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Categories of results in variable Lebesgue spaces theory

Nota del socio Alberto Fiorenza[1,2]

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Abstract – Variable (exponent) Lebesgue spaces represent a relevant research area within the theory of Banach function spaces. Much attention is devoted to look for sufficient conditions on the variable exponent to establish the assertions within the theory. In this Note we try to show the beauty of the research in this field, mainly quoting some known results organized into "categories", each of them characterized by a common typology of conditions on the variable exponent. New results involve the failure of rearrangement-invariant property, the rearrangement of the exponent, and a generalization of a formula known for constant exponents.

Riassunto – Gli spazi di Lebesgue con esponente variabile rappresentano un settore di rilievo nell'ambito della teoria degli spazi funzionali di Banach. Di notevole interesse é la ricerca di condizioni, da imporre alla funzione esponente, sufficienti ad assicurare il verificarsi di determinate affermazioni. In questa Nota ci proponiamo di mostrare il fascino della ricerca in questo settore, segnalando essenzialmente alcuni noti risultati organizzati in "categorie", ognuna delle quali caratterizzata da una comune tipologia di condizioni sulla funzione esponente. I risultati originali sono relativi alla non invarianza per riordinamento, al riordinamento dell'esponente e ad una generalizzazione di una formula nota per esponenti costanti.

1 - A SHORT HISTORY OF VARIABLE LEBESGUE SPACES

The introduction of a number of topological spaces, among which some families of Banach function spaces, contributed indubitably to the development of

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[2]This Note contains an updated and enlarged revision of the introduction of the talk, given by the author, at the AMS-EMS-SPM International Meeting 2015, 10-13 June, Porto, Portugal, entitled *On "essentially variable" variable Lebesgue space problems.*

the Functional Analysis. The continuous need to improve and refine mathematical models gave the opportunity to extend to more general frameworks several classical results of Mathematical Analysis. In this order of ideas a fruitful research area is that one of variable exponent Lebesgue spaces (variable Lebesgue spaces, in short), actually available through several references (see, for instance, Antontsev and Shmarev [10], Cruz-Uribe and the author [27], [31], Diening, Harjulehto, Hästö and Růžička [39], Edmunds, Lang and Méndez [46], Izuki, Nakai and Sawano [72], Kokilashvili, Meskhi, Rafeiro and Samko [76, 77], Lang and Edmunds [85], Meskhi [95], Pick, Kufner, John and Fučík [104], Rădulescu and Repovš [106], Růžička [111]).

Strictly speaking, variable Lebesgue spaces were born in a paper by Orlicz [103] in 1931, where in a remark about functions f such that $|f(x)|^{p(x)} \in L^1(0,1)$, $1 < p(x) < \infty$, he established essentially a kind of sharpness of Hölder's inequality. However, Orlicz is actually known for the spaces called Orlicz spaces, which he also introduced in 1931 in a joint paper with Birnbaum [16] (for the early history of these spaces, see Krasnosel′skiĭ and Rutickiĭ [84]; for treatments on Orlicz spaces, see e.g. also Maligranda [91], Rao and Ren [108], Harjulehto and Hästö [68]).

An important step in the development of the variable Lebesgue spaces came two decades later in the work of Nakano [98, 99] who originated the theory of modular spaces, sometimes referred to as Nakano spaces. A modular space is a topological vector space equipped with a "modular": a generalization of a norm. An important example of a modular space is the function space consisting of all functions f on a (Lebesgue) measurable set $\Omega \subset \mathbf{R}^n$ (in the following we will assume that Ω has positive measure) such that for some $\lambda > 0$,

$$\int_\Omega \Phi\left(x, \frac{|f(x)|}{\lambda}\right) dx < \infty,$$

where $\Phi : \Omega \times [0,\infty) \to [0,\infty]$ is a function such that for almost every $x \in \Omega$, $\Phi(x,\cdot)$ behaves like a Young function (i.e., roughly speaking, a convex function whose graph has the shape "similar" to that one of a power having constant exponent greater or equal than 1). These spaces are referred to as Musielak-Orlicz spaces or generalized (or variable) Orlicz spaces (see e.g. Musielak [97], Harjulehto and Hästö [68]). They contain a number of function spaces as special cases. If $\Phi(x,t) = \Phi(t)$ is just a function of t, they are the Orlicz spaces, and if $\Phi(x,t) = t^p w(x)$, they become the weighted Lebesgue spaces. In [98], Nakano introduced the variable Lebesgue spaces as specific examples of modular spaces: if $\Phi(x,t) = t^{p(x)}$, where $1 \le p(x) < \infty$ is a measurable function on Ω, they are the variable Lebesgue spaces $L^{p(\cdot)}(\Omega)$, which are therefore defined as the set of all measurable functions f on Ω such that for some $\lambda > 0$,

$$\int_\Omega \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx < \infty. \tag{1}$$

$L^{p(\cdot)}(\Omega)$ becomes a *Banach function space* (i.e. a Banach space, whose elements are measurable functions and whose norm verifies some further conditions; see e.g. Bennett and Sharpley [15]) when equipped with the Luxemburg norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}. \tag{2}$$

When $p(\cdot) \equiv p$, p being a constant greater or equal than 1, then $L^{p(\cdot)}(\Omega) = L^p(\Omega)$ and (2) reduces to the classical norm on $L^p(\Omega)$. This notion can be simply adapted to allow $p(x) = \infty$ for a.e. x in a set $\Omega_\infty \subset \Omega$ (see Kováčik and Rákosník [83]), replacing the left hand side in (1) (and the corresponding term in (2)) by

$$\int_{\Omega\setminus\Omega_\infty} \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx + \left\|\frac{f(x)}{\lambda}\right\|_{L^\infty(\Omega_\infty)}.$$

A different way to allow the case $p(x) = \infty$ on some subset of Ω, nicer from the formal point of view, has been suggested in Diening [38] (see also Diening, Harjulehto, Hästö, Mizuta and Shimomura [44]): the positions (1) and (2) *can* remain as they are, the trick is just to make the

$$\text{convention}: \quad t^{+\infty} = \begin{cases} 0 & \text{if} \quad 0 < t \le 1 \\ +\infty & \text{if} \qquad t > 1 \end{cases} \quad ; \tag{3}$$

the resulting norm is slightly different, but the resulting space $L^{p(\cdot)}(\Omega)$ is the same, up to equivalence of norms.

The variable Lebesgue spaces appeared independently in the Russian literature, where they were studied as spaces of interest in their own right. They were introduced by Tsenov [123] in 1961, in the study of a minimization problem. In 1979, Sharapudinov [119] begun to develop the function space theory of the variable Lebesgue spaces on intervals on the real line, introducing the Luxemburg norm (drawing on ideas of Kolmogorov [78]), and showing that when $p(\cdot)$ is bounded, $L^{p(\cdot)}([0,1])$ is separable and its dual space is $L^{p'(\cdot)}([0,1])$, where $p'(\cdot)$ denotes the variable exponent $p'(\cdot) = p(\cdot)/(p(\cdot)-1)$. In [120] he was the first to consider questions that involved the regularity of the exponent function $p(\cdot)$, and introduced the local log-Hölder continuity condition,

$$|p(x) - p(y)| \le \frac{C_0}{-\log(|x-y|)}, \qquad \forall x,y : |x-y| < \frac{1}{2}, \tag{4}$$

that has proved to be of critical importance in the theory of variable Lebesgue spaces. The most influential work is due to Zhikov, who, starting from [124], begun to apply the variable Lebesgue spaces to problems in the Calculus of Variations.

The "modern" period in the study of variable Lebesgue spaces begun with the foundational paper of Kováčik and Rákosník [83] from 1991.

In the early 1990's, Samko and Ross [118, 109] (see also Samko [113, 116]) introduced a Riemann-Liouville fractional derivative of variable order and the corresponding variable Riesz potential. Investigating the behavior of these operators led naturally to the study of convolution and potential operators on the variable Lebesgue spaces: see Samko [114, 115] and Edmunds and Meskhi [48].

In the mid 1990's, functionals with non-standard growth and the $p(\cdot)$-Laplacian were studied by Fan [54, 55], Fan and Zhao [58, 59], Marcellini [92, 93].

Partial differential equations with non-standard growth conditions have been considered, among others, in Fan [56], Harjulehto, Hästö, Lê and Nuortio [69], Mingione [96], Antontsev and Shmarev [10].

Interest in the variable Lebesgue spaces has increased since the 1990's because of their use in a variety of applications. Foremost among these is the mathematical modeling of electrorheological fluids, namely, fluids whose viscosity changes when exposed to an electric field: see Růžička [111, 112], Diening and Růžička [40, 43, 41, 42], Acerbi and Mingione [1, 2, 3].

The variable Lebesgue spaces have also been used to model the behavior of other physical problems. Some examples include quasi-Newtonian fluids (see Zhikov [125]), the thermistor problem (see Zhikov [126]), fluid flow in porous media (see Amaziane, Pankratov and Piatnitski [6], Antontsev and Shmarev [11]), magnetostatics (see Çekiç, Kalinin, Mashiyev and Avci [22]), and the study of image processing (see Blomgren, Chan, Mulet and Wong [17]).

2 - A LIST OF CATEGORIES

Roughly speaking, from the pure mathematical point of view, the attempts to generalize known classical statements to the variable exponent context lead in a natural way to the problem of looking for sufficient conditions on the variable exponent to establish the assertions within the theory. In the following we will quote some known results organized into "categories", each of them characterized by a common typology of conditions on the variable exponent; their number shows the richness of the phenomena encountered by researchers in the theory.

Before the beginning of our presentation, we recall that a kind of exposition into categories appears in Section 1.3 of the book by Diening, Harjulehto, Hästö and Růžička [39]; even if the present Note has some overlap with such reference, the reader here finds a discussion on the results considered and a finer classification, rather than just few collections of results.

This Note is based on the following observation: statements in terms of classical Lebesgue spaces (hence, say, true for constant exponents) may

(i) ... remain true for variable exponents and the extension is trivial

(ii) ... remain true for variable exponents but the proof needs some more effort

(iii) ... remain true only for *certain* variable exponents

(iv) ... are *never* true when exponents are not constant

The theory of variable Lebesgue spaces admits also another category of results, which do not come from the classic theory and their main feature is based specifically on the variability of the exponents:

(v) ... have no interest when exponents are constant

In Section we will shortly describe each of the above categories: we will treat ***(i)-(v)*** in paragraphs 4.1-4.5, respectively. For the completeness of the exposition, we need few prerequisites, which are the object of next Section .

3 - SOME STATEMENTS INVOLVING CLASSICAL LEBESGUE SPACES

For further needs, let us recall few definitions and results. Lebesgue spaces are part of standard knowledge in Mathematical Analysis and are presented in many textbooks and tracts, see e.g. Brezis [18, 19], Castillo and Rafeiro [21], DiBenedetto [35], Okikiolu [102], Pick, Kufner, John and Fučík [104], Rudin [110]. We will use also the notion of decreasing rearrangement, which is e.g. in Bennett and Sharpley [15], Kawohl [73], Korenovskii [81], Leoni [86], Rakotoson [107].

The well known formula (see e.g. DiBenedetto, p.149 [35], Stein p.7 [121], Lieb-Loss p.26 [90], Ambrosio, Fusco and Pallara p.34 [7], Okikiolu p.236 [102])

$$\int_\Omega |f(x)|^p\,dx = p\int_0^\infty t^{p-1}|\{x\in\Omega\,:\,|f(x)|>t\}|\,dt \tag{5}$$

shows that classical Lebesgue spaces are rearrangement-invariant Banach function spaces (see e.g. p.59 in Bennett and Sharpley [15]): in fact, the key property needed to define these spaces is that the norms of every pair of functions f,g equimeasurable, i.e. such that

$$|\{x\in\Omega : |f(x)|>t\}| = |\{x\in\Omega : |g(x)|>t\}| \qquad \forall t\geq 0\,,$$

coincide. As a further consequence of this formula, the norm of every function f in every rearrangement-invariant Banach function space coincides with that one of its decreasing rearrangement, which we denote by f_*. The importance of this class of spaces is the characterization given through the fundamental interpolation theorem: on one hand every rearrangement-invariant Banach function space is an interpolation space between L^1 and L^∞, and on the other every Banach function space which is an interpolation space between L^1 and L^∞ is rearrangement-invariant. Being interpolation space implies, in turn, the boundedness of a wide class of operators acting on them (see e.g. Ch.3 in Bennett and Sharpley [15] for details).

However, even without involving directly the machinery of interpolation theory, (5) can be used directly for proving a result which is a milestone in real-variable Harmonic Analysis. Given a function $f\in L^1_{loc}(\mathbf{R}^n)$, the (uncentered) Hardy-Littlewood maximal function Mf is defined by

$$Mf(x) = \sup_{Q\ni x}\frac{1}{|Q|}\int_Q |f(y)|dy \qquad \forall x\in\mathbf{R}^n\,,$$

where the supremum is taken over all cubes $Q \subset \mathbf{R}^n$ containing x and whose sides are parallel to the coordinate axes. If $f \in L^1_{loc}(\Omega)$, then Mf is defined by extending f to be identically 0 on $\mathbf{R}^n \setminus \Omega$. The following boundedness result holds (see e.g. Stein [121, 122]):

Theorem 1. *If $f \in L^p(\mathbf{R}^n)$, with $1 < p \leq \infty$, then $Mf \in L^p(\mathbf{R}^n)$ and*

$$\|Mf\|_{L^p(\mathbf{R}^n)} \leq C\|f\|_{L^p(\mathbf{R}^n)} \tag{6}$$

where C depends only on p and the dimension n.

It would be impossible to describe in few lines the importance of this classical result, which is a fundamental tool for proving, sometimes in a direct way, the boundedness of many operators in Harmonic Analysis (see e.g. Cruz-Uribe, Martell and Pérez [33], Duoandikoetxea [45], Kokilashvili and Krbec [75], Kokilashvili, Meskhi, Rafeiro and Samko [76, 77], Stein [121, 122]); for our goals we observe that since it provides an alternative proof of the classical Sobolev inequality and its consequences (see e.g. Ziemer [128]), its versions for other Banach function spaces, including variable Lebesgue spaces (see e.g. next Theorem 3), became a tool for extensions of the Sobolev inequality and several other classical results.

Incidentally, even if it is not among the goals of this Note, we notice that Theorem 1 is someway involved also to get packing results in Geometry of fractal sets (see Section 7.5 p. 109 in Falconer [53]).

4 - THE CATEGORIES IN DETAIL

The straightforward generalizations

When making research in Mathematics, sometimes the generalizations of theorems can be proved without any effort: one discovers that a certain argument can work “as it is" because it has in fact a greater validity. Of course formally such new theorems are “better" than the original ones, however, they can be classified as simple remarks, because no new ideas are needed to get the proofs. This is the case, for instance, of the proof of the completeness of variable Lebesgue spaces:

Theorem 2. *If $\Omega \subset \mathbf{R}^n$ and $p(\cdot) : \Omega \to [1, \infty]$ is a measurable function, then $L^{p(\cdot)}(\Omega)$ is complete: every Cauchy sequence in $L^{p(\cdot)}(\Omega)$ converges in norm.*

The proof of Theorem 2 can be found, for instance, in Cruz-Uribe and the author [27] (see Theorem 2.71 therein). It is nice to observe that formally replacing $p(\cdot)$ by p in any line of this reference, one gets the original proof which works for classical Lebesgue spaces.

This kind of results occupies a minor part of the theory of variable Lebesgue spaces; the proofs which can be written almost automatically have, of course, not so much interest.

The case of longer proofs

The existence of much literature involving variable Lebesgue spaces (see e.g. the enormous references list in the book by Cruz-Uribe and the author [27] or in the book by Diening, Harjulehto, Hästö and Růžička [39]) is justified by the fact that the majority of the results cannot be proved as Theorem 2. Maybe the first reason is that the formal substitution of p into $p(x)$ does not work already since the beginning of the theory, namely, with the expression of the norm: the "transformation"

$$\left(\int_\Omega |f(x)|^p\,dx\right)^{\frac{1}{p}} \quad\to\quad \left(\int_\Omega |f(x)|^{p(x)}\,dx\right)^{\frac{1}{p(x)}} \tag{7}$$

does not give, as "output", a number, but a function. Note also that homogeneity is lost: multiplying, for instance, f by 2 in the expression on the right, one does not get the double.

However, the expression of the norm in (2) has been obtained using exactly the formal substitution of p into $p(x)$, which works fine if one, previously, writes

$$\left(\int_\Omega |f(x)|^p\,dx\right)^{\frac{1}{p}}$$

in the different form

$$\inf\left\{\lambda>0:\ \int_\Omega \left|\frac{f(x)}{\lambda}\right|^p dx \le 1\right\}.$$

This trick, which is successful also in the case of Orlicz spaces and more generally in the context of modular spaces (see e.g. Maligranda [91]), has a deep topological validity (see Kolmogorov [78]); however, it reveals once more the usual way to construct generalizations, which is to use some characterization and to "discover" that the equivalent form is adapt to a new framework. A much more known example in this sense is the notion of weak derivative to define Sobolev spaces (see e.g. Brezis [18]).

In some cases the characterization leads to a significant change of the expression which is just apparent. Generalizations follow using the same ideas of the classical argument: just some small extra effort is needed.

For instance, let $1<p<\infty$ and consider the classical Hölder's inequality

$$\int_\Omega |f(x)g(x)|\,dx \le \left(\int_\Omega |f(x)|^p\,dx\right)^{\frac{1}{p}} \left(\int_\Omega |g(x)|^q\,dx\right)^{\frac{1}{q}}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Its standard proof (see e.g. Brezis [18]) starts from the concavity of the logarithm, from which one gets the Young's inequality

$$ab \le \frac{1}{p}a^p+\frac{1}{q}b^q \qquad \forall a\ge 0,\ b\ge 0.$$

Then one replaces a with $|f(x)|$, b with $|g(x)|$, and integrate. The conclusion follows applying the previous argument to $\lambda|f(x)|$ and $\frac{|g(x)|}{\lambda}$, λ being a positive parameter, and finally choosing the "best" λ.

Looking at the corresponding result in the variable case (see e.g. Theorem 2.26 in Cruz-Uribe and the author [27]), when $1 < p(x) < \infty$ the method does not change: the Young's inequality is applied pointwise and therefore one gets

$$ab \leq \frac{1}{p(x)} a^{p(x)} + \frac{1}{q(x)} b^{q(x)} \qquad \forall\, a \geq 0\,,\ b \geq 0\,,\ x \in \Omega\,,$$

where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$ and, as before, one integrates over Ω.

It is clear that we are not in the same situation as in the previous paragraph: the proof does not work *word by word*, because integrating, now, one does not find immediately the norm. The fact that the expression of the norm is different makes the proof a little bit longer (see details in [27]), and some small extra effort must be spent to get the statement in the variable exponent case.

In some applications the effort to pass from the constant case to the variable case must be much higher, even when the exponent is relatively "nice". For instance, this may happen for exponents being continuous until the boundary of the domain: see e.g. El Hamidi [50], where existence results to elliptic systems involving the $p(x)-$laplacian are obtained.

The heart of the theory

Maybe the nicest feature of variable Lebesgue spaces theory is that the extension of results involving classical Lebesgue spaces does not hold for *all* variable exponents, but just for a *certain class* of exponents. It may be a hazard to make general assertions about mathematical theories, but, at least starting from our experience, this category of results includes the majority of the literature in this field. The main features of a result in this category are:

- The result holds for constant exponents
- The result does not hold for all possible variable exponents (and therefore an example is given)
- The result holds for *some* "really variable" (= not constant) exponents
- The class of exponents is studied: it is larger than the classes of exponents, previously known, involved in sufficient conditions for the validity of certain assertions, or it is smaller than the classes of exponents, previously known, involved in necessary conditions. The "full" result characterizes the class of exponents for which a certain assertion holds

A consequence of the features above is that several results do not admit just one variable version, but each of them may admit various "not full" extensions.

An entire “community" may work on the same assertion, producing results which may improve (or simply overlap with) already existing ones.

Let us draw our attention to an example of result in this category. We choose, because of its importance highlighted in Section , Theorem 1: the central problem is therefore that one to study conditions on an exponent $p(\cdot)$ so that the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)}(\mathbf{R}^n)$. The first major result in this direction was due to Diening [36], who showed that it is sufficient to assume that $p(\cdot)$ satisfies the local log-Hölder condition (4), is constant outside of a large ball, and, finally, is bounded (i.e. $\operatorname{ess\,sup} p(\cdot) < \infty$) and bounded away from 1 (i.e. $\operatorname{ess\,inf} p(\cdot) > 1$). This result was generalized by Cruz-Uribe, Neugebauer and the author in [29] (see also Capone, Cruz-Uribe and the author [20] for a simpler proof), where a nearly optimal sufficient condition on the exponent $p(\cdot)$ is given: it was shown that it is sufficient to assume (besides the boundedness and the boundedness away from 1) that (4) holds and $p(\cdot)$ is log-Hölder continuous at infinity, namely, there exist constants p_∞ and C_∞ such that

$$|p(x) - p_\infty| \le \frac{C_\infty}{\log(e+|x|)} \qquad \forall x \in \mathbf{R}^n\,. \tag{8}$$

A generalization which includes the constant case $p(\cdot) \equiv +\infty$ taking into account of the

$$\text{convention :} \quad \frac{1}{+\infty} = 0\,, \tag{9}$$

and involves the log-Hölder continuity both locally (see (4)) and at infinity (see (8)), is the following:

Theorem 3. *Given an open set $\Omega \subset \mathbf{R}^n$, if $p(\cdot) : \Omega \to [p_-, +\infty]$ is such that $p_- > 1$ and such that $1/p(\cdot)$ is log-Hölder continuous both locally and at infinity, then M is bounded on $L^{p(\cdot)}(\Omega)$:*

$$\|Mf\|_{L^{p(\cdot)}(\Omega)} \le C\|f\|_{L^{p(\cdot)}(\Omega)}. \tag{10}$$

The proof of Theorem 3 borrows ideas from several papers. Expositions, eventually with small variants in the assumptions, are e.g. in Diening, Harjulehto, Hästö, Mizuta and Shimomura [44] (see Theorem 1.2 therein), in Cruz-Uribe and the author [27] (see Theorem 3.16 therein), in Cruz-Uribe, Diening and the author [24], in Izuki, Nakai and Sawano [72], in Harjulehto and Hästö [68] (see Corollary 4.5.5 therein), in Diening, Harjulehto, Hästö and Růžička [39] (see Theorem 4.3.8 and Remark 4.3.10 therein). The condition $p_- > 1$ for the maximal operator to be bounded is known to be necessary: this was first proved in Cruz-Uribe, Neugebauer and the author [29, 30] with the additional assumption that $p(\cdot)$ is upper semi-continuous. This hypothesis was removed by Diening [38] (see also Diening, Harjulehto, Hästö, Mizuta and Shimomura [44]). The same references have to be quoted for the case when $\operatorname{ess\,sup} p(\cdot) = \infty$ and the idea of the requirement of the log-Hölder continuity imposed to $1/p(\cdot)$. A very different proof of this theorem when $\operatorname{ess\,sup} p(\cdot) < \infty$, gotten by viewing $L^{p(\cdot)}$

from the perspective of abstract Banach function spaces, was given by Lerner and Pérez [89]. Independently, Nekvinda [100] showed that it was sufficient to assume that $p(\cdot)$ satisfies a somewhat weaker integral decay condition. The log-Hölder conditions are the sharpest possible pointwise conditions (see Pick and Růžička [105] and Cruz-Uribe, Neugebauer and the author [29, 30]) but they are not necessary: see Nekvinda [101], Kopaliani [79] and Lerner [88]. Diening [37] has given a necessary and sufficient condition that is difficult to check but has important theoretical consequences. The importance of these results was reinforced by the work in Cruz-Uribe, Martell, Pérez and the author [28], Cruz-Uribe and Hästö [32], where it was shown that the theory of Rubio de Francia extrapolation could be extended to the variable Lebesgue spaces and generalized Orlicz spaces. This allows the theory of weighted norm inequalities to be used to prove the boundedness of a multitude of operators (such as singular integrals) whenever the maximal operator is.

Of course, especially when full results are missing, in plenty of papers which need the boundedness of the maximal operator, the authors assume it directly (just to quote an example, see Kopaliani and Chelidze [80], where a Gagliardo-Nirenberg inequality with norms having variable exponents is obtained). This solution is reasonable as soon as from the literature it is known at least a set of sufficient conditions for such boundedness, because this way, in principle, any future result giving sufficient conditions can provide a new good set of assumptions. Such policy appears frequently in literature. As a further example, some results in the 2002 paper by the author [61] (see Hästö and Ribeiro [71], Ferreira, Hästö and Ribeiro [60], Harjulehto and Hästö [68] as other references dealing with the same topic) hold assuming the density of smooth functions in variable *Sobolev* spaces and even actually - in spite of various papers on this subject, for instance Cruz-Uribe and the author [26], Edmunds and Rákosník [49], Fan, Wang and Zhao [57], Hästö [70], Samko [117], Zhikov [127], the latest being Kostopoulos and Yannakakis [82] - a full result is still missing (while a full result exists for variable *Lebesgue* spaces, see Edmunds, Lang and Nekvinda [47]). Another paper where density of smooth functions in variable Sobolev spaces plays a key role is by Giannetti [67], who proved a modular version of the Gagliardo-Nirenberg inequality.

We mention now a “full" result in the sense above, due to Diening ([38]): in this case there is no sequence of papers trying to find the exact class of exponents, because it has been found already in the first reference on the topic. Consider the problem to establish the embedding between classical Lebesgue spaces. The result is simple to state and to be proved (see e.g. Theorem 3.10 in Castillo and Rafeiro [21]): for Ω of finite measure, if $1 \leq r \leq p \leq \infty$, then $L^p(\Omega) \subset L^r(\Omega)$ (note that the inclusion as sets is equivalent to the continuous embedding because they are particular Banach function spaces over the same measure space, see Theorem 1.8 in Bennett and Sharpley [15]). Whatever proof of the result for classical Lebesgue spaces is chosen (one may split Ω into the sets where $|f| \leq 1$ and $|f| > 1$, or one may use Hölder’s inequality), the extension can be done without too much effort. The surprise is that while for classical Lebesgue spaces this result solves completely the problem to characterize the embedding (because when Ω has not finite measure, two classical Lebesgue spaces over Ω cannot be compared), in the case of variable Lebesgue spaces this is not true. Namely, the embedding can hold in the case of exponents which become close each other, *very fast at infinity*, in a sense we are going to make precise. This can never happen for different, constant exponents: their distance is always constant. The "full" result for variable Lebesgue spaces (whose proof is quite technical) is the following (see Theorem 2.45 in Cruz-Uribe and the author [27], which is from Diening [38])

Theorem 4. *Given $\Omega \subset \mathbf{R}^n$ and $p(\cdot), q(\cdot) : \Omega \to [1,\infty]$, then $L^{q(\cdot)}(\Omega) \subset L^{p(\cdot)}(\Omega)$ and there exists $K > 1$ such that for all $f \in L^{q(\cdot)}(\Omega)$, $\|f\|_{p(\cdot)} \leq K\|f\|_{q(\cdot)}$, if and only if:*

1. *$p(x) \leq q(x)$ for almost every $x \in \Omega$;*
2. *there exists $\lambda > 1$ such that*

$$\int_D \lambda^{-r(x)}\, dx < \infty, \tag{11}$$

where $D = \{x \in \Omega : p(x) < q(x)\}$ and $r(\cdot)$ is the defect exponent defined by

$$\frac{1}{p(x)} = \frac{1}{q(x)} + \frac{1}{r(x)}.$$

In the same order of ideas we recall the following extension of (6): the inequality

$$\int_{\mathbf{R}^n} Mf(x)^p\, dx \leq c_1 \int_{\mathbf{R}^n} |f(x)|^q\, dx + c_2$$

holds for every $f \in L^q(\mathbf{R}^n)$ and for some positive constants c_1, c_2 independent of f if and only if $1 < p = q$. We stress that it is an extension: in fact, if $1 < p = q$, from the existence of some positive constants c_1, c_2 such that

$$\int_{\mathbf{R}^n} Mf(x)^p\, dx \leq c_1 \int_{\mathbf{R}^n} |f(x)|^p\, dx + c_2 \qquad \forall f \in L^p(\mathbf{R}^n) \tag{12}$$

one gets that also (6) is true, applying (12) to λf, dividing both sides by λ^p, and letting $\lambda \to \infty$. We remark that such homogeneization procedure can be applied in a general context (see e.g. D'Aristotile and the author [34]). The proof of such extension is a consequence of the following "full" result (see Cruz-Uribe, Di Fratta and the author [25])

Theorem 5. *Let $p(\cdot), q(\cdot) : \mathbf{R}^n \to [1,\infty[$. The inequality*

$$\int_{\mathbf{R}^n} Mf(x)^{p(x)}\, dx \leq c_1 \int_{\mathbf{R}^n} |f(x)|^{q(x)}\, dx + c_2$$

holds for every $f \in L^{q(x)}(\mathbf{R}^n)$ and for some positive constants c_1, c_2 independent of f if and only if $L^{q(\cdot)}(\mathbf{R}^n) \subset L^{p(\cdot)}(\mathbf{R}^n)$ and $p(\cdot)$ and $q(\cdot)$ "touch at infinity", namely, for every $E \subset \mathbf{R}^n$ having infinite measure,

$$\operatorname*{ess\,sup}_E\, p(\cdot) = \operatorname*{ess\,sup}_{\mathbf{R}^n}\, p(\cdot) = \operatorname*{ess\,inf}_{\mathbf{R}^n}\, q(\cdot) = \operatorname*{ess\,inf}_E\, q(\cdot)\,.$$

Let us close this paragraph with one more result, where again the class of exponents has been fully characterized. Here the symbol $L_w^{p(\cdot)}(\Omega)$ stands for the weighted version of $L^{p(\cdot)}(\Omega)$, built as in Section using

$$\int_{\Omega\setminus\Omega_\infty}\left|\frac{f(x)}{\lambda}\right|^{p(x)} w(x)dx+\left\|\frac{f(x)}{\lambda}w(x)\right\|_{L^\infty(\Omega_\infty)},$$

where w is a weight, i.e. $w:\Omega\to]0,\infty[$, $w\in L^1_{loc}(\Omega)$.

Theorem 6. *Let $\Omega\subset\mathbf{R}^n$, and let $p(\cdot):\Omega\to[1,\infty]$ be such that $\operatorname{ess\,sup}_{\Omega\setminus\Omega_\infty} p(\cdot)<\infty$. The weight w is noneffective, i.e. $L^{p(\cdot)}(\Omega)=L_w^{p(\cdot)}(\Omega)$, if and only if $w\approx$ constant.*

For the proof and the details about the optimality of the condition $\operatorname{ess\,sup}_{\Omega\setminus\Omega_\infty} p(\cdot)<\infty$ see Krbec and the author [64].

Classic statements are the best ones

One of the features of the previous category of results is that the generalization to the variable setting holds for some non-constant exponents. However, there are results which, if written in terms of variable exponents, hold if and only if the exponents are constant. A collection of results in this category appears explicitly in Section 1.3 of the book by Diening, Harjulehto, Hästö and Růžička [39] already quoted above, and an overlapping collection is someway “hidden" in the Subject Index of the book by Cruz-Uribe and the author [27] (see *non-constant vs. constant* in p. 306 therein).

It is not our goal to make one, maybe more complete, list of results in this category. Just to give an idea of the category, we state and prove the following

Theorem 7. *Let $\Omega\subset\mathbf{R}^n$ and $p(\cdot):\Omega\to[1,\infty]$. The space $L^{p(\cdot)}(\Omega)$ is rearrangement-invariant if and only if $p(\cdot)$ is constant.*

Its proof is in Kováčik and Rákosník [83], where an extra assumption of continuity of the exponent appears, and another proof, where the exponent is assumed just measurable, is in Cruz-Uribe and the author [27] (see Example 3.14 p. 87 therein), but it is wrong. In the case $\Omega=\mathbf{R}^n$, Theorem 7 can be seen as a corollary of Proposition 3.6.1 p. 95 in Diening, Harjulehto, Hästö and Růžička [39], where it is shown that if an exponent $p(\cdot)$ is such that every translation operator maps $L^{p(\cdot)}(\mathbf{R}^n)$ to $L^{p(\cdot)}(\mathbf{R}^n)$, then it must be constant. In fact, suppose that $L^{p(\cdot)}(\mathbf{R}^n)$ is rearrangement-invariant. Since every translation of any $f\in L^{p(\cdot)}(\mathbf{R}^n)$ is equimeasurable with f, its norm equals that one of f. Therefore every translation operator maps $L^{p(\cdot)}(\mathbf{R}^n)$ to $L^{p(\cdot)}(\mathbf{R}^n)$ from which, by the proposition above, $p(\cdot)$ must be constant.

Here we are going to show a direct argument.

Proof.
Fix $E \subset \Omega$ of finite measure on which $p(\cdot)$ is non-constant, so that

$$p_+ := \operatorname{ess\,sup}_E p(\cdot) > \operatorname{ess\,inf}_E p(\cdot) =: p_- \,.$$

Let $p_*(\cdot)$ be the decreasing rearrangement of the restriction of $p(\cdot)$ to E, $p_*(\cdot)$ being defined in $(0,|E|)$. Let $\overline{p} \in (p_-, p_+)$. Since $p_*(\cdot)$ is a decreasing function, then it is limit a.e. of an increasing sequence of step functions, therefore there exists $s_1(\cdot)$ step function such that

$$s_1(t) = \sum_{i=1}^{K_1} \alpha_i \chi_{(t^{(1)}_{i-1}, t^{(1)}_i)} \le p_*(t)\,, \qquad t^{(1)}_0 = 0 < t^{(1)}_1 < \cdots < t^{(1)}_{K_1} = |E|\,,$$

and such that $p_*(t) > \alpha_1 > \overline{p}$ for $t \in (t^{(1)}_0, t^{(1)}_1)$; arguing analogously on $-p_*(\cdot)$, we can get $s_2(\cdot)$ step function such that

$$s_2(t) = \sum_{j=1}^{K_2} \beta_j \chi_{(t^{(2)}_{j-1}, t^{(2)}_j)} \ge p_*(t)\,, \qquad t^{(2)}_0 = 0 < t^{(2)}_1 < \cdots < t^{(2)}_{K_2} = |E|\,,$$

and such that $p_*(t) < \beta_{K_2} < \overline{p}$ for $t \in (t^{(2)}_{K_2-1}, t^{(2)}_{K_2})$. Set

$$\varepsilon = \min\left\{ t^{(1)}_1, t^{(2)}_{K_2} - t^{(2)}_{K_2-1} \right\}\,,$$

and set

$$g_1(t) = t^{-\frac{1}{\alpha_1}} \quad \forall t \in (0,\varepsilon)$$

$$g_2(t) = (|E|-t)^{-\frac{1}{\alpha_1}} \quad \forall t \in (|E|-\varepsilon, |E|)\,.$$

Of course $g_1(\cdot)$ and $g_2(\cdot)$ are equimeasurable.

By Ryff's theorem (see e.g. Theorem 7.5 p. 82 in Bennett and Sharpley [15]), $p = p_* \circ \sigma$ where $\sigma : E \to (0,|E|)$ is a measure-preserving transformation, i.e. a map such that the measure of any subset in $(0,|E|)$ equals the measure of the pre-image in E. Set $f_1 = g_1 \circ \sigma$ and $f_2 = g_2 \circ \sigma$. By Proposition 7.2 p. 82 in Bennett and Sharpley [15], $f_1(\cdot)$ and $g_1(\cdot)$ are equimeasurable and, analogously, $f_2(\cdot)$ and $g_2(\cdot)$ are equimeasurable. Since $g_1(\cdot)$ and $g_2(\cdot)$ are equimeasurable, also $f_1(\cdot)$ and $f_2(\cdot)$ are as well.

On the other hand, for any $\lambda > 0$ the function $(\lambda f_1(\cdot))^{p(\cdot)} = (\lambda g_1 \circ \sigma)^{p_* \circ \sigma} = [(\lambda g_1)^{p_*}] \circ \sigma$ is not integrable, because again by the proposition above, it is equimeasurable with $(\lambda g_1)^{p_*}$ which is not integrable (because for t small we have that $(\lambda g_1(t))^{p_*(t)} > (\lambda t^{-\frac{1}{\alpha_1}})^{\alpha_1} = \lambda^{\alpha_1} t^{-1}$), while the function $(f_2(\cdot))^{p(\cdot)} = (g_2 \circ \sigma)^{p_* \circ \sigma} = [(g_2)^{p_*}] \circ \sigma$ is integrable, because it is equimeasurable with $(g_2)^{p_*}$ which is integrable (because $(g_2(t))^{p_*(t)} < (|E|-t)^{-\frac{\beta_{K_2}}{\alpha_1}}$ for t close to $|E|$, and $\beta_{K_2} < \overline{p} < \alpha_1$). The conclusion is that if $p(\cdot)$ is non-constant, there exist two equimeasurable functions $f_1(\cdot)$ and $f_2(\cdot)$ such that $f_1 \notin L^{p(\cdot)}(\Omega)$, $f_2 \in L^{p(\cdot)}(\Omega)$, and the theorem is proved. $\square$

One more result in this category is a consequence of Theorem 5 above (see Corollary 1.22 in Cruz-Uribe, Di Fratta and the author [25]), which generalizes a result by Lerner [87]:

Theorem 8. *Let $p(\cdot) : \mathbf{R}^n \to [1,\infty[$. The inequality*

$$\int_{\mathbf{R}^n} Mf(x)^{p(x)}\,dx \le c_1 \int_{\mathbf{R}^n} |f(x)|^{p(x)}\,dx + c_2$$

holds for every $f \in L^{p(x)}(\mathbf{R}^n)$ and for some positive constants c_1, c_2 independent of f if and only if $p(\cdot)$ equals a constant $p > 1$ almost everywhere.

We may insert in this category also another result, which holds if and only if the exponent is constant because it seems that it cannot even be stated in the variable setting. The result is formula (5) above, which we recall here:

$$\int_\Omega |f(x)|^p\,dx = p\int_0^\infty t^{p-1}|\{x \in \Omega\,:\,|f(x)| > t\}|\,dt\,.$$

Its generalization to the variable case seems “forbidden": in the book by Diening, Harjulehto, Hästö and Růžička [39] (see the *Warnings!* in p.9; see also p. 4 in Harjulehto and Hästö [68]) it is written that it has no variable exponent analogue, because of course the formula

$$\int_\Omega |f(x)|^{p(x)}\,dx = p\int_0^\infty t^{p(x)-1}|\{x \in \Omega\,:\,|f(x)| > t\}|\,dt$$

has no interest at all: similarly as in the case of (7), on the right hand side one has a function and not a number. Note that since (5) governs the proof of Theorem 1, the proofs of Theorem 3 must be done with much different arguments (and in fact they *use* Theorem 1).

In spite of the considerations above, we wish here to record the following simple formulas.

Proposition 1. *If $\Omega \subset \mathbf{R}^n$ and $p(\cdot) : \Omega \to [1,\infty[$, then for all measurable functions f in Ω the following equalities hold, the first one with the extra assumption $f(x) \neq 0$ for a.e. $x \in \Omega$:*

$$\int_\Omega |f(x)|^{p(x)}\,dx = \int_{\mathbf{R}} e^t|\{x \in \Omega : p(x)\log|f(x)| > t\}|\,dt \tag{13}$$

$$\int_\Omega |f(x)|^{p(x)}\,dx = \int_{\mathbf{R}} e^t|\{x \in \Omega : |f(x)| > e^{\frac{t}{p(x)}}\}|\,dt \tag{14}$$

$$\int_\Omega |f(x)|^{p(x)}\,dx = \int_0^{+\infty} |\{x \in \Omega : |f(x)| > t^{\frac{1}{p(x)}}\}|\,dt \tag{15}$$

Proof.

Proof of (13): for any measurable $g : \Omega \to \mathbf{R}$ we may apply (5) to the positive function $\exp(g(x))$, in the case $p = 1$. After the substitution $\log t = s$, we get

$$\int_{\Omega} \exp(g(x))\,dx = \int_{\mathbf{R}} e^s |\{x \in \Omega : g(x) > s\}|\,ds\,.$$

If $f : \Omega \to \mathbf{R}$ is a.e. nonzero, setting $g(x) = p(x)\log|f(x)|$ in the above equality we get the assertion.

Proof of (14): immediate after (13) for functions f such that $f(x) \neq 0$ for a.e. $x \in \Omega$. If $f(x) = 0$ in a set $\Omega_0 \subset \Omega$, equality (14) holds with Ω replaced by $\Omega \setminus \Omega_0$; however, the same equality is equivalent to the final assertion because the points of Ω_0 do not affect both members.

Proof of (15): apply the substitution $e^t = s$ in (14). $\square$

We note that, in the case $p(\cdot)$ constant, the substitution $t^{\frac{1}{p}} = \tau$ in (15) gives back (5).

Finally, we remark that making the conventions in the spirit of (3), (9), one can extend the validity of (13): for instance, writing

$$\text{convention :} \quad \log 0 = -\infty\,,$$

one can remove the assumption $f(x) \neq 0$ for a.e. $x \in \Omega$ made for (13): in fact, the set $\Omega_0 \subset \Omega$ where $f = 0$ a.e. does not influence both sides of (13) (note that in the right hand side, for any given $t \in \mathbf{R}$, any x such that $f(x) = 0$ would never satisfy the inequality $p(x)\log|f(x)| = p(x)\log 0 = -\infty > t$).

Essentially variable results

This last category concerns results which – in some opposite sense with respect to the previous category – have no interest or no meaning at all when the exponent is constant.

Let us state and prove the following proposition, where the problem of comparability in the sense of inclusion (or, equivalently, the continuous embedding) is considered for two variable Lebesgue spaces whose exponents are linked by the decreasing rearrangement operator.

Proposition 2. *Let $p(\cdot) : (0,1) \to [1,\infty[$. The spaces $L^{p(\cdot)}(0,1)$ and $L^{p_*(\cdot)}(0,1)$ are never comparable, unless $p(\cdot) = p_*(\cdot)$, i.e. unless $p(\cdot)$ is decreasing.*

Proof.
Let us assume that they are comparable. By condition *1.* in Theorem 4, it must be $p(\cdot) \le p_*(\cdot)$ or $p(\cdot) \ge p_*(\cdot)$. If the first option holds and the second one does

not, it would exist a set E, $|E| > 0$, where $p(\cdot) < p_*(\cdot)$. As a consequence, since $p(\cdot) \leq p_*(\cdot)$, it must be $\|p(\cdot)\|_{L^1(0,1)} < \|p_*(\cdot)\|_{L^1(0,1)}$, which is absurd because by Theorem 7 the space $L^1(0,1)$ is rearrangement-invariant. In the other case we can argue similarly, and the conclusion is that both options are true, hence $p(\cdot) = p_*(\cdot)$. □

As in the previous result, next one (see Theorem 3 in Rakotoson and the author [65]) involves, for a function f, the notion of decreasing rearrangement (denoted by f_*). Denoting by f^* the increasing rearrangement of f, all the norms in next chain of inequalities (16) coincide in the case of constant exponents:

Proposition 3. *If $\Omega \subset \mathbf{R}^n$, $p(\cdot) : \Omega \to [1, \infty[$ and $f \geq 0$ in Ω, then*

$$\frac{1}{2(1+|\Omega|)} \|f_*\|_{L^{p^*(\cdot)}} \leq \|f\|_{L^{p(\cdot)}} \leq 2(1+|\Omega|)\|f_*\|_{L^{p_*(\cdot)}} . \tag{16}$$

Other results involving rearrangement of exponents are in Rakotoson, Sbordone and the author [66].

The crucial notion of the last result (see Krbec and the author [62]) is the exponential summability: a function g measurable on $\Omega \subset \mathbf{R}^n$, $|\Omega| < \infty$, is said to belong to the Orlicz space $EXP_a(\Omega)$, $a > 0$, if for some $\lambda > 0$

$$\int_\Omega \exp\left(\lambda |g(x)|^a\right) dx < \infty .$$

Theorem 9. *Let $\Omega \subset \mathbf{R}^n$ be bounded and $p(\cdot) \in EXP_a(\Omega)$ for some $a > 0$. If $f \in L^{p(\cdot)}(\Omega)$, $f \not\equiv 0$, then $p(\cdot)\log(Mf) \in EXP_{a/(a+1)}(\Omega)$.*

It is clear that if $p(\cdot)$ is constant and finite, it is in particular in $L^\infty(\Omega)$ which is contained in every $EXP_a(\Omega)$, $a > 0$. In this case Theorem 9 tells that if $f \in L^p(\Omega)$, $f \not\equiv 0$, then $\log(Mf) \in EXP_{a/(a+1)}(\Omega)$, for every $a > 0$. Note that $a/(a+1) < 1$, hence the result is weaker than $\log(Mf) \in EXP_1(\Omega)$, which is true because, since Ω is bounded, Mf is bounded below by a positive number, and therefore by Theorem 1

$$\int_\Omega \exp\left(p|\log(Mf)|\right) dx < \infty .$$

The weakness of the thesis is readily explained: it is the “price" to pay because of the assumption $p(\cdot) \in EXP_a(\Omega)$, which includes a class of non-constant exponents (in fact, not bounded ones).

We close this Section pointing out that there exist essentially variable results which in the case of constant exponents have interest and meaning, but phenomena are new, the novelty being due exactly to the variability of the exponent. This happens for instance in Mercaldo, Rossi, Segura de León and Trombetti [94], where the authors consider an exponent having just two values, one of them being 1, in a Dirichlet problem involving the $p(x)-$laplacian.

5 - CONCLUSION AND NEW PERSPECTIVES

The categories of results in variable Lebesgue spaces theory presented in this Note could never pretend to be complete or rigorous. Moreover, even if nowadays the field is still very active, recently some new directions of research are becoming of interest among researchers. Just to quote a few of them, the interest in Musielak-Orlicz spaces is actually increasing, especially for their applications, because they are also the natural framework to generalize the conditions coming from variable Lebesgue spaces theory (see e.g. Ahmida, Chlebicka, Gwiazda and Youssfi [4], Baruah, Harjulehto, and Hästö [14], Cruz-Uribe and Hästö [32], Harjulehto and Hästö [68], Hästö[51, 52]); some special Musielak-Orlicz spaces, which are also generalizations of variable Lebesgue spaces, are generated by the functions $\Phi(x,t) = t^p + a(x)t^q$, $p < q$, which give raise to the double phase functional (see e.g. Baroni, Colombo and Mingione [12, 13], Colombo and Mingione [23]); there exist a huge development of variable variants of classical function spaces, for instance the Lorentz spaces with variable exponents (see Kempka and Vybíral [74]), grand variable Lebesgue spaces and their weighted version (see e.g. Kokilashvili, Meskhi and the author [63] and references therein) or the variable exponent Besov and Triebel-Lizorkin spaces (see e.g. Almeida, Diening and Hästö [5]).

Finally, let us mention that in principle, there are even papers where variable exponents appear, but the topic has no connections at all with variable Lebesgue spaces: for instance, in Anatriello, Chill and the author [8] (see also references therein), the norm $\|\cdot\|_{L^{p(x)}}$ which, for a given x, is a norm in the classical Lebesgue spaces, is considered. In Anatriello, Vincenzi and the author [9], starting from a variable exponent which is a simple function (i.e. with a finite number of values), the function spaces themselves "vary", and norms on product of quasinormed spaces which are roots of polynomials are considered.

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