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# Free boundary minimal surfaces: a survey of recent results

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*Key words:* free boundary minimal surfaces, Steklov eigenvalues, Morse index, effective estimates.

**Abstract –** We present a wide-spectrum overview of some recent developments in the theory of free boundary minimal surfaces, with special emphasis on the problem of compactness under mild curvature conditions on the ambient manifold.

**Riassunto –** Presentiamo un ampio resoconto di alcuni sviluppi recenti nella teoria delle superfici minime a frontiera libera, con particolare riferimento al problema della compattezza sotto deboli ipotesi sulla curvatura sulla varietà ambiente.

## 1 - INTRODUCTION

Let $\Sigma$ be a smooth manifold of dimension $k \geq 2$, let $(X,g)$ be a smooth Riemannian manifold of dimension $d \geq 3$, and let $\varphi : \Sigma \to X$ be a proper immersion satisfying $\varphi(\Sigma) \cap \partial X = \varphi(\partial\Sigma)$ (i. e. the boundary of $\Sigma$ is contained in the boundary of $X$, and there is no interior point of $\varphi(\Sigma)$ touching $\partial X$): we say that $\varphi : \Sigma \to X$ is a free boundary minimal immersion if it is a critical point for the $k$-dimensional area functional in the category of relative cycles, namely under all compactly supported variations $(\varphi_t)$ subject to the constraint that $\varphi_t(\partial\Sigma) \subset \partial X$. Considering the first-variation formula it is easily seen that this happens if and only if $\varphi(\Sigma)$ has zero mean curvature and meets the ambient boundary orthogonally. We shall mostly be interested in the case when the map $\varphi$ is an embedding, in which case we shall be talking about free boundary minimal submanifolds, and (with some abuse of language) identify the map in question with its image $\varphi(\Sigma)$.

Besides the self-evident geometric significance, which can be traced back at least to Courant [23, 24], free boundary minimal hypersurfaces also naturally arise in partitioning problems for convex bodies, in capillarity problems for fluids and, as has significantly emerged in recent years, in connection to extremal

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metrics for Steklov eigenvalues for manifolds with boundary (see primarily the works by Fraser-Schoen [33, 34, 36] and references therein).

A good point to start our discussion is provided by the unit ball in $\mathbb{R}^3$: in this case we have, modulo isometries, two simple examples of free boundary minimal surfaces. On the one hand, we have flat disks passing through the origin, while on the other we have the so-called *critical catenoids*, which are defined as the only catenoids centered at the origin and meeting the unit sphere orthogonally. Going beyond these classical examples, and producing free boundary minimal surfaces of topological type different from that of the disk or the annulus turns out to be a rather delicate task, that was only accomplished in recent years. In that respect, we mention the work by Fraser and Schoen [36] (genus zero and any number of boundary components), by Folha, Pacard and Zolotareva [29] (genus zero or one and any sufficiently large number of boundary components), by Ketover [50] and Kapouleas and Li [43] (arbitrarily large genus and three boundary components) and by Kapouleas and Wiygul [44] (arbitrarily large genus and one boundary component). In higher dimension, namely for $B^n \subset \mathbb{R}^{n+1}$ with $n \geq 4$, infinite families of examples have been found, via equivariant methods, by Freidin, Gulian and McGrath [37].

For general Riemannian manifolds, possibly subject to additional curvature conditions, in addition to older works mostly appealing to the parametric approach (cf. [23, 24, 30, 38, 47–49, 76, 78] and references therein) we have witnessed the implementation, for relative cyles, of powerful constructions like the min-max à la Almgren-Pitts or the degree-theoretic approach à la White: in that respect one should mention the work by Li [51], Li-Zhou [52], De Lellis-Ramic [25] and Maximo-Nunes-Smith [61]. In fact, there is good reason to believe that the min-max theory by Marques and Neves (see in part. [45,54,56,58,60]) should be pushed to the same impressive summits that have been achieved in the closed case, so to lead to a Weyl law for the (free boundary) volume spectrum, and to general density and equidistribution results.

Motivated by this variety of existence results, one is naturally lead to investigate some fundamental geometric questions which have to do with (what might be called) *the ensemble of free boundary minimal surfaces* inside a given Riemannian manifold $(X, g)$:

1. When $(X, g)$ is a space form, can one classify all free boundary minimal immersions having a pre-assigned topological type (namely: for fixed $\Sigma$)?

2. Under what curvature conditions $(X, g)$ is the class of its free boundary minimal embeddings having a pre-assigned topological type *compact* in the sense of smooth, graphical one-sheeted convergence?

3. Are there universal bounds, only depending on the ambient manifold, relating the topological invariants of any free boundary minimal surface to its geometric data like e. g. area, spectral invariants, Morse index?

Each of these questions is already highly meaningful in the aforementioned special case of the unit ball in $\mathbb{R}^3$. To fix the ideas, one could ask, for instance, whether (1)' the flat (equatiorial) disks are, in fact, the only contractible free boundary minimal surfaces in $B^3$, whether (2)' the space of free boundary minimal surfaces of genus five and three boundary components is strongly compact in the sense above, and whether (3)' the Morse index of any free boundary minimal surface is bounded from above and/or below by a linear function of its relative first Betti number.

The scope of this article is to describe various recent results related to the three questions above (and ramifications thereof), informally present some key ideas that come up in the corresponding proofs, and suggest a few related open problems.

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## 2 - TOPOLOGICAL UNIQUENESS RESULTS

The first general problem we wish to discuss has to do with classifying special geometric objects: when $(X, g)$ is a simple model space, like e. g. the unit ball $B^3$ or the hemisphere $S^3_+$ can we produce a complete list of all free boundary minimal immersions of fixed topological type? Besides the 'cheap' results one can obtain, in very special cases, via reflection methods (hence by reduction to the complete, or to the compact case), this question turns out to be very subtle. The first significant result, addressing the most basic of all such problems, was obtained by J. Nitsche [64] relying on a Hopf differential argument:

**Theorem 1.** *The only free boundary minimal immersions of a disk in the Euclidean unit ball are totally geodesic.*

The proof of this statement clearly resembles the one proposed by Almgren in 1966 [3] in the closed setting, in order to prove that the only minimally immersed two-spheres in the round three-sphere must be equatorial. This correspondence, i. e. the parallel between the theory of closed minimal surfaces in round $S^3$ and of free boundary minimal surfaces in $B^3$, turns out to be extremely rich, and surprisingly inspirational in the development of the subject. Within this framework, a fundamental conjecture was proposed in 2014 by Fraser and Li [32], to be thought in analogy with the theorem by Brendle [10] classifying the Clifford torus as the only minimally embedded torus in the sphere (as predicted by Lawson in 1970).

**Conjecture 2.** *The only free boundary minimal embeddings of an annulus in the Euclidean unit ball are, modulo isometries, reparametrizations of the critical catenoid.*

This result has very recently been claimed by Nadirashvili and Penskoi [63], and is currently under scrutiny. Although there are some serious concerns about the correctness of the proposed argument, it is interesting to remark that the authors show how confirming this conjecture would allow to prove a classification result for overdetermined elliptic boundary value problems on spherical domains, namely to determine all couples $(\Omega, v)$ for $\Omega \subset S^2$ a smooth simply-connected domain and $v$ a smooth function solving a problem of the form

$$\begin{cases} \Delta v = -\lambda v & \text{in } \Omega, \\ v = \alpha & \text{on } \partial\Omega, \\ |\nabla v| = \beta & \text{on } \partial\Omega \end{cases}$$

for constants $\lambda, \alpha, \beta \in \mathbb{R}$. A similar result for scaling-invariant domains in $\mathbb{R}^3$ and degree one homogeneous functions, related to earlier work by Caffarelli, Jerison and Kenig [12], would also follow.

A different task concerns, instead, the generalization of Nitsche's theorem to higher dimension and/or codimension. If we stick to $k = 2$ (notation as in the Introduction) but allow for any $d \geq 3$, the question has been very satisfactorily addressed in yet another contribution by Fraser and Schoen [35]: the only free boundary minimal immersions of a disk in the Euclidean unit $n$-dimensional ball are totally geodesic.

This may sound like a very plausible and expected statement, but it is quite remarkable that, on the other hand, for any $d \geq 4$ there is plenty of non-trivial minimal immersions $\varphi : S^2 \to S^d$ (see Calabi [13]). Thereby, we face an interesting *broken symmetry* with respect to the parallelism described above.

*Remark* 3. In $B^4$ there are free boundary minimally embedded Möbius bands, hence free boundary minimally immersed annuli. Thus the conclusion of the conjecture above cannot possibly hold in higher codimension (at least not without additional assumptions).

*Remark* 4. The uniqueness results above, for disk-type free boundary minimal surfaces, actually holds true in the larger category of *branched* free boundary minimal immersions.

On the other hand, in the codimension one case i. e. if we take $k = d - 1$ and let $d \geq 4$ then the uniqueness problem for free boundary minimal immersions of simply-connected domains is completely open. Based on the analogy with the closed case, and specifically on the (abundant) existence of minimal hyperspheres in round $S^4$ that are *not* totally geodesic [41, 42], we are inclined to believe that rigidity should *not* hold. More precisely, there may be a chance of suitably desingularizing the cone, centered at the origin, over a non-equatorial minimal $(d-2)$-dimensional minimal hyperpshere in $S^{d-1}$ to obtain a smooth free boundary minimal disk with the desired properties, at least for certain values of the integer $d$.

*Remark* 5. For an extension of Theorem 1 to the case of capillary surfaces, i. e. surfaces with constant mean curvature and a constant contact angle (not necessarily $\pi/2$) the reader may wish to consult Ros and Souam [70].

*Remark* 6. We still do not know whether for any (topological type of) compact surface with boundary $\Sigma$ there exists a minimal immersion $\varphi : \Sigma \to B^3$. This is an interesting and challenging gap in the existence theory. For instance, it would be good to know whether there exist examples of free boundary minimal surfaces in $B^3$ with one boundary component and positive, but low genus.

**3 - COMPACTNESS RESULTS**

As a straightforward consequence of the classification result presented in the previous section, we notice that in $B^3$ the space of free boundary minimal disks is parametrized by a group of isometries (in fact by $SO(3)$) and, therefore, is compact with respect to the appropriate notion of convergence. The purpose of this section is to study such a problem in general Riemannian domains, where classification results cannot be expected. For the sake of simplicity and notational convenience, we will now restrict to free boundary minimal *embeddings* and work in codimension one.

Generalizing to the free boundary setting a foundational result by Choi and Schoen, Fraser and Li proved the following theorem:

**Theorem 7.** *[32] If $(X,g)$ is a compact three-dimensional Riemannian manifold of non-negative Ricci curvature and convex boundary, then the space*

$$\mathfrak{M}_{\gamma,\rho} = \{\Sigma \in \mathfrak{M}(X) \ : \ genus(\Sigma) = \gamma, \#boundary\ components(\Sigma) = \rho\}$$

*is strongly compact, in the sense of subsequential smooth graphical convergence with unit multiplicity.*

In higher dimension, a conclusion of type cannot possibly hold because of the following two classes of counterexamples:

*i)* Given $m,n \geq 2$ such that $m+n < 8$, by work of Freidin-Gulian-McGrath [37], there exists an infinite family of distinct, free boundary minimal hypersurfaces in the Euclidean unit ball of dimension $m+n$, all having the topological type of $D^m \times S^{n-1}$ and converging (in the sense of varifolds) to a singular limit.

*ii)* The 'second principal family' constructed by Hsiang in 1983 provides infinite examples of free boundary minimal hypersurfaces all with the same topology (namely that of $D^2 \times S^1$) inside the upper hemisphere $S^4_+$, but the limit of these hypersurfaces is singular.

Therefore, we have counterexamples both in the case when either the Ricci tensor vanishes on the interior and the boundary is strictly convex, or the Ricci tensor is positive on the interior and the boundary is weakly convex. This evidence being provided, our idea (which goes back to [74] and [6]) was to approach the compactness problem from a somewhat different perspective, with less emphasis on the topological type and more on analytic bounds.

In order to introduce the fundamental notion of Morse index of a free boundary minimal hypersurface, let us consider a normal section $v \in \Gamma(N\Sigma)$ and compute the second variation of the area functional:

$$\begin{aligned} Q^{\Sigma}(v,v) :&= \int_{\Sigma} \left( |\nabla^{\perp} v|^2 - (Ric_X(v,v) + |A|^2|v|^2) \right) + \int_{\partial\Sigma} \mathrm{II}_{\partial X}(v,v) \\ &= -\int_{\Sigma} g(v, L_{\Sigma}(v)) + \int_{\partial\Sigma} \left( g(v, \nabla^{\perp}_{\tau} v) + \mathrm{II}_{\partial X}(v,v) \right) \end{aligned}$$

for $L_{\Sigma} v := \Delta^{\perp}_{\Sigma} v + Ric^{\perp}_X(v,\cdot) + |A|^2 v$. If we consider the eigenvalue problem

$$\begin{cases} L_{\Sigma}(v) + \lambda v = 0 & \text{on} \quad \Sigma, \\ \nabla^{\perp}_{\tau} v = -(\mathrm{II}_{\partial X}(v,\cdot))^{\sharp} & \text{on} \quad \partial\Sigma. \end{cases} \qquad (*)$$

standard analytic results ensure the existence of a *discrete spectrum*: we have a complete basis of $L^2$ normal sections, say $\{v_j\}$, and an associated sequence $\{\lambda_j\}$ with $\lambda_j \to +\infty$ solving $(*)$. Thereby, one defines the Morse index of $\Sigma$ as

$$index(\Sigma) = \#\left\{\lambda_j \text{ eigenvalue } : \lambda_j < 0\right\},$$

and the Morse nullity as

$$nullity(\Sigma) = \#\left\{\lambda_j \text{ eigenvalue } : \lambda_j = 0\right\}.$$

We shall say that $\Sigma$ is stable if it has zero Morse index, and is unstable otherwise.

These definitions being given, one can informally rephrase the argument by Fraser-Li as follows: on the one hand a bound on the topology implies a uniform area bound, hence a weak form of convergence (as encoded in the context of *Geometric Measure Theory*), while on the other hand a bound on the topology implies a uniform bound on the Morse index, hence a subsequential convergence in the sense of laminations (cf. Colding-Minicozzi [22]). Roughly speaking, the combination of the two things allows to gain convergence to a smooth, free boundary minimal hypersurface, in the sense of smooth graphical convergence (possibly with multiplicity) away from finitely many points where *necks* may form, and whose number is controlled by the (uniform) index bound in question. This form of subsequential convergence for classes of the form

$$\mathfrak{M}_p(\Lambda,\mu) := \{\Sigma \in \mathfrak{M}(X) \ : \ \lambda_p(\Sigma) \geq -\mu \text{ and } \mathscr{H}^n(\Sigma) \leq \Lambda\}.$$

(which include, as a special case, sets of free boundary minimal hypersurfaces with uniformly bounded area and index) is the content of Theorem 2 and Theorem 5 in [8]. A fundamental point of our analysis was then the construction of Jacobi fields for the limit object, which in fact have a sign in the case when the limit is two-sided (i. e. when it has trivial normal bundle) and convergence happens with multiplicity $m \geq 2$. As a result of our study, we obtained the following strong compactness theorem:

**Theorem 8.** *Let $2 \leq n \leq 6$ and $(X^{n+1}, g)$ a compact Riemannian manifold satisfying either of the following two assumptions*

1. *$Ric_X \geq 0$ with $\partial X$ strictly convex, or*
2. *$Ric_X > 0$ with $\partial X$ weakly convex and strictly mean convex.*

*Then the corresponding class $\mathfrak{M}_p(\Lambda, \mu)$ is sequentially compact for geometric convergence and thus $\mathfrak{M}_p(\Lambda, \mu)$ consists of* finitely many *diffeomorphisms classes.*

Here and below, we have employed the convenient phrase '*geometric convergence*' to refer to smooth, graphical convergence with multiplicity one.

In the so-called *bumpy* case, by which we mean that all free boundary minimal hypersurfaces (in the given ambient manifold $(X, g)$) do not have non-trivial Jacobi fields and the same is true for any finite covering thereof, one actually gets a much stronger finiteness result:

**Theorem 9.** *Let $2 \leq n \leq 6$ and $(X^{n+1}, g)$ a compact Riemannian manifold such that $\partial X$ is stricly mean convex. Suppose that for all $\Sigma \in \mathfrak{M}(X)$ and $\widetilde{\Sigma} \in \widetilde{\mathfrak{M}}(X)$ there exist no non-trivial Jacobi fields over $\Sigma$ or $\widetilde{\Sigma}$. Then $|\mathfrak{M}_p(\Lambda, \mu)| < \infty$.*

*Remark* 10. This fact has a very interesting consequence: in the setting above the whole class $\mathfrak{M}(X)$ of free boundary minimal hypersurfaces is *countable*. The corresponding assertion in the closed case (which follows from [6, 74]) plays a key role in the proof of the density theorem in [45], which first settled Yau's 1982 conjecture in the generic case.

Obviously, this result poses the problem of understanding how restrictive the non-degeneracy assumption actually is. This is the object of the following *bumpy metric theorem* in the free boundary context:

**Theorem 11.** *Let $X^{n+1}$ be a smooth, compact, connected manifold with non-empty boundary, and $q$ denote a positive integer $\geq 3$, or $\infty$.*

*Let $\mathscr{B}^q$ be the subset of metrics $g$ in $\Gamma^q$ defined by the following property: no compact smooth manifolds with boundary that are $C^q$ properly embedded as free boundary minimal hypersurfaces in $(X, g)$, and no finite covers of any such hypersurface, admit a non-trivial Jacobi field. Then $\mathscr{B}^q$ is a comeagre subset of $\Gamma^q$.*

The statement above is the free boundary counterpart of the bumpy metric theorem obtained by B. White in 1991 (for finite $q$) and in 2015 (for $q = \infty$). In simple terms, this theorem ensures that the notion of bumpyness given above is indeed *generic* in a set-theoretic sense (cf. Baire's category theorem).

*Remark* 12. B. White also proved, see [79] that if $\partial X$ is mean convex then: $X$ contains no closed smooth and embedded minimal hypersurface if and only if there exists some $C = C(X)$ such that for all free boundary minimal hypersurfaces $\Sigma$ it holds that

$$\mathscr{H}^n(\Sigma) \leq C \mathscr{H}^{n-1}(\partial \Sigma).$$

Motivated by this result, one can define

$$\mathfrak{M}_p^{\partial}(\Lambda,\mu) := \{\Sigma \in \mathfrak{M}(X) \ : \ \lambda_p(\Sigma) \geq -\mu \text{ and } \mathscr{H}^{n-1}(\partial\Sigma) \leq \Lambda\}.$$

If $2 \leq n \leq 6$ and $(X^{n+1}, g)$ is a compact Riemannian manifold with mean convex boundary and contains no closed minimal hypersurface, then Theorem 8 and Theorem 9 hold for the class $\mathfrak{M}_p^{\partial}(\Lambda,\mu)$ as well (hence also under a uniform index bound together with a uniform bound on the boundary mass).

## 4 - BUBBLING ANALYSIS AND QUANTIZATION

If one weakens the curvature condition that the Ricci curvature be positive, the conclusion of the compactness theorem by Choi-Schoen [19] is no longer true: indeed, Colding and De Lellis have constructed in [21] examples of three-dimensional manifolds (of positive *scalar* curvature) containing sequences of closed minimal surfaces of any pre-assigned fixed genus but arbitrarily large Morse index (and with a *lamination* limit). Although not yet in the literature, there is little doubt that a similar phenomenon (i. e. a similar lack of compactness, in spite of fixing the topological type) holds in the free boundary case as well: thus one cannot expect an *unconditional* result to hold. In this section, we will instead describe how *conditional* compactness results can be recovered, and how degenerations can be understood and described.

Embracing the same perspective as in the previous section, let us consider in a compact $(X^3, g)$ a sequence of free boundary minimal surfaces with uniform bounds on the area and the Morse index (the topological type will come later into play). We already described how, at this level of generality (and without any curvature condition assumption) the sequence in question will subsequentially converge to a smooth limit, the convergence being smooth and graphical (with integer multiplicity $m \geq 1$) away from finitely many points where necks may form. The simplest local model to keep in mind is provided by the homothetic rescalings of a catenoid in $\mathbb{R}^3$ (in the free boundary case other basic examples are given by free boundary half-catenoids, either vertically or horizontally cut).

The way to proceed, and refine the *global* analysis described in the previous section, is to get a precise, qualitative and quantitative, understanding of what happens during the convergence process near the points where curvature concentration occurs. This has been done in [5] (which follows the study of the closed case, that was previously done in [4, 11]). The first result there is a quantization identity for the total curvature functional $\mathscr{A}(\cdot)$, the integral of the $n$-th power of the length of the second fundamental form.

**Theorem 13.** *Let $2 \leq n \leq 6$ and $(X^{n+1}, g)$ be a compact Riemannian manifold with strictly mean convex boundary. For fixed $\Lambda, \mu \in \mathbb{R}_{\geq 0}$ and $p \in \mathbb{N}_{\geq 1}$, suppose that $\{\Sigma_k\}$ is a sequence in $\mathfrak{M}_p(\Lambda,\mu)$. Then there exist a $\Sigma \in \mathfrak{M}_p(\Lambda,\mu)$, $m \in \mathbb{N}$ and a finite set $\mathscr{Y} \subset X$ with cardinality $|\mathscr{Y}| \leq p-1$ such that, up to subsequence, $\Sigma_k \to \Sigma$ locally smoothly and graphically on $\Sigma \setminus \mathscr{Y}$ with multiplicity m. Moreover there exists a finite number of non-trivial bubbles or half-bubbles $\{\Gamma_j\}_{j=1}^J$ with*

*$J \leq p-1$ and*

$$\mathscr{A}(\Sigma_k) \to m\mathscr{A}(\Sigma) + \sum_{j=1}^{J} \mathscr{A}(\Gamma_j), \quad (k \to \infty).$$

*For k sufficiently large, the hypersurfaces $\Sigma_k$ of this subsequence are all diffeomorphic to one another (hence the class $\mathfrak{M}_p(\Lambda, \mu)$ is finite modulo diffeomorphisms).*

*Remark* 14. The statement above is somewhat less general than what we proved in [5]: the assumption that the boundary of the ambient manifold be strictly mean convex is un-necessary, but avoids a digression on possibly improper limit surfaces.

Theorem 13 has a few straightforward geometric implications, which we present here as a list of remarks:

- In ambient dimension three (corresponding to $n = 2$), the total curvature of any bubble is an integer multiple of $8\pi$ (cf. [65, 66]) and thus the total curvature of any half-bubble is an integer multiple of $4\pi$: hence Theorem 13 implies that for a sequence of surfaces that eventually satisfy $\mathscr{A}(\Sigma_k) \leq 4\pi - \delta$ for some $\delta > 0$, the set $\mathscr{Y}$ must be empty and the convergence to $\Sigma$ is smooth and graphical everywhere (but possibly with higher multiplicity, which however will not happen if the limit is two-sided);
- In the very setting of the theorem, there exist:
  - a constant $C = C(p, \Lambda, \mu, X, g)$ such that the total curvature of any element in $\mathfrak{M}_p(\Lambda, \mu)$ is bounded from above by $C$.
  - a constant $I = I(p, \Lambda, \mu, X, g)$ such that the Morse index of any element in $\mathfrak{M}_p(\Lambda, \mu)$ is bounded from above by $I$.

The very last assertion in Theorem 13, about the unconditional finiteness of the diffeomorphisms types represented in $\mathfrak{M}_p(\Lambda, \mu)$ is justified by the following fine local description result:

**Theorem 15.** *With the setup as in Theorem 13, for each $y \in \mathscr{Y}$ there exist a finite number of point-scale sequences $\{(p_k^i, r_k^i)\}_{i=1}^{J_y}$ where $\sum_{y \in \mathscr{Y}} J_y \leq p-1$ with $\Sigma_k \ni p_k^i \to y$, $r_k^i \to 0$, and finite numbers of non-trivial bubbles and half-bubbles $\{\Gamma_i\}_{i=1}^{J_y}$, such that the following is true.*

- *For all $i \neq j$, we have*

$$\frac{r_k^i}{r_k^j} + \frac{r_k^j}{r_k^i} + \frac{dist_g(p_k^i, p_k^j)}{r_k^i + r_k^j} \to \infty.$$

  *Taking normal coordinates centered at $p_k^i$, then $\widetilde{\Sigma}_k^i := \frac{\Sigma_k}{r_k^i}$ converges locally smoothly and graphically, away from the origin, to a disjoint union of* *finitely many (half-)hyperplanes and at least one non-trivial bubble or half-bubble. The convergence to any non-trivial component of the limit occurs with multiplicity one.*

- *Given any other sequence $\Sigma_k \ni q_k$ and $\rho_k \to 0$ with $q_k \to y$ and*

$$\min_{i=1,\dots J_y}\left(\frac{\rho_k}{r_k^i}+\frac{r_k^i}{\rho_k}+\frac{dist_g(q_k,p_k^i)}{\rho_k+r_k^i}\right)\to\infty$$

*then taking normal coordinates at $q_k$, we obtain that $\widehat{\Sigma}_k := \frac{\Sigma_k}{\rho_k}$ converges to a collection of parallel (half-)hyperplanes.*

*When $n=2$, any blow-up limit of $\widetilde{\Sigma}_k^i$ is always connected. The convergence is locally smooth, and of multiplicity one. Moreover we always have*

$$(\dagger)\qquad \frac{dist_g(p_k^i,p_k^j)}{r_k^i+r_k^j}\to\infty.$$

Notice that condition $(\dagger)$ ensures that one can *separate* the bubble regions, so that in a certain sense there is no interaction between different regions of high curvature.

Let us focus on the case of ambient dimension three: in that case, one can rely on the Gauss-Bonnet theorem, and on the varifold convergence of the boundaries to rewrite the quantization identity in the form

$$\chi(\Sigma_k)=m\chi(\Sigma)+\sum_{j=1}^{J}(\chi(\Gamma_j)-b_j),$$

where $\chi(\Gamma_j)$ denotes the Euler characteristic of $\Gamma_j$ and $b_j$ denotes the number of its ends.

Let us now see a simple application of this machinery. Since one can fully classify bubbles and half-bubbles of Morse index less than two, we were able to obtain novel geometric convergence results for sequences of free boundary minimal surfaces of low index. To shorten the statement it is convenient to introduce the following notation:

$$\mathfrak{M}(\Lambda,I):=\{\Sigma\in\mathfrak{M}(X)\ :\ index(\Sigma)\leq I \text{ and } \mathscr{H}^n(\Sigma)\leq\Lambda\}.$$

**Theorem 16.** *Let $(X^3,g)$ be a compact Riemannian manifold, with non-empty boundary $\partial X$. Assume that:*

a) *either the scalar curvature of $(X,g)$ is positive and $\partial X$ is mean convex with no minimal component;*

b) *or the scalar curvature of $(X,g)$ is non-negative and $\partial X$ is strictly mean convex.*

*Then, for any $\Lambda>0$ the following assertions hold:*

1. *The class $\mathfrak{M}(\Lambda,0)$ is sequentially compact in the sense of smooth multiplicity one convergence. Similarly, any subclass of $\mathfrak{M}(\Lambda,1)$ of fixed topological type is sequentially compact, in the sense of smooth multiplicity one convergence, for all given topological types except those of the disk and of the annulus. In particular, we obtain unconditional sequential compactness for any class of non-orientable surfaces of given topological type.*

2. *Let $\{\Sigma_k\}$ be a sequence of disks (respectively: annuli) in $\mathfrak{M}(\Lambda,1)$. Then:* either *a subsequence converges smoothly, with multiplicity one, to an embedded minimal disk (respectively: annulus) of index at most one* or *there exists a subsequence converging smoothly, with multiplicity two and exactly one vertically cut catenoidal half-bubble (respectively: exactly one catenoidal bubble), to a properly embedded, free boundary stable minimal disk. As a result, if $X$ contains no stable, embedded, minimal disks then strong compactness holds.*

*All conclusions still hold true without assuming any a priori upper area bound if $X$ is simply connected and, in case b), if moreover there is no* closed *minimal surface in $X$.*

*Remark* 17. It would be interesting to know if the *degenerations* listed above actually occur, i. e. to provide a construction of degenerating disks or annuli of Morse index one (and uniformly bounded) area with a point of bad convergence. There is a chance this may be achieved via a (rather subtle) gluing/desingularization scheme.

We refer the reader to [5] for a list of other interesting applications of the quantization identity.

## 5 - ESTIMATES INVOLVING THE MORSE INDEX

An interesting aspect of our analysis of the limit behaviour of sequences of index one free boundary minimal surfaces (Theorem 16) is that, in many interesting cases, an a priori area estimate is not needed for one can prove it using a variation of the Hersch trick. In fact, one gets an effective estimate: if we set $\rho := \inf R$ and $\sigma := \inf H$, both assumed to be non-negative numbers, then any index one, orientable free boundary minimal surface $\Sigma$ satisfies

$$\frac{\rho}{2}\mathscr{H}^2(\Sigma)+\sigma\mathscr{H}^1(\partial\Sigma)\leq 2\pi(8-\#\text{boundary components}(\Sigma))\leq 16\pi.$$

An estimate of this type is not known for surfaces of higher index, i. e. for free boundary minimal surfaces whose Morse index is bounded from above by a given integer $k$. Nevertheless, an *ineffective* estimate can be proven via refined surgery techniques following the work by Chodosh-Ketover-Maximo for the closed case [17]. These results, and their applications, are the object of a forthcoming article by the author and Franz. However, such a priori area estimate are only one of the two key ingredient needed to extend Theorem 16 to the case of higher index. The second one, which compensates for a lack of classification results for bubbles of index $\geq 4$, is a general index-topology inequality for complete minimal surfaces in $\mathbb{R}^3$ due to Chodosh-Maximo [18]. Their work fits in a very active area of research, which we shall now briefly describe with special focus on the free boundary setting. We will present a network of results in this spirit, which fit in a general program of the author of comparing different *notions of complexity* for minimal submanifolds.

Except in very few, special circumstances (including the critical catenoid, see [26, 62, 75]) the Morse index of a free boundary minimal hypersurface is unlikely to be computable, or even to be effectively estimated. Thus, it is of some importance to develop more general methods to bound it, from above and/or from below, in terms of other data of the surface in question, like e. g. its topology. We shall describe the topology of a manifold with boundary by means of its (real) homology groups. As it is well-known, in the most basic case of orientable surfaces with boundary the topological type can be completely described by means of two numbers, namely the genus and the number of boundary components of the surface in question.

There are some general results about the geometry and topology of stable and index one compact free boundary minimal surfaces in general three-manifolds whose boundary satisfies some convexity assumption. For example, by a variation of the Schoen-Yau rearrangement trick it is known that stable compact two-sided free boundary minimal surfaces in mean convex domains of three-manifolds with non-negative scalar curvature must be topological disks or totally geodesic annuli. Moving one step further, Cheng, Fraser and Pang showed in [16] that there exists an explicit upper bound on the genus and the number of boundary components of index one compact two-sided free boundary minimal surfaces in such manifolds. Related results about the topology of free boundary volume-preserving stable CMC surfaces in strictly mean convex domains of the three-dimensional Euclidean space were obtained by Ros in [69]. Going beyond the low index regime, but adding suitable curvature assumptions on the ambient manifold, in [9] we obtained general linear inequalities for the Morse index. For the sake of simplicity, anad to avoid an unnecessary digression on the methods we had introduced in [8] for the closed case, we shall simply stick to the special case of compact, regular domains in $\mathbb{R}^3$.

**Theorem 18.** *Let $\Omega^{n+1}$ be a strictly mean convex domain of the $(n+1)$-dimensional Euclidean space, $n \geq 2$. Let $\Sigma^n$ be a compact, orientable, properly embedded free boundary minimal hypersurface in $\Omega$. Then*

$$index(\Sigma) \geq \frac{2}{n(n+1)} dim H_1(\Sigma, \partial\Sigma; \mathbb{R}).$$

The dimension of the homology group $H_1(\Sigma, \partial\Sigma; \mathbb{R})$ can be explicitly computed in terms of the homology groups of $\Sigma$ and $\partial\Sigma$. In particular, we can obtain an estimate for the index in terms of the number of boundary components. Furthermore, in the special case of free boundary minimal surfaces ($n = 2$), the estimate also involves the genus of the surface and can in fact be upgraded to the more general scenario when the ambient domain is only *weakly* mean convex. This requires an *ad hoc* argument, and relies on a result of Ros [68].

**Theorem 19.** *Let $\Omega^3$ be a mean convex domain of the three-dimensional Euclidean space. Let $\Sigma^2$ be a compact, orientable, properly embedded free boundary minimal surface in $\Omega$ with genus $\gamma$ and $\rho \geq 1$ boundary components. Then*

$$\mathit{index}(\Sigma) \geq \frac{1}{3}(2\gamma + \rho - 1).$$

The conclusion of this theorem coincide with the one obtained by Ros and Vergasta [71] in the special case of index one free boundary minimal surfaces in strictly convex domains of $\mathbb{R}^3$ (which contain no stable free boundary minimal surfaces).

*Remark* 20. In the case of the unit ball in $\mathbb{R}^3$, one should be able to improve the estimate above in analogy with Theorem 1.3 in [72], concerning the index of closed minimal surfaces in the round three-dimensional sphere. In terms of *absolute* lower bounds (i. e. estimates not brining topology into play) Fraser and Schoen [36] have proven that if $\Sigma^n \subset B^{n+1}$ (a free boundary minimal hypersurface in the unit ball of $\mathbb{R}^{n+1}$) then either $\Sigma^n$ is a flat disk (whose index is one) or its Morse index is at least $n+2$.

*Remark* 21. The above theorem can be used to understand the behaviour of the index of some known examples of free boundary minimal surfaces constructed in the unit ball in $\mathbb{R}^3$. In particular, it implies that the examples in any of the families of free boundary minimal surfaces obtained in [36], [29], [43] and [44] have arbitrarily large Morse indices.

*Remark* 22. All the main results in [5] actually hold for properly *immersed* free boundary minimal hypersurfaces. The necessary modifications to our proofs are of purely notational character.

*Remark* 23. For (strictly) two-convex domains of the Euclidean space one can actually improve the estimate in Theorem 18 replacing, in the right-hand side, $\dim H_1(\Sigma, \partial\Sigma; \mathbb{R})$ by means of $\max\{\dim H_1(\Sigma, \partial\Sigma; \mathbb{R}),\ \dim H_{n-1}(\Sigma, \partial\Sigma; \mathbb{R})\}$.

The ideas behind the proof of Theorem 18 (hence Theorem 19) have their roots in earlier work by Ros [68], Savo [72] and, as mentioned above, by Ambrozio, Sharp and the author [8] in the study of a well-known conjecture by Schoen on the index of closed minimal hypersurfaces inside three-dimensional manifolds of positive Ricci curvature. Let us explain it, very briefly, in the simpler case of Theorem 19.

Roughly speaking, given a harmonic one-form $\omega \in \mathscr{H}^1_T(\Sigma, g)$ subject to suitable boundary conditions, one considers its projections with respect to some orthonormal basis $\{\theta_i\}$ of $\mathbb{R}^3$ and proves, via a direct calculation, an identity for the mean value $\sum Q^\Sigma(u_i, u_i)$ where $Q^\Sigma$ is the Jacobi form arising in the second variation of the area of $\Sigma$ and $u_i = \langle \omega, \theta_i \rangle$. Thereby, the geometric assumption that the domain be strictly mean convex ensures that this sum must be *negative*.

Now, let us assume that $\Sigma^2$ has index $k$, and denote by $\{\phi_q\}_{q=1}^{\infty}$ an $L^2$-orthonormal basis of eigenfunctions of the Jacobi operator of $\Sigma$ satisfying the Robin boundary conditions $(*)$ (see Section ). Let then $\Phi$ denote the linear map defined by

$$\begin{array}{llcl} \Phi: & \mathscr{H}^1_T(\Sigma, g) & \to & \mathbb{R}^{3k} \\ & \omega & \mapsto & [\int_\Sigma \langle \omega, \theta_i \rangle \phi_q], \end{array}$$

where $q$ varies from 1 to $k$. Clearly, by linear algebra

$$dim \mathscr{H}^1_T(M,g) \leq dim Ker(\Phi) + 3k$$

Since $\mathscr{H}^1_T(\Sigma, g) \simeq H_1(\Sigma, \partial\Sigma; \mathbb{R})$, and thus both spaces have dimension $2\gamma + \rho - 1$, the result will follow once we prove injectivity of the map $\Phi$.

Let then $\omega$ be an element of the kernel of the map $\Phi$. This means that all functions $u_i$ are orthogonal to the first $k$ eigenfunctions, namely $\phi_1, \ldots, \phi_k$. Since $index(\Sigma) = k$, we must have

$$Q(u_i, u_i) \geq \lambda_{k+1} \int_\Sigma u_i^2 \, d\mathscr{H}^2 \geq 0, \quad \text{for } i = 1, 2, 3,$$

by the variational characterization of the eigenvalues for problem $(*)$. On the other hand, we have already observed that the assumption $H^{\partial\Omega} > 0$ implies that the above inequality can only possibly hold if $|\omega|$ vanishes identically on $\partial M$. But then $\omega = 0$ on $\Sigma$ by suitably applying the maximum principle. Hence, if the domain is strictly mean convex, $\Phi$ has trivial kernel, and the conclusion follows.

General estimates of the type above are not known in higher codimension, or under weaker curvature assumptions. However, we remark that some interesting results on the index of free boundary minimal submanifolds of higher codimension have been proven in [31] and [36], Theorem 3.1.

Besides the index estimates provided above, we further know, thanks to recent work [53] by V. Lima, extending to the free boundary setting the results by Eijiri-Micallef [28] and Cheng-Tysk [15], that a uniform bound both on the area and on the topology of a sequence of orientable free boundary minimal surfaces implies a uniform bound on the Morse index. We note that when $n = 2$ and one considers convex domains in Euclidean space the theorem by V. Lima can be regarded as a partial converse to Theorem 19 above.

Lastly, let us mention some very recent work by Aiex and Hong [1] presenting index estimates in the spirit of [8, 9] for *constant mean curvature* surfaces in three-dimensional manifolds, both in the closed and in the free boundary case (see [14] for earlier contributions in the case of mean convex domains in $\mathbb{R}^3$). Of course, in the stable case these estimates recover much more basic results concerning the topology of isoperimetric surfaces in Euclidean bodies.

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