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Implicitization of Parametric Hypersurfaces via Points

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Abstract - Given a parametric polynomial representation of an algebraic hypersurface S in the projective space we give a new algorithm for finding the implicit cartesian equation of S . The algorithm is based on finding a suitable finite number of points on S and computing, by linear algebra, the equation of the hypersurface of least degree that passes through the points. In particular the algorithm works for plane curves and surfaces in the ordinary three-dimensional space. Using C++ the algorithm has been implemented on an intel Pentium running Linux. Since our algorithm is based only on computations of linear algebra it reveals very efficient if compared with others that do not use linear algebra for the computations.

Riassunto - Data una rappresentazione parametrica polinomiale di una ipersuperficie algebrica S nello spazio proiettivo presentiamo un nuovo algoritmo per trovare l'equazione cartesiana implicita di S . L'algoritmo si basa sul trovare un opportuno numero finito di punti su S e calcolare, usando l'algebra lineare, l'equazione della ipersuperficie di grado minimo che passa per i punti. In particolare, l'algoritmo si applica alle curve piane e alle superfici dello spazio tridimensionale ordinario. Usando il software C++ l'algoritmo è stato implementato su un Intel Pentium con sistema operativo Linux. Poiché il nostro algoritmo si basa solo su calcoli di algebra lineare, si mostra molto efficiente se paragonato con altri che non usano l'algebra lineare per i calcoli.

1 - INTRODUCTION

The hypersurface implicitization problem has attracted many authors, also recently and in the case of ordinary curves and surfaces has applications to

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Computer aided Geometric Design and Geometric Modelling. The problem consists in determining an implicit representation of an irreducible parametric algebraic hypersurface in the projective n -dimensional space (in particular a plane curve or a surface). It has been classically faced by elimination theory and can be solved by the computation of Gröbner bases or by the computation of resultants. New methods have been recently introduced in (Abbot *et al.*, 2016). All these methods have been implemented in various softwares of Computer Algebra (see Abbot *et al.*, 2016). For doing these computations on the computer the hypersurface has to be parametrically represented by polynomials with coefficients rational numbers (or more simply integers) and the result is an equation whose coefficients are rational numbers (or integers). Another possible representation of the coefficients of the parametric polynomials is in the set of integers modulo a prime number. In this paper, we introduce a new alternative method that reconducts the implicitization of S to the computation of the equation of the hypersurface that contains a suitable set of points on S . The algorithm works both on rational (or integers) numbers and on integers modulo a prime number. The algorithm is based on the fact that, if we find an appropriate set T of general points on S , a polynomial F vanishes on the polynomials that represent parametrically S if and only if F vanishes on T . This gives rise to a homogeneous linear system of equations. A non null solution of the system (which can be found very easily by Gauss reduction) gives the coefficients of the implicit equation of S . The algorithm of this paper applies to plane curves and surfaces in the ordinary space. The notations and results tacitly assumed in this article can be found in (Atiyah and Macdonald, 1969) and in (Hartshorne, 1977).

2 - COMPUTING THE IMPLICIT EQUATION OF A PARAMETRIC HYPERSURFACE BY POINTS

Let $\mathbf{P}^n$ the n -dimensional projective space over a field $\mathbf{k}$.

Definition 1.1 We say that an irreducible (algebraic) hypersurface S is parametric if there exists a rational map h from $\mathbf{P}^m$ to $\mathbf{P}^n$ given by homogeneous polynomials f_i in m variables of the same degree r whose image is a dense subset of S .

Since S is a hypersurface, it is also the set of solutions of a homogeneous polynomial F in $n+1$ variables. We consider the polynomial F of least degree d that vanishes on S . The corresponding equation $F=0$ is said to implicitize the parametric hypersurface S and F is said to be the polynomial defining S . F is unique modulo a constant term of the field $\mathbf{k}$.

Definition 1.2 (Orecchia, 1981) A set S of $N = (d+m)!/m!d!$ points of $\mathbf{P}^n$ is in generic position (or simply the points are in generic position) if it is not contained in a projective hypersurface of degree $< d+1$.

Example 1.3 Six points that do not lie on a conic and ten points that do not lie on a quadric surface are in generic position.

Lemma 1.4 Let S be a parametric hypersurface given by homogeneous polynomials of the same degree r . Let d be a fixed integer, and P_i be N points of $\mathbf{P}^n$ in generic position such that $Q_i = h(P_i)$ form a set T of N points of $\mathbf{P}^n$. If F' is a homogeneous polynomial of n variables with degree $d' < d+1$ then F' vanishes on S if and only if F' vanishes on T .

Proof. Clearly, since T is a subset of S a polynomial vanishing on S vanishes on T . We prove the converse by contradiction. Let F' be a homogeneous polynomial of degree d' vanishing on the set of points T and assume that there exists a point P such that $F(h(P))$ is not null. Then $F'(f_0, \dots, f_n)$ is a non-zero polynomial of degree $d'r < dr+1$ vanishing on the points P_i and this contradicts the assumption of generic position.

By Lemma 1.4 the computation of the equation that defines S is reconducted to the computation of the equation that vanishes on a finite set of points. Before starting the computation one has to find a set of points in generic position. Almost all sets of N points are in generic position (Geramita and Orecchia, 1981).

3 – THE ALGORITHM

By the results of section 2, the problem of determining the implicit equation of a parametric hypersurface S is reconducted to the problem of finding a polynomial F that vanishes on a suitable set of points on S . In this section we construct an algorithm for solving this problem using only linear algebra.

Let T be a set of points Q_i and d be a positive integer. An homogeneous polynomial F of degree d vanishes on T if and only if, for any i , $F(Q_i) = 0$. This gives a homogeneous linear system with coefficients in $\mathbf{k}$ in n indeterminates. Denote by B the set of terms of degree d in n the indeterminates, ordered with respect to any term ordering. The set B is a basis of the $\mathbf{k}$ vector space of the homogeneous polynomials F of degree d . Consider the matrix M whose generic element is the evaluation of the terms of B at the points Q_i . All the previous results allow to formulate the following algorithm. Let S be a hypersurface.

ALGORITHM

INPUT: Degree d and coefficients of the polynomials f_i .

OUTPUT: Implicit equation $F=0$ of S .

1. Set $d = 1$.

2. Computation, in degree d , of the set T of points of Lemma 1.4

3. Computation of the rank r' of the matrix M .
- 4 If r' is not maximal goto 5 else set $d=d+1$ and goto 2.
5. Compute a solution X of the null space of M and stop.

X gives the coefficients of the implicit equation $F=0$ of S .

Our algorithm was implemented in C++ on an Intel Pentium running Linux.

We have tested the algorithm for the computation of the implicit equation of hypersurfaces on various examples. Our results coincide with the ones obtained by other authors which compute the implicit equation with other methods. Since our algorithm is based on computations of linear algebra it reveals very efficient if compared with others that do not use linear algebra for the computations.

Remark 2.2 *The problem of implicitizing a parametric space curve was tackled in (Albano et al., 2000).*

7 – REFERENCES

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