La Matematica nella Società e nella Cultura

Rivista dell'Unione Matematica Italiana

Maria Dedò, Simonetta Di Sieno

The Mathematics Laboratory: An Outline of Contents and Methodologies

La Matematica nella Società e nella Cultura. Rivista dell'Unione Matematica Italiana, Serie 1, Vol. **6** (2013), n.2, p. 321–342.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=RIUMI_2013_1_6_2_321_0>

La Matematica nella Società e nella Cultura. Rivista dell'Unione Matematica Italiana, Unione Matematica Italiana, 2013.

La Matematica nella Società e nella Cultura
Rivista dell'Unione Matematica Italiana
Serie I, Vol. VI, Agosto 2013, 321-342

The Mathematics Laboratory: An Outline of Contents and Methodologies

MARIA DEDÒ - SIMONETTA DI SIENO

The widely-held opinion of the ineffectiveness of mathematics teaching in Italian schools has certainly not abated in recent years. This judgment is not only based on more or less accurate enquiries that compare the outcomes of mathematics teaching in Italy with those of other countries, but also reflects the sense of impotence that afflicts many teachers when they try to reconcile their own expectations and those of the students, demands from the working world, the status and image of mathematics as a discipline, the amount of time required to make lessons learned enduring ones, and the unrelenting rhythm imposed by scholastic institutions.

The society in which students will find themselves working once they leave school will undeniably be a complex one, with a great variety of processes that follow one another in extremely rapid sequence, and so, in order not to become passive, defenseless victims of change, young people in school need to acquire the tools that allow them to comprehend and interpret reality, and are thus useful in daily life. Therefore, not only have society's expectations regarding the mathematical skills of those who finish school radically mutated, but such skills are moreover difficult to identify *a priori*, because different professions require different mathematical skills; in some cases it is not even possible at this moment to predict how these will change in the course of the next few years. It is thus clear that mathematics teaching has to adjust to these changing conditions, and such an adjustment requires not so much a substitution of one chapter of mathematics for another, but rather, as Kra says, '(it makes sense) to expose everyone to as vast a mathematical landscape as possible ([see [11]).

To do this, the truly crucial point – that which "makes the difference" – is in the teaching (it is no coincidence that Kra's article is entitled 'Teachers are the Key'). It is up to the mathematics teacher to design and carry out, in close collaboration with his or her colleagues and in a way that makes the most of his or her specific skills, educational/training itineraries that promote the students' skill in reasoning, rational thought and critical capacity, develop the capacity to address and solve problems, call on operative capabilities, and stimulate autonomy and personal creativity and the assumption of responsibility.

One of the characteristics that typifies mathematics is the preparation of a strategy. In this present context, that characteristic represents an important tool, and a key to unlock other kinds of knowledge. To obtain results that do not vanish in the space of the few weeks between one test of learning and the next, the teachers must make firm decisions that regard the *corpus* of the discipline, as well as other significant choices regarding the area of interpersonal relationships: teachers must – to guarantee the effectiveness of their work – be in tune with the motivations, interest and previous knowledge of the students, and thus also make coherent choices regarding how to communicate with and manage the class as a group.

The experience of the last ten years of experimentation regarding laboratories carried out by the 'matematita' Centre[1] shows that in this context introducing laboratory methods in pre-university teaching represents an interesting possibility. Here we will describe the activities of the past decade.

The laboratory method calls for students to experiment with solutions to problems – either those found in everyday life, or

[1] 'Matematita' (a play on the words *matematica*, mathematics, and *matita*, pencil) is an Interuniversity Research Centre for Informal Communication and Learning of Mathematics. To test its proposals, the Centre carries out many activities aimed at preuniversity students: exhibitions, online activities and also on-site workshop activities carried out in the Department of Mathematics of the Università degli Studi di Milano; see http://www.matematita.it.

taken from within mathematics itself – using knowledge, mathematical and otherwise, learned in school, connecting knowledge acquired in various milieus and thus showing which of these have truly become a part of their personal patrimony (which is naturally different with respect to simply showing that they know how to repeat a given statement). A mathematics teaching hinged on laboratory-like activities allows the youngsters to work together with their classmates towards a common objective, expressing high levels of efficiency and correctness, making it possible to make the most of abilities that are often undervalued or even unrecognised, and guaranteeing a greater involvement in the process of learning itself, and consequently a greater effectiveness of that learning, above all over a long period.

At the same time, for teachers laboratories can constitute a gymnasium for training and continual professional development, both as they prepare the laboratory itself, through the need to conjoin weighty themes and absorbing methodologies, and as they lead the students in the various activities. Laboratories are also quite useful in carrying out and observing the constructive interpersonal dynamics mentioned earlier, which are very difficult to identify in the usual classroom setting.

In what follows, after having set out and analysed what we mean by 'laboratory', we will go into detail about the various aspects that we have just mentioned here, distinguishing the phase of preparation (which involves only teachers), the phase of realisation, and the final phase of evaluation of the laboratory. In particular, regarding the second phase, we will discuss the role played here by different aspects that typify the discipline (from rigour to language, from error to the manipulation of concrete objects or virtual animations), and especially the different roles that these play in a laboratory activity with respect to a 'frontal lesson', that is, one in which the teacher explains and the students listen. This allows us to show that it is not by chance that certain potentially positive mechanisms manifest themselves more readily in this way of working.

What is meant by 'laboratory'?

There are many meanings assigned to the term 'laboratory' as it is used in the literature regarding mathematics teaching, both in manuals for use by teachers and in current scholastic practice. Given that there is no unanimous agreement for exactly what a laboratory is, it will be useful here to premise our discussion with our 'definition' of laboratory, which will thus be the meaning of the term as used in this present article.

In order for us to classify a didactic activity as proper to a 'laboratory', it is necessary that:

- students have an active role in it: they are not merely listeners, but must operate in a concrete way, working in small groups and discussing the problem or activity among themselves, in order to construct their own knowledge;
- teachers must play the role of active guides who observe and listen, respond to questions as they arise, are capable of indicating a fruitful path and directing students away from one that is less profitable, and above all, who help the youngsters – at the end of the journey – to sum up the activity that they have carried out.

Then there are other elements as well that are useful for identifying a laboratory (for example, the use of materials that can be manipulated, the use of technology, etc.). However, such elements are not so much 'necessary' to call a given activity a 'laboratory' as they are elements which can rather naturally lead to the 'undoing' of a frontal lesson, and thus facilitate the kind of dynamics typical of a laboratory in our definition of the term.

This is certainly not the first time in all these years of the considerations of mathematicians and teachers of mathematics that the laboratory method appears as the option of choice for teaching/learning. It suffices to think not only of relatively recent figures such as Emma Castelnuovo (b. 1913) and Vittorio Checcucci (1918-1991), but of those much further back in time such as Giovanni Vailati (1863-1904), who campaigned for a school in which students were not constrained ‘to learn theories before knowing the facts to which these refer’ and to ‘hear words repeated before being in possession of concrete elements they can sense from which, by abstraction, their meaning can be obtained’ ([16], our trans.).

Before the laboratory: the training of the teacher/guide

The proposal of a laboratory is configured as an effective working method for learning mathematics, regardless of scholastic level; it is a method that makes it possible at least in part to overcome the difficulties present in learning mathematics, and offers a key for all students to an experience of mathematics. All of this requires first of all an adequate training of the teachers who intend to offer this experience to their students. As Giuliano Spirito has written, a laboratory teaching

> involves the need for the teacher to have a bit more attention for didactics, a bit more capacity for conducting, and even a bit more clarity of exposition (that clarity that comes from a profound meta-reflection on disciplinary nodes). But in reality, a good laboratory teaching requires much more again: it requires an in-depth rethinking of the hierarchy of contents that one wishes to transmit and a detailed reflection – I would say a problem-by-problem reflection – on the ways, times, conceptual nodes, cognitive obstacles to learning. On the other hand, this is the only way that it becomes possible to make lofty ambitions (that is, to allow the students to know the foundational acquisitions of mathematical knowledge, those that make mathematics beautiful even before it is useful) coexist with didactic effectiveness (that is, to obtain that each student achieves the maximum results to which his potential allows him to arrive) [15, our trans.].

It is thus necessary for teachers to be trained to this end, with attention placed on both conducting the class as a group and the choice of the mathematical concepts involved (see [9] and [17]). For questions of the first kind (which regard both the setting and the pedagogical project, and go from the organisation and the use of the space to the determination and use of time, as well as the formulation and negotiation of rules of behaviour, and to the identification of roles), we refer the reader to the wealth of literature about this. Instead, in what follows here we will describe how to construct (learn to construct and teach to construct) meaningful mathematical itineraries to be followed in the laboratory, focussing attention on the significance of contents, the identification of instruments and teaching aids for laboratory activities and with respect to an explicit leading thread.

In our work at the 'matematita' Centre we have found that the teachers/guides have not always had the experience, in their own scholastic careers (including university and post-university), of teaching by problems (a method of learning that appertains not only to mathematics but was indeed born for courses in medicine; see [2], [3] and [14]); in any case not all of them have managed to avoid a totality of frontal lessons. This necessitates a detailed kind of training that is neither simple nor capable of being simplified. Thus, before all else, those participating in training are invited to attempt to solve some problems, without any immediately previous preparation, and to address them together with other people, most of the times in a small group. Generally speaking, the problems must not have ready solutions, nor must they involve the need to resort to sophisticated techniques; rather they must require a re-reading (perhaps not trivial) of results that can even be quite 'elementary'. Above all they must require the use of those ingredients of common sense and correct reasoning that form the basis of any lasting acquisition of a mathematical concept (see [5]).

The fact that the participants have sufficient time to dedicate to these problems and that they can compare and contrast ideas with the others in the small group almost always leads to the solution to the problem they are assigned. This makes it possible to discuss which environmental conditions and which relationships between students and teachers, and among the students themselves, lead to the creation of a shared and meaningful mathematical activity. Additionally, this activity of seeking solutions makes it possible to reproduce in a natural way the same phases that a mathematical researcher goes through: the need to solve a problem which is not even known for sure to have a solution; the need to analyse the problem using 'common sense' in order to invent a solution, a path to follow; losing time in inconclusive or erroneous reasoning before finding the correct route; the re-reading of the results attained in order to find their generalisation (that is, the production of new 'problems'); the recounting of the results found to the scientific community (see [1]).

This process is not unique to groups of teachers! It is truly amazing to see, in the accounts that teachers sometimes send us of activities that have taken place in the classroom, that what the teacher notes as the most meaningful stages in the work of their students (including children of the early grades of primary school) are, *mutatis mutandis*, precisely those that we have just outlined, that is, the most important stages gone through by a mathematician who is engaged in solving a research problem.

The problems most suitable to this phase of the teachers' first encounter with the laboratory method are those which are, for one reason or another, most disorienting. For example, we sometimes propose a problem that is apparently innocuous, such as the following, known as the 'Monty Hall problem':

> There are three closed doors. Behind one is a Ferrari, while behind each of the other two is a box of chocolates. You are invited to choose a door and take home whatever is hidden behind it (and let us suppose that you are more interested in the car than in the candy). You choose a door and, before you open it, a friend opens one of the other two and shows you that behind it is a box of chocolates. Now you are again allowed to choose the door you believe hides the Ferrari. Do you stay with the door you chose first, or do you change doors? Why?

Quite often the teachers think it is completely natural to answer that it makes no difference whether you change doors or stay with the first choice, but this is wrong. Why? What has not been taken into account? Often the person who tries to analyse the situation, without being distracted by the context, understands that in two out of three cases it is better to change the initial choice, but is so stupefied by this fact that he can hardly be convinced. This leads naturally to a reflection on how important the effect of disorientation is for a solution that doesn't 'come from above', but requires being proven beyond the shadow of a doubt. If we are not directly engaged in the solution of a problem, we lose the capacity to evaluate the complexity of the question being dealt with, and thus the difficulty of the task: everything is flattened and all solutions are the same. Let us recall that a proof (regardless of level, from the answer to the 'why' of primary school children to the first attempts of older students at formal justifications) serves no purpose whatsoever if first is not created the need for something to be proved! Thus, the search for a 'strange' problem and the effect of surprise is certainly NOT aimed at reducing mathematics to the level of gambling, but to the contrary, wants to use surprise to drive a person to ask himself why what happens, happens and thus create the need for a proof.

Having direct experience with the mechanisms used to address a problem in a situation that is to all effects a laboratory, with the difficulties of interpreting a situation, constructing hypotheses for its solution, and communicating the results obtained, thus becomes a good way to understand the value of an analogous experience on the part of students. Attention is focussed in a natural way on the attempt to construct good, analogous occasions for astonishment, disorientation and surprise. Where shall we fish for such problems? What relationship should they have with the curriculum? Should they touch on the themes of the usual scholastic problems, or on 'alternative' themes that are not normally explored? Should itineraries be conceived for all students, only for the 'really good' ones, or only for those having trouble? What risks or advantages are there for the construction of a rigorous language on the part of the students? These are all questions that the teacher must ask himself in order to propose a laboratory that is consistent with the general choices that have been made with the class.

When the aspiring teachers/guides have chosen a theme within a determined disciplinary area, they are asked to construct laboratory sessions around it for a certain series of classes. It becomes evident from the very first concrete proposals that the activities related to themes that are quite restricted and of scant cultural content, are often either trivial reformulations of problems/exercises already seen a thousand times, so that it is not difficult to imagine that the students will be hardly interested, if at all, or simply exercises camouflaged by questions. Nor does the fact that the exercises are presented in the form of games change the situation: for example, games with coins to learn tenths and hundredths intrigue the youngest students, but are immediately regarded with condescension by the older ones. Instead, those who succeed in identifying a central node in mathematics, an important question to work on, have an easier time constructing sensible and engaging laboratory activities that often open different horizons, leading to the consequent and obvious need to decide/choose whether or not to pursue them. The fundamental unity of mathematics soon intervenes to establish hierarchies of importance in the activities aimed at illustrating (for example) the rational numbers: all hopes fade of separating numbers from shapes, geometry from arithmetic, and so forth (see, for example, [4]).

One piece of evidence that we have derived experimentally from our work with laboratories over the past ten years is the fact that the choice of a theme for a laboratory is a question that cannot be considered in view of a single segment of scholastic work, but must instead be addressed while taking into account as much as possible the students' overall scholastic itinerary, from the first grades of primary schools through the end of pre-university levels.

This is not at all counter to the characteristics of the discipline we are interested in: to the contrary, in mathematics there are various conceptual nodes which both teachers of primary schools and those of secondary schools must come to terms with, and for which the forms and methodologies of presentation are very closely related to the different ages of the students. To be sure, the objectives that can be set for a pre-teen in middle school who grapples with a certain question of mathematics are different from those set for a youngster in an early grade of primary school. However, common to both is the need to lead the student, regardless of age, to a direct experience of the results and methods of mathematics. Also quite similar are the ways of accompanying the students to having this experience. The same conceptual nodes are also subtended in the teaching of mathematics in secondary schools, especially in the first two years, in different ways according to the specificity of the course and the kind of schools, but in an important and meaningful way for all, with difficulties and points of incomprehension that are very similar.

It doesn't appear useful to us in this phase to propose a differentiation between the first two years in schools that place an emphasis on mathematics (such as schools for the sciences) and those that don't (such as schools for the arts): our experience in these years with secondary schools has shown that the gap between teachers and students at this age is completely analogous in the two types of schools; in both cases it is often the fruit of an acceleration on the road to the formalisation of the mathematical sciences to the detriment of an acceptable comprehension of their results and their methods. Evidently there is a difference between the disciplinary acquisitions in schools that emphasise mathematics and those that don't, but this occurs beginning with the contents that are meaningful from a disciplinary point of view and is not due to technicalities overcome and/or completely useless, even though these are traditionally believed to comprise aspects that are generally valuable for education, aspects that are in fact inexistent.

Attention to meaningful contents also means that it is sometimes possible to introduce, perhaps in the background and not as the protagonists of the laboratory sessions, problems that are genuinely 'difficult'; it is interesting for youngsters to have the opportunity to see that there are also problems of this level, problems that not only they don't know how to solve, but that their teachers may not know how to solve, and perhaps no one in the world is capable of solving either. It would give a completely flat (and false!) idea of the discipline if they were to become convinced, implicitly, that mathematics only serves to solve trivial problems (such as some of those same old exercises found in some textbooks!).

Going back to the itinerary for training those who will lead the laboratories, during the final phase the teachers/guides are asked to turn over the proposal they have constructed to a colleague for experimentation and to try it out for themselves. The instructions contained in the outline for the youngsters, the hypotheses made regarding the reactions and behaviour of the students, the reactions of colleagues, the difficulties that these may encounter in the work: all of these are important elements for understanding what a laboratory session is or should be. It does not matter then whether the teachers limit themselves to using laboratory kits prepared by others, because – as experience has taught us – having understood 'how it works' impedes an acritical use of the various proposals and increases the desire to intervene directly in the construction of integrations and/or completely new additions. In this second phase there clearly emerge a series of intimately connected questions that arise from the observation of the students' work in the laboratory. We will illustrate these in what follows.

During the laboratory: the role of rigour

The laboratory appears precisely as the apotheosis of substantial rigour and the annulment of formal rigour. Elio Fabri, speaking about the teaching of physics, writes:

> ... the criterion for rigour is not that of the exact definitions in the first chapter of a book. Rigour signifies clarity in the meaning of the individual steps, it means saying explicitly that the concepts are specified as one gradually goes forward, that the validity of principles and theories is reinforced when one sees the entire significance, that there are no single laws proven by single experiments, but that the entire construction stands as a whole, and that the whole is confirmed by the facts. Obtaining that the student understands and remembers all of this is more important than individual notions, rules, experimental data. This takes time, but it is time well spent, even if one must sacrifice some part of the traditional treatment. To convince ourselves of this we need only have the honesty to ask ourselves how much of what is done in a course with the pretext of completeness is effectively remembered, even after only a year, by the average student: one will necessarily arrive at the conclusion that completeness without clarity of comprehension is wasted effort. Naturally, this does not mean that notions, rules, experimental data must not be known and employed: but that they should be known and employed in view of an aim that is quite precise and not as an end in themselves (see [10], our trans).

Even though many years have passed since their publication, Fabri's words are still valid today; we can't speak for physics, but they are certainly true for mathematics. The often disconcerting outcomes that are achieved when notions are presented to students (including university students in the sciences) regarding fundamental themes with which they should have greater confidence are only a signal of the gap that exists between teaching and learning, an alarm signalling the evident lack of awareness of the work that is carried out by the students.

It is precisely the act of focussing attention on the problem to be solved – especially if that problem is absorbing, either due to its very nature, or because discussing it and grappling with it together with others has made it so – that leads in a natural way to concentrating on the actual difficulties, on the conceptual node about which the problem is constructed. Then technique will enter as well – obviously in the laboratory, this represents a necessary stage (see [7]). The youngsters themselves are the first to see that technique, knowingly used, is sometimes exactly what makes it possible to take a step forward in the comprehension and management of a problem. However, such a technique is never simply an artifice, or an end in itself, one that can always be used in the same way and seems conceived to 'train' a circus performer than to teach a concept.

What gives strength to this situation, and establishes the foundations for a learning that lasts, is the fact that rigour is never separated from meaning: it is obviously very different for the student to take possession of a technique that he himself has chosen as most suitable for controlling of a certain situation (taking advantage of the occasion to refresh his knowledge of its function, which he might not recall perfectly), than it is for him to see in this technique merely an artful ad hoc use in exercises or problems.

It is precisely through laboratory activity that it becomes possible to melt away, in a natural manner, some of the inconsistencies that might otherwise disconcert our students. Rigour is never an absolute concept, but is something which one approaches by successive approximations. This means, among other things, that the answer to a question such as, 'is it rigorous enough or not?' leads to a comparison between the statement in question and the context in which one is working. Inevitably we communicate to the students, consciously or unconsciously, information regarding rigour that is different, and sometimes even contradictory: there is a time when all commas must be in their proper place, and there is a time when it is necessary to understand, more or less, 'how it works'. There is nothing wrong with this inconsistency, which is effectively a reflection of the various stages that exist when we try to understand, acquire, systematise, and communicate mathematics. However, it is necessary that the 'rules' are agreed upon with the students, and that the inconsistencies are explicit. If there are only traces of them but they are not clear, then we only communicate bewilderment (not to mention when this is interpreted as something that penalises the errors of the students and pardons those of the teacher!).

The laboratory also appears to be a privileged place for exploring the possibilities and the difficulties that are tied to language. We know full well that the technical language of mathematics can constitute a difficulty, and is sometimes truly insurmountable for many youngsters. There are concrete difficulties here that the teacher can neither cancel nor ignore, in the attempt to make life easier for his students. However, he can – indeed, he must! – seek to arrive at the heart of the obstacle, so that the actual effort is spent on the genuine difficulties and not on artificial technicalities. It is precisely for these reasons that the laboratory is a particularly suitable context for addressing such problems: laboratory activities lead naturally to focussing attention on difficulties of substance, and of sense, and render it spontaneous to not be distracted by difficulties of form.

We often use terms whose meanings are taken for granted, but equally often we become aware of how necessary it is to repeat concepts over and over, dwelling on the meaning and the current use of the term in order to compare it in a second moment with the meaning that the term has in the specific language of mathematics.

It often occurs in the laboratory that we begin with a familiar language, with the use of terms that come from everyday language, and which seem apt for describing even the mathematical situation we find ourselves in, but then we pass to a rigorous language when we see that, in order to understand each other and to communicate, it is necessary to establish the meaning of the term in that context.

As Domenico Luminati and Italo Tamanini write:

> In the case of mathematics, a fundamental role in comprehension is played by the formalisation of the language to express it. Intuition, imagination and common sense, without an adequate dose of formalism, can easily lead to a wrong turn and making mistakes (see [12], our trans.).

During the laboratory: the role of error

The laboratory is the reign of informal mathematics, the one we do in overalls and not in jacket and tie. Naturally, the role played by error is also quite different here as we work together to understand something, with respect to when we are trying to clean up, systematise, formalise and communicate what we have learned. There are obvious reasons why we make greater use of error during the first phase than the second, but there is also a more intrinsic motive: during the phase of research, errors are useful, indeed, they are precious!

This is a fact that the teacher begins to deal with as soon as he begins to study the possibility of having his students work in a laboratory. More generally, from the very beginning he must come to grips with the role that can be usefully assigned to the informal level of learning in mathematics. Everyone agrees that informal learning can constitute a fundamental prerequisite for any later, more formalised acquisition of knowledge, but is that all? In reality it is necessary to go a step further, and determine what successive choices are consistent with these premises. If we hold that the informal phase is a useful and necessary first step, then it must be made clear to the students as well, as part of the educative process, that error (at least in this phase) constitutes a normal tool for knowing, something that can occur not only within a group of students, but also to the teacher who is playing along with them (or pretending to play along with them), something that needs to be focussed on in order to use its potential to fullest advantage for learning. Viewed in this way, error is a valuable resource. It is fundamental to create for the learner a space in which there is no fear of making a mistake, which is one of the most powerful enemies of learning (for a more detailed analysis of the theme of error, see, for example, [8]).

In our opinion, this attitude is so fundamental that it might even constitute a 'definition' of the laboratory: a type of activity where making mistakes is not only permitted, it is necessary; where error is neither repressed, nor erased, but is encouraged and discussed in order to profit as much as possible from the analysis of it, leading us to work with the mechanisms of reasoning, both ours and those of our students. This also leads us suggest that laboratory activity in itself should NOT be the direct subject of evaluation, but should be seen as the basis of learning, the outcomes of which – if such are required – can then rightly be the object of valuation. We believe this to be the only way to prevent students from falling back on mechanisms that are the fruit of the fear of making mistakes.

During the laboratory: the role of discussion

Another characteristic of laboratories is the fact that youngsters, working in a group, are 'forced' to discuss ideas with each other and to make an effort to express their views of a given problem to their companions. This is a fundamental point: as a vast number of studies have pointed out, there is an actual lag between the moment of comprehension of a given concept or problem and the moment in which that concept can be considered to have been acquired, that is, to the point where the learner is capable of narrating it, to himself and to others.

This lag between comprehension and acquisition is not so much just a temporal lag, as it is a lag in substance: in fact, the passage from one level to the next is not at all automatic, and thus it is necessary for the teacher to provide *ad hoc* activities to help each student acquire this knowledge. The discussions within the small group force the students to participate in these activities of metacognitive consolidation, which they otherwise might consider useless and even boring, especially for those who think, 'anyway I've already understood how it works'.

It should be noted that, outside of a laboratory setting, it truly rarely happens that students have an occasion to talk to each other about topics in mathematics; communication (about mathematics) is almost never 'between peers', but is almost always communication between those who know and those who don't. All the aspects that we have discussed here – rigor, language, error – assume very different values in communication between peers and communication between those who are on different levels. In peer communication, it is first of all necessary to understand each other. It is precisely this necessity for mutual understanding that leads to putting the accent on rigour of substance, and searching for an unambiguous language; this in turn leads in a very natural way to understanding that an error (appropriately discussed) can sometimes allow us to take a step forward in the comprehension of a problem.

Above all, informal discussion between peers leads the students to 'understand that they have understood', if they have understood. This is not a mere play on words: how many times have our students given us the impression of having more or less acquired a given concept, while they themselves do not see clearly that they have acquired it! Sometimes (and this aspect is strongly evident is some situations, for example, with students in the degree program for the Science of Primary Education), the profession of teaching consists not so much in presenting new concepts, as in making the interlocutor aware that he already knows it.

Naturally, in this regard, a fundamental stage of the laboratory is the final one in which the entire class, guided by the teacher, sums up the work carried out. It is in this phase that the teacher can perform a valuable service, by making evident those moments which have constituted the most significant passages, discussing the differences between the approaches taken by the different groups so that the students acquire in practice the idea that there is never a single route to solving a problem. At the same time light can be shed on both the *quid* that allowed them to make a qualitative leap in the direction of abstraction, and on the role played by technique, for example, a good notation that made it possible for them to easily manage a problem that would otherwise have seemed extremely difficult.

During the laboratory: concrete and abstract; real and virtual

The laboratory (not only of mathematics, but also of the more common laboratories of physics and other scientific disciplines) is often identified with a room apart, different from the normal school classroom, where there are objects to manipulate and experiments can be made. We said at the beginning that objects of this sort can be useful, although they do not in themselves characterise laboratory activities for mathematics. The use of concrete objects to represent examples of abstract problems and concepts in mathematics is quite a delicate issue, one with enormous potential, but which should also be used with a degree of caution.

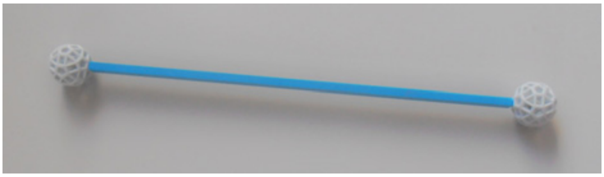
Meanwhile, not all problems and arguments lend themselves to modelling through the use of concrete objects, and on the other hand, this turns out to be useful (indeed, valuable) only when it grows naturally out of a (realistic!) contextualisation of the problem: artificial examples (stemming from a desire to interject the concrete at any cost) turn out to be not only useless, but detrimental. Think of those textbooks that propose finding the volume of a saucepan in the form of a triangular prism with a base whose side measures one metre, or a umbrella holder that is a cylinder whose opening is in the shape of a quadriangular pyramid. To be sure, if this is concrete, let's stay in the abstract!

Further, even in cases where a concrete modelling is particularly suited to illustrating the problem being discussed, it is necessary to keep in mind that the concrete object is never the abstract concept that it intends to illustrate. Although this statement may seem trivial, we believe it is important to reiterate it, because we (first of all!) tend to forget it, not because we don't know it, but because the clearer the concept (for example, a sphere) is in our minds, the more we project it onto the concrete object used to represent it (a soccer ball), sometimes to the point where we don't even see the concrete attributes (material, weight, colour, the seams between pentagons and hexagons that make up its faces, etc.), but only see the idealisation of the concrete attributes that it represents. This obviously cannot and must not happen to someone who doesn't know the abstract concept that we wish to represent with that concrete object (see [6]).

This experience of 'seeing only the ideal' is a little like when we take a photograph because a certain thing (for example, the flight of a seagull) has struck us as being perfectly beautiful: we don't even notice that the seagull is too small, or too far away, or even that there is an ugly rubbish bin in the foreground; nor are we aware of what chord in our imagination has been struck so that what we saw was so beautiful and important that it filled our entire visual field. However, when we compare the mental visual field of our imagination to the real visual field of the photo we have taken, we see that the photo is really terrible, because it contains the many things we hadn't noticed, which are the things that anyone else who didn't have our reasons for seeing only the seagull would have seen.

We should point out that misunderstandings of this kind (applied to the communication of mathematics via concrete objects) occur more easily in (NON-peer) communication between teacher and students, that is, in the communication between the one who already has in mind a mental image of the abstract concept to which the model refers, and the one who on the other hand is constructing it. Analogous misunderstandings occur less frequently in the context of the communication between peers that is the norm in the laboratory.

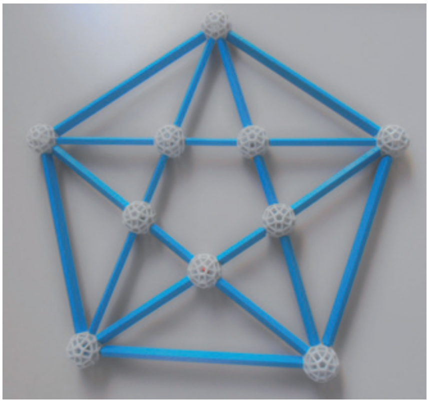
It is primarily up to the teacher to bear this problem in mind when he is choosing the material that will accompany the laboratory activity. For example, we already know what a segment is, and thus we don't hesitate at all to see a segment in the object shown below:

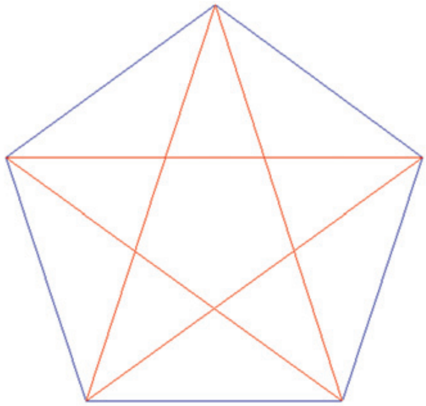


We might not even pay any mind at all to the characteristics that prevent this object from actually being a segment. Instead, we must remind ourselves of this when we decide on the materials that might work in a given laboratory setting: it might make sense, for instance, to think of using this kind of material with older students in secondary school, but prefer other materials for primary school children.

Finally, it is necessary to always bear in mind that the objective of laboratory activity is the acquisition (more or less solid depending on the age of the student and the type of problem) of a certain abstract concept. It is thus necessary to take care that the manipulation of concrete objects is configured as a tool aimed at giving substance and meaning to the concept we are constructing, and not one that instead risks substituting the abstract concept itself (see, in this regard, Russo's polemic in [13]). On the other hand, when such care is taken, it is precisely the acquisition of the abstract concept that benefits from the manipulation of concrete objects.

For example, let us think of the dynamics between abstract and concrete that arise when we ask (with students old enough, in secondary school) what we might mean by 'length of a segment' in reference to a material object such as that shown on the left:





It is a fine discovery when we see that the material object (which is constructed in quite a refined way, from a mathematical point of view) presumes that the 'length' of a given stick signifies the distance between the centres of the two small balls attached to the ends of the stick itself. This notion of length is precisely what allow us, when we have sticks whose lengths are in certain ratios (for example, the golden ratio), to carry out with the concrete objects the geometric constructions that can be made with corresponding abstract segments (shown on the right).

Similarly, interaction with what different technologies place at our disposal today is – like the manipulation of concrete objects – a resource that can be useful in a laboratory activity, but again does not necessarily characterise it. There can be laboratories that use technology and its virtual features, and there can be laboratories that don't use such tools, just as there are ways of interacting with technology in the classroom that constitute laboratory-like activities, and others that have nothing at all in common with a laboratory. In order to be able to speak of laboratory activities – we mentioned earlier that we consider this a defining characteristic – it is necessary that the role of youngsters in interacting with the virtual is active, not passive; we might even say 'really' active, to underline the fact that it is not sufficient for the student in front of a screen to merely click and then watch when something pre-programmed happens that he can only observe. Rather, a genuinely interactive context occurs when the animation provides a kind of environment in which the user can carry out experiments on his own and save the results. It is naturally even better when the students can themselves create virtual objects.

After the laboratory: evaluation

We have already presented the reasons why we believe that laboratory activity should not be used as the criterion for evaluating individual students. Too often the Italian school is one reduced to its job of evaluating; instead, we feel that school should remain before all else a school that teaches!

Instead, it's a different story regarding the evaluation of the laboratory itinerary carried out with the class as a group: this is an element of great importance that should be addressed, making use of the instruments that make it possible to measure the effectiveness of that itinerary. We will not go into detail here as to how such instruments are constructed, but will underline some of the questions that should be borne in mind in their preparation.

First of all, any kind of evaluation must necessarily provide for a 'measure' of effectiveness in the medium- or long-term. In fact, it makes no sense at all to evaluate an element that is too narrowly delimited, both because learning in the short-term is inevitably fragile, and because all that we have discussed regarding the laboratory situation points instead to a overall vision of the itinerary.

Another element that should be pointed out is the opportunity (or necessity?) for collective discussion among teachers in order to remain mutually up to date regarding a laboratory activity and to evaluate the effectiveness of the work carried out. In our experience it in fact happens rather frequently that a discussion of this kind generally leads a number of the teachers to change their point of view, sometimes radically, perhaps because it makes them aware of some of the aspects that are more difficult to quantify which they were not conscious of at first. By way of example, we can cite the level to which students themselves assume responsibility for what they learn, which is obviously crucial for learning to become fixed and grow over time, and which is surely one of the privileged aspects of laboratory activities.

Conclusions

As we mentioned in the introduction, the themes discussed in this article grew out of ten years of activity that the 'matematita' Centre has carried out with and for schools. We think it is helpful at this point to provide some statistics:

- the laboratories which have taken place in the Department of Mathematics of the Università degli Studi di Milano have involved more than 1,000 classes, for a total of more than 25,000 students and more than 2,000 teachers;
- the games offered online by the website 'Quaderno a quadretti' (http://www.quadernoaquadretti.it) have involved between 3,500 and 4,500 primary school and secondary school classes, for a total of more than 100,000 students and between 1,500 and 2,000 teachers;
- the laboratory kits loaned to schools (an activity that began only five years ago), have involved more than 500 classes, for a total of more than 12,000 students and some 1,000 teachers.

These are the numbers (to which are added our experience with the numerous training courses for teachers offered during these years) that add weight to the positive – sometimes really enthusiastic! – feedback of the many, many teachers who have tried out these activities, and who have testified, over a distance of years, to the fact that youngsters who have been exposed to itineraries of this kind encounter many fewer difficulties during national examinations than their companions who have not had this advantage.

Obviously, the laboratory method is not the only way to teach mathematics, but at this particular moment in time, it shows itself to be a particularly effective way to do it.

BIBLIOGRAPHY

[1] M. Cazzola, *L'insegnamento della matematica: una didattica meta-cognitiva,* in O. Albanese, P.A. Doudin, D. Martin (a cura di), *Metacognizione ed educazione,* Franco Angeli, Milano, 2003.

[2] M. Cazzola, *Problem-based Learning and Mathematics: Possible Synergical Actions*, in L. Gòmez-Chova, D. Martì Belenguer, I. Candel Torres (editors), *Proceedings of ICERI 2008 Conference,* (Madrid, 2008), Valencia, IATED.

[3] M. Cazzola, *Problem-based Learning and Teacher Training in Mathematics: the Role of the Problem*, in M. Tzekaki, M. Kaldrimidou, H. Sakonidis (editors), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education* (Salonicco, 2009).

[4] E. Colombo - M. Dedò, *Uguali? Diversi! La matematica delle classificazioni*, Mimesis, Milano, 2013. ISBN 978-88-5751-744-5

[5] M. Dedò, *Più matematica per chi insegna matematica*, La Matematica nella Società e nella Cultura, *Bollettino dell'Unione Matematica Italiana,* (8), 4-A, 2001, pp. 247-275.

[6] M. Dedò, *Matemática informal: ¿una contradicción?*, in R. Mallavibarrena (editor), *Escuela de Educación Matemática "Miguel de Guzmán": enseñar divulgando*, Secretaria General Tecnica, 2010, pp. 47-72. ISBN 978-84-369-4937-7.

[7] M. Dedò, *Rigour in Communicating Maths: a Mathematical Feature or an Unnecessary Pedantry?*, in E. Behrends, N. Crato, J.F. Rodrigues (editors), *Raising Public Awareness of Mathematics*, Springer, 2012, pp. 339-358.

[8] M. Dedò - L. Sferch, *Right or Wrong? That is the Question*, Notices of the Amer. Math. Soc., vol. 59, n. 7, 2012, pp. 924-932, http://www.ams.org/notices/201207/rtx120700924p.pdf

[9] S. Di Sieno, *"Doing Mathematics": a Crucial Step in Mathematical Teachers Training,* in Jianpan Wang, Binyan Xu (editors), *Trends and Challenges in Mathematics Education*, East China Normal University Press, Shangai, 2004, pp. 365-375.

[10] E. Fabri, *Che cos'è il rigore logico in fisica?,* in AAVV, *Guida al laboratorio di fisica,* Zanichelli, Bologna, 1995, http://www.df.unipi.it/~fabri/rigore/rigore00.htm

[11] I. Kra *(Math) Teachers are the Key*, Notices AMS, vol. 59, n. 4, 2012, pp. 556-557, http://www.ams.org/notices/201204/rtx120400556p.pdf

[12] D. Luminati - I. Tamanini, *La matematica a casa tua,* L'insegnamento della matematica e delle scienze integrate, vol. 28 A n. 1, 2005, pp. 33-60.

[13] L. Russo, *Segmenti e bastoncini*, Feltrinelli, Milano, 1998.

[14] J.R. Savery, *Overview of Problem-based Learning: Definitions and Distinctions*, The Interdisciplinary Journal of Problem-based learning, vol. 1, n. 1, 2006.

[15] G. Spirito, *A proposito di metodologie, a proposito di contenuti* in http://www.quadernoaquadretti.it/scuola/riflessioni/spirito_08.pdf

[16] G. Vailati, Recensione di *C. Laisant. La mathématique: philosophie, enseignement,* in "Giovanni Vailati. Scritti", a cura di M. Quaranta, Forni, Bologna, 1987.

[17] H. Wu, *The Mis-education of Mathematics Teachers,* Notices of the Amer. Math. Soc., vol, 58, n. 3, 2011, http://www.ams.org/notices/201103/index.html

Maria Dedò - Simonetta Di Sieno
Dipartimento di Matematica, Università degli Studi di Milano
Via Cesare Saldini 50, Milano
E.mail: maria.dedo@unimi.it simonetta.disieno@unimi.it