure 6. While chaotic orbits starting between the central and 2:1 resonant island may fill up the complete chaotic region of the phase space on short times, motion

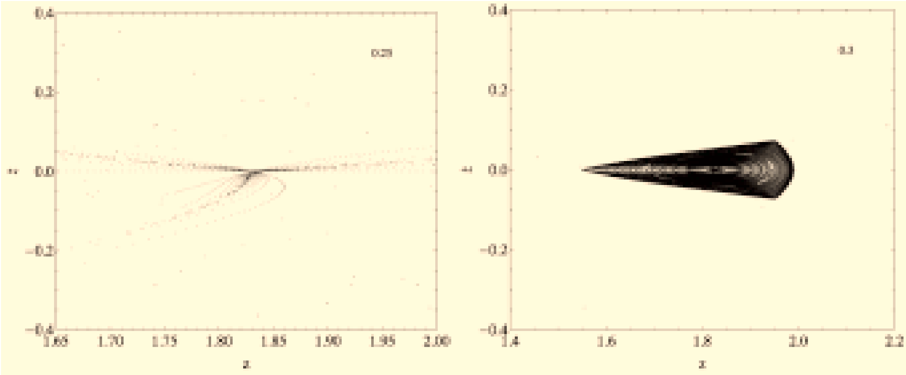

Figure. 8: Left: phase portrait modulo 2π for the Sitnikov problem with $e = 0.25$ – the 2:1 resonant orbit becomes unstable. Right: the case $e = 0.3$ – the 2:1 resonant island reappears.

starting on invariant curves stays bound (at least for a while). We also see, in Figure 6 the appearance of higher order resonant islands that are separated from the 2:1 resonant island that, however, disappear for increasing e.

The same fate happens to the 2:1 resonant island at $e = 0.25$ as shown in Figure 8 (left): we show the region close to the positive part of the island and find that the stability of the central point has been reversed and the small invariant curves that previously formed the resonant island have disappeared. It is interesting to notice that resonant islands may reappear for increasing parameter e as it is illustrated in Figure 8 (right) where we show the phase space around the exact 2:1 resonance on a very fine grid of initial conditions for $e = 0.3$.

The presence of invariant curves in phase space is strongly related to the stability of motion of the third body. We demonstrate this by means of a parametric study in Figure 9: we perform numerical integrations of (1) in the space $z(0) \times e$ with $0 \leq z(0) \leq 3$, with $\dot{z}(0) = 0$, and $0 \leq e \leq 0.3$. The color code indicates the stability time (in percentage of the most stable one) of the third body. Here, yellow (light) marks stable regions and the blue (dark) regions lead to an escape of the body on relatively short times, while red regions mark the so-called sticky orbits that resemble stable orbits for very long time until they finally escape due to the presence of weak chaos (see, e.g. Dvorak et al.,

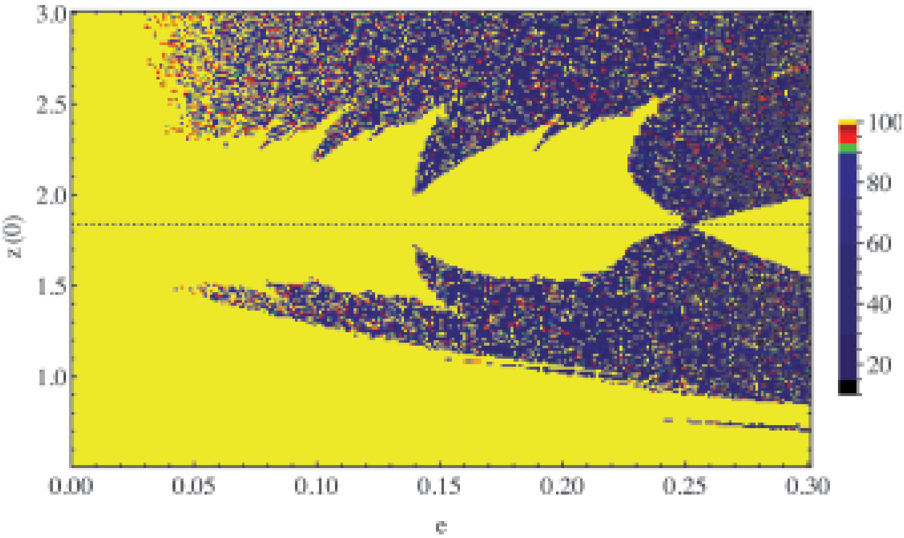

Figure. 9: Parameter study (e vs. $z(0)$ for $\dot{z}(0) = 0$): stable orbits (yellow, top of color legend), sticky orbits (red, second on top of color legend), unstable orbits (blue, bottom of color legend). See also Dvorak and Lhotka (2013).

1998). We notice the evolution of the extent of the 2:1 resonance (located at $z \simeq 1.848\ldots$ and indicated by a dashed line) with increasing e: the stability regime (related to the extent of invariant curves around exact resonance) increases up to $e \simeq 0.15$, decreases, increases again up to $e \simeq 0.25$, where it disappears, but reappears again for slightly larger e.

Many numerical studies have been performed for the Sitnikov problem that provide a quite complete picture of the phase space. We only list a few of them: a complete parameter study of the classical Sitnikov problem can be found in Dvorak (2007). The author provides detailed surfaces of sections in the parameter space e, and initial condition space. A shrinking of the main island with increasing e, and the dynamics of the 2:1 periodic orbit is demonstrated: the corresponding island disappears (reappears) by means of (inverse) pitchfork bifurcations. Furthermore, the role of sticky orbits, and escape channels on the dynamics are discussed. In Jiménez-Lara and Escalona-Buendía (2001) the authors calculate the symmetry lines of the Sitnikov problem and their dependency on the parameters using stroboscopic maps. They find families of periodic orbits and their bifurcations. Bifurcation diagrams and patterns of bifurcations are found too. A systematic numerical study of the dynamical problem can also be found in Dvorak (1993). The author provides in this valuable article a parameter study, and shows the existence of invariant curves that exist up to a certain value of the initial conditions. A numerical study, that is based on Poincaré surface of sections and Lyapunov characteristic exponents is provided in Liu et al. (1991a,b). Here, the authors support a relation between the chaotic region and the eccentricity of the primary's orbit.

The list is not complete, see references to (and within) for further information.

4. – Stability & approximate solutions

Numerical studies of the problem have shown to be a perfect strategy to investigate the phase space and space of parameters of the system in great detail. However, numerical studies are lacking of the possibility to relate the system parameters and initial conditions with basic properties of the solution, like its (non-)linear stability, or the fundamental oscillation periods. In this section we therefore summarize the result of some relevant studies that have been made to investigate the problem from an analytical point of view. Most of the studies are based on some kind of approximation of the original equation of motion, i.e. series expansion of (1).

The Taylor series expansion of the right hand side of (1) takes the form:

$$-\frac{z}{\left(r(t)^2+z^2\right)^{3/2}} = -\sum_{k=0}^{\infty}\binom{-\frac{3}{2}}{k}\frac{z^{2k+1}}{r(t)^{2k+3}}\ , \tag{13}$$

that is a convergent series for $|z/r(t)| < 1$. Up to first order in z the linearized equation of motion, valid for $|z| << 1$ is given by:

$$\ddot{z} + g(t;e)z = 0\ . \tag{14}$$

Here the function g turns out to be $g(t;e) = 1/r(t)^3$ with $r(t;e)$ given up to 4-th order in the eccentricity e:

$$r(t;e) = \frac{1}{2} - \frac{e}{2}\cos(t) + \frac{e^2}{4}[1 - \cos(2t)] + \frac{3e^3}{16}[\cos(t) - \cos(3t)] + \frac{e^4}{6}[\cos(2t) - \cos(4t)]. \tag{15}$$

The coefficient $g(t;e)$ in the second order linear differential equation (14) turns therefore out to be a periodic coefficient in time. This type of equation can therefore easily be investigated using Floquet theory (see, e.g. Lhotka, 2004; Hagel and Lhotka, 2005). On its basis it can be shown that the central equilibrium point is linearly stable as long as the trace of the transfer matrix R, that defines the mapping from $(z(2k\pi), \dot{z}(2k\pi))$ to $(z(2(k+1)\pi), \dot{z}(2(k+1)\pi))$ is in modulus strictly smaller than 2. The trace of R up to high order in e has been derived in Lhotka (2004). Up to 8-th order in e it is of the form:

$$Tr(R) = 2\cos\left(\sqrt{2}\pi\left[4. + 0.677e^2 + 0.375e^4 + 0.259e^6 + 0.198e^8 + \ldots\right]\right) \tag{16}$$

The critical points in e turn out to be close to $\pm 0.544\ldots$, ± 0.859, ± 0.966 at which $Tr(R)$ reaches ± 2. At these points nonlinear contributions in z from (13) may harm the stability of motion of the third body. Using a different approach (Martinez Alfaro and Chiralt, 1993) the authors find two sequences of critical values of e, and in addition values at which the central equilibrium becomes unstable (e.g. at $e = 0.8558625\ldots$).

We notice that a nonlinear stability analysis of the central equilibrium for small z and e has been done in Di Ruzza and Lhotka (2011), where the authors implement a suitable change of variables to obtain a high order normal form of (1). The nonlinear stability character of the regime close to the central equilibrium has been shown on the basis of the non-normal form terms, i.e. in a small domain close to $z = \dot{z} = 0$ and $e \leq 0.05$.

In Lhotka (2004), Hagel and Lhotka (2005) the authors derive an approximate solution of the full nonlinear Sitnikov problem that reproduces the qualitative dynamics in the domain $-0.2 \leq z(0) \leq 0.2$ and $0 \leq e \leq 0.4$ with great precision. The solution is obtained using the combination of advanced perturbation techniques, like Floquet theory, Courant Snyder transformation, and Poincaré-Lindstedt methods. At lowest degree in Fourier harmonics the solution of the Sitnikov problem after Lhotka (2004) is given by:

$$z(t) = A(t)\cos\left(\sqrt{8}\ \Phi(t)\right) . \tag{17}$$

Here, $A(t) = A(t; e, z(0), \dot{z}(0))$, $\Phi(t) = \Phi(t; e, z(0), \dot{z}(0))$ are periodic functions in time that depend on the parameter e and initial conditions $z(0)$, $\dot{z}(0)$. Let N be the order of truncation. Then the amplitude and phase functions are given by:

$$\begin{aligned} A(t) &= \sum_{k=0}^{N} a_k[e, z(0), \dot{z}(0)]\cos(kt) , \\ \Phi(t) &= \phi_0[e, z(0), \dot{z}(0)]\, t + \sum_{k=1}^{N} \phi_k[e, z(0), \dot{z}(0)]\sin(kt) . \end{aligned} \tag{18}$$

We provide the a_k, ϕ_k with $N = 7$ valid for the choice of parameter $e = 0.2$, and initial conditions $z(0) = 0.2$, $\dot{z}(0) = 0$ in Table 1. The comparison of the solution obtained from (17) with a numerical solution obtained from (1) is shown in Figure 10: the two different solutions (numerical=dashed, red, analytical=thick, black) perfectly agree in shape for times $0 \leq t \leq 20\pi$ (left plot). After long enough time, the approximation error in the phase, due to higher order terms, starts accumulating and the analytical solution only qualitatively reproduces Sitnikov's orbit (right). Higher order harmonics of the solution of the Sitnikov problem have been derived. The corresponding amplitude

Table 1: Coefficients (units $\times 10^{-7}$) of an approximate solution of the Sitnikov problem, given by (17), for $e = 0.2$, $z(0) = 0.2$, $\dot{z}(0) = 0$.

k :	0	1	2	3	4	5	6	7
a_k :	2413148	−362246	−40730	−6481	−1196	−254	−54	–
	1	$\cos(t)$	$\cos(2t)$	$\cos(3t)$	$\cos(4t)$	$\cos(5t)$	$\cos(6t)$	–
ϕ_k :	8950379 t	2759084	316590	51557	9434	1927	502	113
	–	$\sin(t)$	$\sin(2t)$	$\sin(3t)$	$\sin(4t)$	$\sin(5t)$	$\sin(6t)$	$\sin(7t)$

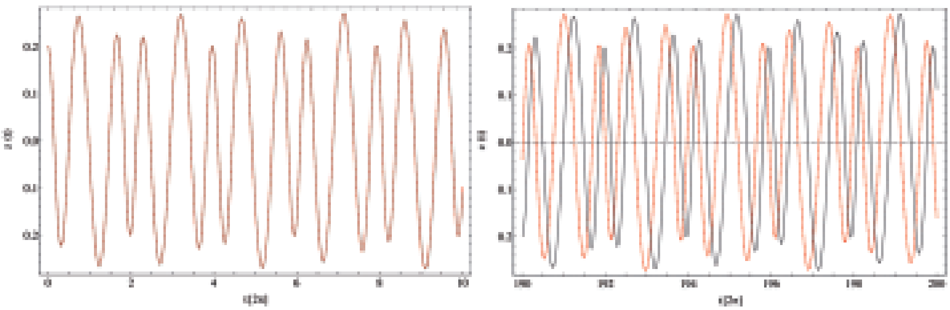

Figure. 10: Comparison of a 7-th order analytical (black, thick) and numerical (dashed, red) solution for $e = 0.2$, $z(0) = 0.2$, $\dot{z}(0) = 0$ for times between 0 and 20π (left) and for times 380π and 400π (right).

and phase functions at high order and for different $z(0)$, $\dot{z}(0)$ have been published in Hagel and Lhotka (2005).

We list some few additional analytical studies for further reading: A conformal mapping is derived and used in Liu and Sun (1990). On its basis, the authors are able to show the existence of a hyperbolic invariant set. They also measure the stochasticity of the mapping in terms of Kustaanheimo-Stiefel entropy. Good agreement is demonstrated between the analytical determinations and the results of numerical simulations. A mapping and a first integral of motion for the Sitnikov problem are derived in Hagel and Trenkler (1993) by making use of computer algebraic methods, that were originally designed for high energy particle accelerators. The results are based on the formulation of the problem in terms of an approximate polynomial differential equation, using Chebyshev approximation techniques.

An analytic approach is used to obtain approximate analytic solutions of the Sitnikov problem in Hagel (1992). The author uses a transformation that reduces the linear part of the equation of motion to the type of a harmonic oscillator. In this setting it is possible to derive an approximate integral of motion that is then used to explicitely find the solution of the problem. Regular solutions near the 3:2 resonant orbit are derived in Jalali and Pourtakdoust (1997) using a rotating coordinate system and the averaging method. Approximate solutions are found by the authors by means of Jacobian elliptic functions. The authors also provide a study of the breakdown of regular motion due to chaos for certain values of the eccentricity of the primary bodies. The stability of motion of the third body when the primaries are oblate and radiating is studied in Kalantonis et al. (2008). Using perturbation theory based on Floquet theory the authors compute the stability of motion, they provide critical orbits at which families of periodic orbits bifurcate. It is shown that the family of straight line oscillations only exists for identical primary bodies. The known result, that in a first order approximation the Sitnikov problem has the form of a Hill-type equation (linear second order equation with time dependent, periodic coefficients), is used in Kalas and Krasil'Nikov (2011) to show that the stability of the center equilibrium solution depends on the eccentricity of the primaries e. The authors find that the center is stable for almost any e, with the exception of a discrete set of e values that accumulates at $e = 1$. Perturbation theory based on normal form theory implemented in the case of this simple system can be found in Di Ruzza and Lhotka (2011), Floquet theory and Courant Snyder theory has been used to find approximate solutions of the nonlinear system in Hagel (2009); Hagel and Lhotka (2005); Faruque (2003, 2002). Analytical studies may also be found in Jalali and Pourtakdoust (1997); Wodnar (1995); Hagel and Trenkler (1993); Hagel (1992). This list is not complete, but should give the reader a starting point for further information on the problem.

5. – Generalizations & conclusions

Various generalizations and extensions of the Sitnikov problem can be found in literature. The system has already been investigated for the case of unequal masses of the primaries in Perdios and Markellos (1988). The mass effect of the third body on the motion of the primaries has been investigated Dvorak and Sui Sun (1997). The point mass character of the primaries has been replaced by oblate bodies in Pandey and Ahmad (2013); Douskos et al. (2012); Kalantonis et al. (2008). While the motion of the primaries out of the inertial reference plane has been subject of Dvorak and Sui Sun (1997); Perdios and Markellos (1988), initial conditions of the third body off the line of motion has been studied in great detail in

Perdios and Kalantonis (2012); Bountis and Papadakis (2009); Soulis et al. (2008, 2007); Belbruno et al. (1994). The number of primaries has been increased in Bountis and Papadakis (2009); Soulis et al. (2008), it was replaced by a continuous ring in Bountis and Papadakis (2009). Additional non relativistic forces have already been included in Kovács et al. (2011).

The Sitnikov problem has shown to be a testbed for various analytical and numerical techniques, probably due to the reason that it is a very simple toy model of celestial mechanics, while it already contains all different kind of complexity that is found in planetary dynamics: sticky and chaotic orbits have been investigated in full detail in Kovács and Érdi (2009); Hevia and Ranada (1996), escape and diffusion channels are scientific subject in Dvorak (2007). The fractal structure of the phase and parameter space is has been exploited in Kovács and Érdi (2007); Dvorak (1993), bifurcations & families of periodic orbits have been found and analyzed in Perdios (2007); Perdios and Kalantonis (2006). Chaos indicators and the usage of stroboscopic maps and surfaces of section have been tested, e.g. in Kovács and Érdi (2007); Dvorak (2007); Jiménez-Lara and Escalona-Buendía (2001); Liu et al. (1991a). A symplectic mapping for the Sitnikov problem has been derived in Liu and Sun (1990), specialized numerical integration techniques to investigate shadowing orbits are subject in Urminsky (2010).

The Sitnikov problem serves as a toy model of celestial dynamics that is comparable to the role of the standard map in physics. No planetary system has been observed so far that is situated in such a special kind of configuration (still, many binary star systems have been observed and may host celestial bodies that perform possible vertical motions). For this reason, our gedankenexperiment will probably stay a theoretical construct in our mind for ever.

To this end, we would like to advertise additional review works on the Sitnikov and related problems that may be of interest to the reader: Dvorak and Lhotka (2014), Dvorak and Lhotka (2013), Castriotta (2012), Dzhanoev et al. (2009), Lacomba et al. (2002), Corbera and Llibre (2002), García and Pérez-Chavela (2000), Chesley (1999), Martínez Alfaro and Chiralt Monleón (1991), to name a few.

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I thank Rudolf Dvorak & Johannes Hagel for introducing me to the Sitnikov problem when I was a student. The simplicity of its mathematical formulation paired together with the complexity of its different possible kinds of motions is still appealing to me nowadays. I also thank the editorial board and anonymous reviewers for a thorough reading of the manuscript and many very helpful suggestions related to the gedankenexperiment.

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Appendix: Sitnikov's problem from Newton's law of gravity

In this Section we derive equation (1) from Newton's law of gravity. We denote by m_1, m_2, m_3 the masses of the primaries (m_1, m_2) and the third body (m_3), respectively. Let the origin of an inertial coordinate system coincide with the center of gravity. Let $\vec{r}_i$ be the position vector of the body with mass m_i and $\Delta_{i,j}$ denote the scalar distance between body with mass m_i from the body with mass m_j (with $i \neq j$ and $i, j = 1, 2, 3$). The force, due to gravity, $\vec{F}_{ij}$ acting on the body of mass m_i due to the body with mass m_j is given by Newton's law of gravity:

$$\vec{F}_{ij} = -G \sum_{i \neq j, j=1}^{3} m_i m_j \frac{(\vec{r}_i - \vec{r}_j)}{\Delta_{i,j}^3} \ . \tag{19}$$

Here, G is the gravitational constant. Let the vector $\vec{r}_k$ be parametrized by components (x_k, y_k, z_k) with $k = 1, 2, 3$. Then, the scalar $\Delta_{i,j}$ in Euclidean space is simply given by:

$$\Delta_{i,j} = \sqrt{\left(x_i - x_j\right)^2 + \left(y_i - y_j\right)^2 + \left(z_i - z_j\right)^2} \ . \tag{20}$$

Let the accelerations be given by $\ddot{\vec{r}}_k = (\ddot{x}_k, \ddot{y}_k, \ddot{z}_k)$ with $k = 1, 2, 3$. Then the system of equations of motions for the three bodies is given by:

$$\ddot{\vec{r}}_1 = -G \left(\frac{m_2(\vec{r}_1 - \vec{r}_2)}{\Delta_{1,2}^3} + \frac{m_3(\vec{r}_1 - \vec{r}_3)}{\Delta_{1,3}^3} \right) \ , \tag{21}$$

$$\ddot{\vec{r}}_2 = -G \left(\frac{m_1(\vec{r}_2 - \vec{r}_1)}{\Delta_{1,2}^3} + \frac{m_3(\vec{r}_2 - \vec{r}_3)}{\Delta_{2,3}^3} \right) \ , \tag{22}$$

$$\ddot{\vec{r}}_3 = -G \left(\frac{m_1(\vec{r}_3 - r_1)}{\Delta_{1,3}^3} + \frac{m_2(\vec{r}_3 - \vec{r}_2)}{\Delta_{2,3}^3} \right) \ . \tag{23}$$

At initial time $t = t_0$ the initial conditions are

$$\vec{r}_k(t_0) = (x_k(t_0), y_k(t_0), z_k(t_0)) \ , \ \dot{\vec{r}}_k(t_0) = (\dot{x}_k(t_0), \dot{y}_k(t_0), \dot{z}_k(t_0)) \ , \tag{24}$$

with $k = 1, 2, 3$.

The number of degrees of freedom of the above system of differential equations is 9. If we assume that the third body does not influence the motions of the primary bodies, then the 6 equations of motion for the primaries become uncoupled from the remaining 3 that describe the motion of the third body. We remark, that the two-body problem defined by the primaries alone is integrable. Their motion lies in an invariant plane that we choose as the basic reference plane of our system. Let $m \equiv m_1 = m_2$, and the x-axis of the inertial coordinate system passing through the apocenters of the primary bodies. Then the orbits of the primaries are antisymmetric with respect to each other as follows:

$$\vec{r}_1 = -\vec{r}_2 = (x_1, y_1, 0) \ . \tag{25}$$

It is therefore sufficient to deal with the reduced set of differential equations to describe the motion of the primaries:

$$\ddot{x}_1 = -\frac{Gmx_1}{4\left(x_1^2 + y_1^2\right)^{3/2}} \tag{26}$$

$$\ddot{y}_1 = -\frac{Gmy_1}{4\left(x_1^2 + y_1^2\right)^{3/2}} \ . \tag{27}$$

Moreover, if we restrict the motion of the third body to the inertial z-axis that is normal to the orbital plane of the primaries, (23) reduces to:

$$\ddot{z}_3 = -\frac{2Gmz_3}{\left(x_1^2 + y_1^2 + z_3^2\right)^{3/2}} \ . \tag{28}$$

We notice that (28) is coupled with the equations of motion of the primaries only in terms of their radial distance from the common barycenter:

$$r_1^2 = x_1^2 + y_1^2 \ . \tag{29}$$

Let us drop – for simplicity – indices from now on: we therefore denote r_1, x_1, y_1 by r, x, y and z_3 by z. We first solve the Kepler problem in polar coordinates to find the solution $r = r(t)$. Let (r, ϕ) denote polar coordinates related to (x, y). The respective equations of motion in terms of r, ϕ are then given by

$$\ddot{r} - r\dot{\phi}^2 = -\frac{2Gm}{r^2} \ , \tag{30}$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right) = 0. \tag{31}$$

We make use of the conservation of the angular momentum c, and rewrite (30) in terms of

$$\ddot{r} - r\left(\frac{c}{r^2}\right)^2 = -\frac{2Gm}{r^2} \ , \tag{32}$$

together with (Dvorak and Lhotka, 2013):

$$c = r^2\dot{\phi} = \sqrt{2Gma(1-e^2)} \ . \tag{33}$$

Here, a is the semi-major axis of one of the primary bodies, and e is its orbital eccentricity. We notice, that the solution $r(t)$ can be found from (32) independently from (28). In fact, by making use of Kepler's first law, the solution in polar coordinates is of the form:

$$r = \frac{a(1-e^2)}{1 + e\cos(\phi)} \ . \tag{34}$$

Here, the motion law of the angle ϕ can also be deduced from Kepler law I & II:

$$\dot{\phi} = \sqrt{\frac{2Gm}{a^3(1-e^2)^3}} \left(1 + e\cos(\phi)\right)^2 \ , \tag{35}$$

that reduces the problem to find $r(t)$ from $\phi(t)$ by making use of a first order differential equation instead of (32). We are left to substitute $r(t)$ in (28), and finally obtain the equation of motion for the third body:

$$\ddot{z} + \frac{2Gmz}{\left(r(t)^2 + z^2\right)^{3/2}} = 0 \ . \tag{36}$$

Setting $2Gm = 1$ we obtain (1). It is a common practice to set $t_0 = 0$, $a = 1$ (or $2a = 1$) such that units of mass, length, and time correspond to the total mass of the primaries, the semi-major axis (or distance between the primaries), and one revolution period of the primary bodies.

We remark that in the context of the assumptions that have been made, the solutions of (36) are solutions of (21)-(23) for special initial conditions with the symmetric properties that have been described above. Equation (1) therefore describes special solutions of the spatial, elliptic restricted three-body problem.

Christoph Lhotka
Space Research Institute, Austrian Academy of Sciences,
Schmiedlstrasse 6, A-8042 Graz (Austria)
E-mail: christoph.lhotka@oeaw.ac.at