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Low-Dimensional Pure Braid Group Representations Via Nilpotent Flat Connections

ALBERTO BENVEGNÙ - MAURO SPERA

Abstract. – *In this note we discuss low-dimensional matrix representations of pure braid group (on three and four strands) obtained via holonomy of suitable nilpotent flat connections. Flatness is directly enforced by means of the Arnol'd relations. These explicit representations are used to investigate Brunnian and "nested" Brunnian phenomena.*

1. – Introduction

In this note we discuss pure braid group low dimensional matrix representations – confining ourselves to the three and four strand cases – obtained differential-geometrically via holonomy of suitable nilpotent flat connections. Our contribution mainly consists in elaborating on quite concrete and special instances of the abstract general framework developed in [20, 21, 22, 23] (see also [2, 25, 24, 6, 28, 26]).

Our matrix representations also stem from ideas in [30] (and also [29]).

Indeed, one of our basic motivation is to give a *concrete* application of the above general formalism, aimed at finding easy-to-compute *topological invariants* which can at least partially distinguish (pure) braids with three or four strands, especially those exhibiting a Brunnian like character, namely, those which become trivial after removing some strands therefrom in an arbitrary way (see below for precise definitions). The study of such braids, interesting in itself, is also motivated by the general issue "quantum entanglement vs topological entanglement", aimed at relating knot theory with quantum mechanical states and measurements thereon [3, 19, 7], though we shall not discuss this topic in the present note. Moreover, nilpotent connections are noteworthy for their manageability, at least in principle (the Chen series giving their parallel transport is indeed a finite sum) and for their relationship with Vassiliev's finite order invariants.

The general theory already tells us that the Kohno monodromy representations exhaust all unipotent representations on P_n, ([22], Theorem 1.2.6, see also [2], Théorème 1); nevertheless, as far as we can see, explicit calculations are unavoidable if one aims at getting more detailed information.

The basic idea underlying the paper is extremely simple: since the n-strand pure braid group P_n is the fundamental group of the configuration space $Conf(n, \mathbb{C})$ of n distinct points on the (complex) plane, we try to manufacture a flat nilpotent connection *à la* Knizhnik-Zamolodchikov-Kohno (KZK) (see e.g. [20, 21, 22, 23, 28]) by imposing that its curvature just involves the Arnol'd relations (see below), so it vanishes; in this way the so-called "infinitesimal braid relations" will be then automatically fulfilled. We find a many-parameter solution, including Heisenberg group type representations, using a geometrically flavoured "linearization" method. Then the main job rests in computing its parallel transport – yielding the sought for representation – and this involves Chen's iterated path integrals. It is then clearly enough to calculate the latter around the (Artin) generators: this is already a non trivial task, which can be achieved in principle by resorting to hyperlogarithms and their monodromy [32]; in our setting, it will be enough to compute suitable *double* iterated integrals.

Many authors (besides the previously cited) have tackled some of the problems discussed in the present paper, among others [8, 9, 10, 11, 16] have been particularly inspiring for us. We eventually obtain fully explicit *families* of pure braid invariants by a systematic approach resting on a vivid differential geometric principle.

The paper is organized as follows. In the preliminary Section 2 we collect our basic technical tools, and review Chen's theory, pure braid groups and hyperlogarithms. In Section 3 we further elaborate on the basic ideas involved in the paper. In Sections 4 and 5 we construct our connections and compute their parallel transport (for $n = 3, 4$ respectively). In Section 6 we investigate "nested" Brunnian type phenomena via our representations. Subsequently (in Section 7), we discuss the Heisenberg type representations hinted at above, providing an alternative direct algebraic interpretation thereof and we construct further explicit representations of P_3, building again on the differential geometric techniques employed throughout the paper. The final Section 8 is devoted to concluding remarks and outlook. In particular, we show that the linearization method previously devised does not lead directly to non trivial solutions for $n \geq 5$, due to the excessive growth of constraints stemming from the Arnol'd relations.

2. – Preliminaries

2.1 – *Chen's iterated path integrals and nilpotent connections*

Chen's iterated path integrals provide an extremely general and flexible technical tool usefully employed throughout mathematics (see [14, 15] for a comprehensive account). Here we just recall some basic facts concerning the simplest of them, mostly following [31, 29, 30].

Let M be a smooth manifold and $\gamma : [0,1] \to M$ be a smooth path therein, with velocity field $\dot{\gamma}$. Let $v_1, \ldots, v_m$ be (real or complex) 1-forms, with $v_i(t_i) := v_i(\gamma(t_i), \dot{\gamma}(t_i))$, $i = 1, \ldots, m$, and denote by $\triangle^m$ the standard m-simplex in $\mathbb{R}^m$:

$$\triangle^m := \{(t_1, \ldots, t_m) \in \mathbb{R}^m \mid 0 \leq t_1 \leq t_2 \leq \ldots \leq t_m \leq 1\}$$

Define the *(Chen) iterated path integral*

$$\int_\gamma v_1 \ldots v_m := \int_{\triangle^m} v_1(t_1) \ldots v_m(t_m)\, dt_1 \ldots dt_m$$

Equivalently, setting $\gamma^t : [0,1] \ni s \mapsto \gamma(ts) \in M$, we may also write down, recursively:

$$\int_\gamma v_1 \ldots v_m \quad = \quad \int_\gamma \left(\int_{\gamma^t} v_1 \ldots v_{m-1} \right) v_m$$

In particular one has ([31])

$$\int_{\gamma^{-1}} v_1 \ldots v_r = (-1)^r \int_\gamma v_r \ldots v_1 \qquad (*)$$

The general formula ([31])

$$\int_\gamma v_1 v_2 + \int_\gamma v_2 v_1 = \int_\gamma v_1 \int_\gamma v_2 \qquad (**)$$

will be used throughout the paper, together with the following one, valid if $\int_\gamma v_1 = 0$ or $\int_\gamma v_2 = 0$, coming directly from it and from $(*)$, for $r = 2$:

$$\int_\gamma v_1 v_2 = -\int_{\gamma^{-1}} v_1 v_2$$

Let us consider, on the trivial $\mathbb{C}^n$-bundle over M, a *nilpotent connection* 1-form $\mathbf{v}$ given by

$$\mathbf{v} = \begin{pmatrix} 0 & v_1 & v_{12} & \cdots & v_{12\ldots n} \\ 0 & 0 & v_2 & \cdots & v_{2\ldots n} \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & v_n \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

(*Hain-Tavares connection* [31, 17]) with corresponding *curvature form*

$$\Omega = d\mathbf{v} + \mathbf{v} \wedge \mathbf{v} = \begin{pmatrix} 0 & w_1 & w_{12} & \cdots & w_{12\ldots n} \\ 0 & 0 & w_2 & \cdots & w_{2\ldots n} \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & w_n \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

where

$$\begin{array}{rcl} w_1 & = & dv_1 \\ w_{12} & = & v_1 \wedge v_2 + dv_{12} \\ & \cdots & \\ w_{12\ldots n} & = & v_1 \wedge v_{2\ldots n} + v_{12} \wedge v_{3\ldots n} + \cdots + dv_{12\ldots n}. \end{array}$$

Then the *holonomy* (or parallel transport) of the connection along a path γ is given by

$$\rho(\gamma) = \begin{pmatrix} 1 & \int\limits_\gamma u_1 & \int\limits_\gamma u_{12} & \cdots & \int\limits_\gamma u_{12\ldots n} \\ 0 & 1 & \int\limits_\gamma u_2 & \cdots & \int\limits_\gamma u_{2\ldots n} \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & \int\limits_\gamma u_n \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

where

$$\begin{array}{rcl} \int\limits_\gamma u_1 & = & \int\limits_\gamma v_1 \\ \int\limits_\gamma u_{12} & = & \int\limits_\gamma v_1 v_2 + v_{12} \\ \int\limits_\gamma u_{123} & = & \int\limits_\gamma v_1 v_2 v_3 + v_{12} v_3 + v_1 v_{23} + v_{123} \\ & \cdots & \\ \int\limits_\gamma u_{12\ldots n} & = & \int\limits_\gamma v_1 v_2 \ldots v_n + \cdots + v_{12\ldots n} \end{array}$$

It is a general fact that if a connection is *flat*, i.e. its curvature vanishes, then homotopic paths between the same endpoints yield the same holonomy. In particular, a flat connection induces a representation of the fundamental group of the underlying manifold. This will be crucial for the sequel.

2.2 – *Pure braid groups*

This subsection is meant to provide a minimal background on pure braid groups and to establish notation. For a full account see e.g. [5, 27, 13, 18, 23].

The Artin braid group B_n is the group generated by $n-1$ generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and the *braid relations*

$$\sigma_i \sigma_j = \sigma_j \sigma_i$$

for all $i,j = 1,2,\ldots,n-1$ with $|i-j| \geq 2$, and

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

for $i = 1,2,\ldots,n-2$.

The *pure* (or *coloured*) *braid group* P_n is the kernel of the natural projection $\pi : B_n \to S_n$ where S_n is the symmetric group:

$$P_n = Ker(\pi : B_n \to S_n)$$

The n-strand pure braid group on P_n is the fundamental group of the configuration space $Conf(n, \mathbb{C})$ of n distinct points on the (complex) plane. We take its base point at $(1,2,\ldots,n) \in Conf(n,\mathbb{C})$.

The pure braid group P_n is generated by the $n(n-1)/2$ elements $\{A_{ij}\}_{1 \leq i < j \leq n}$ subject to the so-called Artin relations, which will be written down explicitly for $n = 3, 4$ below. One sets, $A_{ij} = A_{ji}$, for $i \neq j$. The generators A_{ij} can be represented (up to isotopy) by downward directed *geometric braids* such that strand i (starting from and ending at $z = i$) winds *clockwise* around strand j or conversely (this is consistent with the above convention). We do not need their expression in terms of the σ's. In what follows we shall blur the distinction between a geometric braid and the element of the braid group it represents. Concretely (and with an abuse of language) the product $b_1 \cdot b_2$ is given by juxtaposition, drawing b_2 below b_1.

The Artin relations for P_3 read

$$\begin{aligned} A_{12}^{-1} A_{23} A_{12} &= A_{13} A_{23} A_{13}^{-1} \\ A_{12}^{-1} A_{13} A_{12} &= A_{13} A_{23} A_{13} A_{23}^{-1} A_{13}^{-1} \end{aligned}$$

whereas the centre of P_3 is generated by $\Delta_3^2 = A_{12} A_{13} A_{23}$.

The Artin relations for P_4 read, in turn

$$\begin{aligned}
1)\quad A_{12}^{-1} A_{34} A_{12} &= A_{34} \\
2)\quad A_{23}^{-1} A_{14} A_{23} &= A_{14} \\
3)\quad A_{12}^{-1} A_{23} A_{12} &= A_{13} A_{23} A_{13}^{-1} \\
4)\quad A_{12}^{-1} A_{24} A_{12} &= A_{14} A_{24} A_{14}^{-1} \\
5)\quad A_{13}^{-1} A_{34} A_{13} &= A_{14} A_{34} A_{14}^{-1} \\
6)\quad A_{23}^{-1} A_{34} A_{23} &= A_{24} A_{34} A_{24}^{-1} \\
7)\quad A_{12}^{-1} A_{13} A_{12} &= A_{13} A_{23} A_{13} A_{23}^{-1} A_{13}^{-1} \\
8)\quad A_{23}^{-1} A_{24} A_{23} &= A_{24} A_{34} A_{24} A_{34}^{-1} A_{24}^{-1} \\
9)\quad A_{13}^{-1} A_{24} A_{13} &= A_{14} A_{34} A_{14}^{-1} A_{34}^{-1} A_{24} A_{34} A_{14} A_{34}^{-1} A_{14}^{-1}
\end{aligned}$$

The centre of P_4 is generated by $\Delta_4^2 = A_{12} A_{13} A_{14} A_{23} A_{24} A_{34}$.

The (co)homology ring of the coloured braid group (namely, that of $Conf(n,\mathbb{C})$) is isomorphic to the exterior graded ring generated by one-dimensional elements $\omega_{ij} = \omega_{ji}$ $1 \le i \ne j \le n$ satisfying the Arnol'd relations [4] (i, j, k distinct)

$$I_{ijk} := \omega_{ij} \wedge \omega_{jk} + \omega_{jk} \wedge \omega_{ki} + \omega_{ki} \wedge \omega_{ij} = 0$$

Concretely, one takes the logarithmic 1-forms

$$\omega_{ij} := \frac{1}{2\pi\sqrt{-1}}\; d\;\log(z_i - z_j) = \frac{1}{2\pi\sqrt{-1}}\,\frac{d\,(z_i - z_j)}{z_i - z_j}$$

(for $z_i \ne z_j$, $\sqrt{-1} = +i$). Thus there are $\binom{n}{3}$ independent Arnol'd relations.

Specifically, the Arnol'd identity for P_3 reads

$$\mathbb{I}_1 := \omega_{12} \wedge \omega_{23} + \omega_{23} \wedge \omega_{31} + \omega_{31} \wedge \omega_{12} = 0$$

whereas the Arnol'd identities for P_4 read, in turn

$$\begin{array}{lcrcl}
\mathbb{I}_1 & := & \omega_{12} \wedge \omega_{23} + \omega_{23} \wedge \omega_{31} + \omega_{31} \wedge \omega_{12} & = & 0 \\
\mathbb{I}_2 & := & \omega_{12} \wedge \omega_{24} + \omega_{24} \wedge \omega_{41} + \omega_{41} \wedge \omega_{12} & = & 0 \\
\mathbb{I}_3 & := & \omega_{13} \wedge \omega_{34} + \omega_{34} \wedge \omega_{41} + \omega_{41} \wedge \omega_{13} & = & 0 \\
\mathbb{I}_4 & := & \omega_{23} \wedge \omega_{34} + \omega_{34} \wedge \omega_{42} + \omega_{42} \wedge \omega_{23} & = & 0
\end{array}$$

Let us also recall for completeness the definition of the *holonomy algebra* $\mathcal{P}_n$ (see e.g. [28]), generated (over $\mathbb{C}$), by elements t_{ij}, $i, j = 1, 2, \ldots, n$, $\;i < j$, fulfilling the so-called *infinitesimal pure braid relations* :

$$t_{ij} = t_{ji}$$

$$[t_{ik}, t_{ij} + t_{jk}] = [t_{ij}, t_{ik} + t_{jk}] = 0, \quad i, j, k \;\text{ all distinct}$$

$$[t_{ij}, t_{hk}] = 0 \;\text{ if }\; i, j, k, h \;\text{ all distinct}$$

It is actually the universal enveloping algebra of the Lie algebra generated by the t's subject to the infinitesimal braid relations. It is well known that t_{ij} can be depicted as a set of n parallel vertical strings together with a horizontal string connecting string i with string j (with the product defined by juxtaposition).

2.3 – *Hyperlogarithms*

Hyperlogarithms are ("many-valued") holomorphic functions on the complex plane with n distinct points $a_j, j = 1, \ldots, n$ removed, defined by

$$F_n\left(\begin{matrix} a_1,\ldots,a_n \\ b_1,\ldots,b_n \end{matrix}\middle| z\right) := \int_{b_n}^{z} \cdots \left(\int_{b_2}^{t_3} \left(\int_{b_1}^{t_2} \frac{dt_1}{t_1-a_1} \right) \frac{dt_2}{t_2-a_2} \right) \cdots \frac{dt_n}{t_n-a_n}$$

Starting at a point z with the value $F_n\left(\begin{matrix} a_1,\ldots,a_n \\ b_1,\ldots,b_n \end{matrix}\middle| z\right)$ on the main branch and going around a_j (and not around a_k, $k \neq j$) counterclockwise, one reaches the new value

$$F_n\left(\begin{matrix} a_1,\ldots,a_n \\ b_1,\ldots,b_n \end{matrix}\middle| z\right) + \Delta_j F_n$$

where for $1 \leq j \leq n$

$$\Delta_j F_n = 2\pi\sqrt{-1}\, F_{j-1}\left(\begin{matrix} a_1,\ldots,a_{j-1} \\ b_1,\ldots,b_{j-1} \end{matrix}\middle| a_j\right) F_{n-j}\left(\begin{matrix} a_{j+1},\ldots,a_n \\ a_j,\ldots,a_j \end{matrix}\middle| z\right)$$

(*monodromy*). In the following we shall use the following special cases

$$\begin{aligned} \Delta_1 F_1 &= 2\pi\sqrt{-1} \\ \Delta_1 F_2 &= 2\pi\sqrt{-1}\,(\log(z-a_2) - \log(a_1-a_2)) \\ \Delta_2 F_2 &= 2\pi\sqrt{-1}\,(\log(a_2-a_1) - \log(b_1-a_1)) \end{aligned}$$

(see [32] for details) to calculate suitable Chen iterated integrals. Arguments are chosen in $[0, 2\pi)$). For example, and in view of future use, let us evaluate $\int_{A_{12}} \omega_{12}\omega_{13}$, letting for instance "1" wind around "2" *clockwise* (see also Section 4). Consider the hyperlogarithm

$$F_2\left(\begin{matrix} 2,3 \\ 1,1 \end{matrix}\middle| z\right) = \frac{1}{(2\pi\sqrt{-1})^2} \int_1^z \left(\int_1^{t_2} \frac{dt_1}{t_1-2} \right) \frac{dt_2}{t_1-3}$$

(slight abuse of notation) whose monodromy at $z = 1$ after completion of a circuit winding *clockwise* around $a_1 = 2$, but not around $a_2 = 3$, equals

$$-\Delta_1 F_2 = -\frac{1}{(2\pi\sqrt{-1})^2} \cdot 2\pi\sqrt{-1}\,[\log(1-3) - \log(2-3)] = \frac{\sqrt{-1}}{2\pi}\,\log 2$$

(using $\log(-x) = \log x + \sqrt{-1}\,\pi$ for $x > 0$). But this is precisely the iterated integral in question:

$$\int_{A_{12}} \omega_{12}\omega_{13} = \frac{\sqrt{-1}}{2\pi}\,\log 2$$

with the above convention. The interpretation of iterated integrals in terms of monodromies of hyperlogarithms will be crucial in the sequel. We notice that suitable *polylogarithms* play a crucial role in working out the Kontsevich integral for links, whereby one obtains most interesting identities between multiple zeta functions ([26]). However, we need the above different type of calculation.

3. – The basic idea

This section is devoted to elaboration of the main idea of the present work. Recall, in general, the abstract Kniznik-Zamolodchikov-Kohno (KZK) connection ([20, 21, 22, 23]

$$\mathbf{v} = \sum_{i<j} t_{ij}\omega_{ij}$$

defined on $Conf(n, \mathbb{C})$, with the t's fulfilling the infinitesimal braid relations and the ω's fulfilling, in turn, Arnol'd's relations. The KZK connection is flat, namely

$$d\mathbf{v} + \mathbf{v} \wedge \mathbf{v} = \mathbf{v} \wedge \mathbf{v} = \mathbf{0}$$

Then, its parallel transport, defined by a time-ordered exponential involving Chen integrals

$$\rho(\gamma) = T \exp \int_{\gamma} \mathbf{v}$$

(γ being a path in $Conf(n, \mathbb{C})$), gives rise to a representation (call again it ρ) of P_n via the holonomy algebra $\mathcal{P}_n$ discussed above. This is the crucial ingredient in Kontsevich's universal knot invariant construction [24, 6, 28, 26]. In what follows we aim at finding concrete *nilpotent matrix* valued Hain-Tavares connections [31], enforcing flatness via the Arnol'd relations (in this manner, the infinitesimal braid relations for the ensuing t's will then be automatically fulfilled). The Chen series then becomes a finite sum. This idea stems from [30]. In the subsequent sections we shall pursue such a programme for P_3 and P_4, also explaining why the specific method we use does not produce non trivial representations for P_n, $n > 4$.

4. – Representations of P_3

We are looking for $1-$forms v_k

$$v_k := t_k^{12}\omega_{12} + t_k^{13}\omega_{13} + t_k^{23}\omega_{23} \qquad k = 1, 2, 3 \qquad t_k^{ij}(= t_k^{ji}) \in \mathbb{C}$$

such that $v_1 \wedge v_2 = \lambda(\omega_{12} \wedge \omega_{23} + \omega_{23} \wedge \omega_{31} + \omega_{31} \wedge \omega_{12})$ with $\lambda \in \mathbb{C}$. We take the following connection 1-form

$$\mathbf{v} = \begin{pmatrix} 0 & v_1 & v_3 \\ 0 & 0 & v_2 \\ 0 & 0 & 0 \end{pmatrix}$$

with curvature

$$\Omega = d\mathbf{v} + \mathbf{v} \wedge \mathbf{v} = \begin{pmatrix} 0 & 0 & v_1 \wedge v_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

to be set equal to zero, this leading to the conditions:

$$t_1^{12} t_2^{23} - t_2^{12} t_1^{23} \quad = \quad t_1^{23} t_2^{31} - t_2^{23} t_1^{31} \quad = \quad t_1^{31} t_2^{12} - t_2^{31} t_1^{12}$$

Upon intepreting 1-forms as geometric vectors (so long as their coefficients are real), together with their wedge products (which, in turn, become ordinary vector products), the above condition tells us that the two vectors $(t_k^{12}, t_k^{13}, t_k^{23})$, $k = 1, 2$ lie on the plane $x + y + z = 0$:

$$t_k^{12} + t_k^{23} + t_k^{31} = 0 \qquad k = 1, 2$$

(slight abuses of language and obvious notation), so long as they are real. Hence we may replace quadratic conditions by *linear* ones. The geometric picture persists algebraically for complex t's as well. So we get parametric solutions (with $\alpha, \beta, \gamma, \delta, x_{12}, x_{23}, x_{31}$ complex; we also set $x_{ij} = x_{ji}$ throughout):

$$\begin{aligned} v_1 &= \alpha\,(\omega_{12} - \omega_{13}) + \beta\,(\omega_{12} - \omega_{23}) \\ v_2 &= \gamma\,(\omega_{12} - \omega_{13}) + \delta\,(\omega_{12} - \omega_{23}) \\ v_3 &= x_{12}\,\omega_{12} + x_{23}\,\omega_{23} + x_{31}\,\omega_{31} \end{aligned}$$

or

$$\begin{aligned} v_1 &= (\alpha + \beta)\,\omega_{12} - \alpha\,\omega_{13} - \beta\,\omega_{23} \\ v_2 &= (\gamma + \delta)\,\omega_{12} - \gamma\,\omega_{13} - \delta\,\omega_{23} \\ v_3 &= x_{12}\,\omega_{12} + x_{23}\,\omega_{23} + x_{31}\,\omega_{31} \end{aligned}$$

with the only condition $\alpha\delta \neq \beta\gamma$ in order to avoid trivialities in

$$v_1 \wedge v_2 = -(\alpha\delta - \beta\gamma) \cdot \mathbb{I}_1$$

In order to calculate the holonomy

$$\rho(b) = \begin{pmatrix} 1 & \int_b v_1 & \int_b v_1 v_2 + v_3 \\ 0 & 1 & \int_b v_2 \\ 0 & 0 & 1 \end{pmatrix}$$

for a generic pure braid b written as a word in the Artin generators, we must use the following easily established results involving hyperlogarithms (cf $(**)$ in Subsection 2.1, together with Subsection 2.3).

$$\int_{A_{ij}} \omega_{kh} = -\delta_{(ij),(kh)}$$

$$\int_{A_{ij}} (\omega_{ij})^n = \frac{(-1)^n}{n!}$$

Also, upon moving "1" around "2" *clockwise*:

$$\int_{A_{12}} \omega_{12} = -1 \qquad \int_{A_{12}} \omega_{12}\omega_{12} = +\frac{1}{2}$$

$$\int_{A_{12}} \omega_{12}\omega_{13} = +\frac{\sqrt{-1}}{2\pi} \log 2 \qquad \int_{A_{12}} \omega_{13}\omega_{12} = -\frac{\sqrt{-1}}{2\pi} \log 2$$

Moving "1" around "3" *clockwise*:

$$\int_{A_{13}} \omega_{13} = -1 \qquad \int_{A_{13}} \omega_{13}\omega_{13} = +\frac{1}{2}$$

$$\int_{A_{13}} \omega_{12}\omega_{13} = +\frac{1}{2} \qquad \int_{A_{13}} \omega_{13}\omega_{12} = -\frac{1}{2}$$

Moving "2" around "3" *clockwise*:

$$\int_{A_{23}} \omega_{23} = -1 \qquad \int_{A_{23}} \omega_{23}\omega_{23} = +\frac{1}{2}$$

$$\int_{A_{23}} \omega_{12}\omega_{23} = +\frac{\sqrt{-1}}{2\pi} \log 2 \qquad \int_{A_{23}} \omega_{23}\omega_{12} = -\frac{\sqrt{-1}}{2\pi} \log 2$$

We sketch some details of the argument leading to them; a similar, more involved computation will be needed in the sequel.

Let us consider for instance the iterated integral

$$\begin{aligned}\int_b v_1 v_2 &= \alpha\gamma \int_b A^2 + \alpha\delta \int_b AB + \beta\gamma \int_b BA + \beta\delta \int_b B^2 \\ &= \frac{1}{2}\alpha\gamma\Big(\int_b A\Big)^2 + \alpha\delta \int_b AB + \beta\gamma \int_b BA + \frac{1}{2}\beta\delta\Big(\int_b B\Big)^2\end{aligned}$$

with $A = \omega_{12} - \omega_{13}$, $B = \omega_{12} - \omega_{23}$ (closed 1-forms). It is globally a homotopy invariant quantity. The first and fourth summands are homotopy invariant. In view of the arbitrariness of the coefficients, the single iterated integrals $\int_b AB$ and $\int_b BA$ are homotopy invariant as well. One of them, say $\int_b AB$, can be calculated directly, the other follows via ($*$ $*$). Specializing further $b = A_{12}$, we have to use a concrete path of integration, either moving "1" around "2" (clockwise) or conversely. *This choice has to be kept throughout the computation of the single summands entering* $\int_{A_{12}} AB$. The latter is possibly a bit tedious but straightforward (the iterated integrals not involving encircling of singularities vanish) and yields the above results.

Thus we eventually find

$$\rho_3(A_{12}) = \begin{pmatrix} 1 & -\alpha-\beta & \frac{1}{2}(\alpha+\beta)(\gamma+\delta) + (\alpha\delta - \beta\gamma)\frac{\sqrt{-1}}{2\pi}\log 2 + x_{12} \\ 0 & 1 & -\gamma-\delta \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho_3(A_{13}) = \begin{pmatrix} 1 & +\alpha & \frac{1}{2}\alpha\gamma + \frac{1}{2}(\alpha\delta - \beta\gamma) + x_{13} \\ 0 & 1 & +\gamma \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho_3(A_{23}) = \begin{pmatrix} 1 & +\beta & \frac{1}{2}\beta\delta + (\alpha\delta - \beta\gamma)\frac{-\sqrt{-1}}{2\pi}\log 2 + x_{23} \\ 0 & 1 & +\delta \\ 0 & 0 & 1 \end{pmatrix}$$

The central element reads,

$$\rho_3(\Delta_3^2) = \begin{pmatrix} 1 & 0 & x_{12}+x_{13}+x_{23} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

But, since the x_{ij}'s are arbitrary, one gets the following

THEOREM 1. – (i) *There exists a 7-complex parameter family of* 3×3 *nilpotent representations* ρ_3 *of* P_3 *reading, on Artin's generators:*

$$\rho_3(A_{12}) = \begin{pmatrix} 1 & -\alpha-\beta & X_{12} \\ 0 & 1 & -\gamma-\delta \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho_3(A_{13}) = \begin{pmatrix} 1 & +\alpha & X_{13} \\ 0 & 1 & +\gamma \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho_3(A_{23}) = \begin{pmatrix} 1 & +\beta & X_{23} \\ 0 & 1 & +\delta \\ 0 & 0 & 1 \end{pmatrix}$$

(ii) *The central element reads, in turn, with respect to the new parameters:*

$$\rho_3(\varDelta_3^2) = \begin{pmatrix} 1 & 0 & -\alpha\delta - \beta\delta - \alpha\gamma + X_{12} + X_{13} + X_{23} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

REMARKS – 1. It is important to notice that when computing the holonomy of a generic braid b, the representation matrices for the generators entering the word giving b must be written in the *reverse* order: in a product $b_1 \cdot b_2$, b_1 comes first, so $\rho(b_1)$ must accordingly act first. It is readily checked, retrospectively (by hand or by a computer algebra system, e.g. Mathematica®) that the Artin relations are fulfilled with the above convention.

2. Strictly speaking, in view of the arbitrary character of x_{ij}, the computation of double iterated integrals turns out to be unnecessary in this case. However we carried it out since it will be nevertheless needed below.

3. The above representaton is actually a Heisenberg group one (see also Section 7).

We can also construct 4×4 nilpotent matrix representations of P_3 in the following manner. One starts from a nilpotent connection form

$$\mathbf{v} = \begin{pmatrix} 0 & v_1 & v_4 & v_6 \\ 0 & 0 & v_2 & v_5 \\ 0 & 0 & 0 & v_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with curvature form

$$\varOmega = \begin{pmatrix} 0 & 0 & v_1 \wedge v_2 & v_1 \wedge v_5 + v_4 \wedge v_3 \\ 0 & 0 & 0 & v_2 \wedge v_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and holonomy

$$\rho_4(b) = \begin{pmatrix} 1 & \int\limits_b v_1 & \int\limits_b v_1v_2 + v_4 & \int\limits_b v_1v_2v_3 + v_1v_5 + v_4v_3 + v_6 \\ 0 & 1 & \int\limits_b v_2 & \int\limits_b v_2v_3 + v_5 \\ 0 & 0 & 1 & \int\limits_b v_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Upon enforcing flatness via Arnol'd's relations, one arrives at

$$\begin{aligned} v_1 &= (\alpha + \beta)\,\omega_{12} - \alpha\,\omega_{13} - \beta\,\omega_{23} \\ v_2 &= (\gamma + \delta)\,\omega_{12} - \gamma\,\omega_{13} - \delta\,\omega_{23} \\ v_3 &= (\xi + \eta)\,\omega_{12} - \xi\,\omega_{13} - \eta\,\omega_{23} \\ v_4 &= (\sigma + \tau)\,\omega_{12} - \sigma\,\omega_{13} - \tau\,\omega_{23} \\ v_5 &= (\zeta + \lambda)\,\omega_{12} - \zeta\,\omega_{13} - \lambda\,\omega_{23} \\ v_6 &= x_{12}\,\omega_{12} + x_{13}\,\omega_{13} + x_{23}\,\omega_{23} \end{aligned}$$

wherefrom one gets the following

THEOREM 2. – *There exists a 13-complex parameter family of* 4×4 *nilpotent representations* ρ_4 *of* P_3 *reading, on Artin's generators:*

$$\rho_4(A_{12}) = \begin{pmatrix} 1 & -\alpha - \beta & \frac{1}{2}(\alpha+\beta)(\gamma+\delta)+(\alpha\delta-\beta\gamma)\frac{\sqrt{-1}}{2\pi}\log 2-(\sigma+\tau) & X_{12} \\ 0 & 1 & -\gamma - \delta & \frac{1}{2}(\gamma+\delta)(\eta+\xi)+(\gamma\eta-\delta\xi)\frac{\sqrt{-1}}{2\pi}\log 2-(\zeta+\lambda) \\ 0 & 0 & 1 & -\xi - \eta \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho_4(A_{13}) = \begin{pmatrix} 1 & +\alpha & \frac{1}{2}\alpha\gamma+\frac{1}{2}(\alpha\delta-\beta\gamma)+\sigma & X_{13} \\ 0 & 1 & +\gamma & \frac{1}{2}\gamma\xi+\frac{1}{2}(\gamma\eta-\delta\xi)+\zeta \\ 0 & 0 & 1 & +\xi \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho_4(A_{23}) = \begin{pmatrix} 1 & +\beta & \frac{1}{2}\beta\delta+(\alpha\delta-\beta\gamma)(-\frac{\sqrt{-1}}{2\pi}\log 2)+\tau & X_{23} \\ 0 & 1 & +\delta & \frac{1}{2}\delta\eta+(\gamma\eta-\delta\xi)(-\frac{\sqrt{-1}}{2\pi}\log 2)+\lambda \\ 0 & 0 & 1 & +\eta \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(ii) *The central element reads, in turn*

$$\rho_4(\Delta_3^2) = \begin{pmatrix} 1 & 0 & 0 & U \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$U = -(\eta+\xi)(-\frac{1}{2}\beta\gamma+\sigma) - \alpha\lambda - \beta\lambda - \alpha\zeta - \eta\tau + \frac{\sqrt{-1}}{2\pi}\log 2 \cdot (\alpha(\gamma\eta - \delta\xi) + \delta(\alpha\eta - \beta\xi)) + X_{12} + X_{13} + X_{23}$$

One gets more refined representations than the preceding ones, also upon comparison with the Burau one (which is faithful for $n = 3$ and not faithful for $n \geq 5$, [12]).

Important remark. The above 7-parameter and 13-parameter families of representations exhaust the unipotent representations of P_3 in view of Kohno's general theory [20, 21, 22] (see also [2]). The fundamental theorem of Kohno [22], Theorem 1.2.6 – see also Aomoto's Théorème 1 in [2] – shows in particular that every unipotent representation of the pure braid group can be realised as a monodromy representation. A simple general argument for constructing unipotent representations of P_3 can be outlined as follows. One looks for $n \times n$-nilpotent matrices t_{12}, t_{13}, t_{23} fulfilling the infinitesimal braid relations: therefore, two of them can be chosen arbitrarily, the third one can be chosen to be central. In total we have $2 \times n(n-1)/2 + 1 = n(n-1) + 1$ parameters, yielding 7 and 13 for $n = 3, 4$, respectively. Of course one has to solve the iterated integrals involving the Artin generators if one looks for concrete formulae. In this section we have provided *fully explicit* instances of this construction. That this is possibly a non trivial task is better shown in the following section.

5. – Representations of P_4

In this section we extend the previous method and discuss 4×4-nilpotent matrix representations of the 4-strand pure braid group P_4. We start from a nilpotent connection form:

$$\mathbf{v} = \begin{pmatrix} 0 & v_1 & v_4 & v_6 \\ 0 & 0 & v_2 & v_5 \\ 0 & 0 & 0 & v_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with

$$v_k = t_k^{ij}\omega_{ij} \qquad k = 1, \ldots, 6 \qquad i < j$$

having curvature form:

$$\Omega = \begin{pmatrix} 0 & 0 & v_1 \wedge v_2 & v_1 \wedge v_5 + v_4 \wedge v_3 \\ 0 & 0 & 0 & v_2 \wedge v_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The zero curvature condition, *together with the stronger requirement*

$$v_1 \wedge v_5 = 0, \quad v_4 \wedge v_3 = 0 \qquad (***)$$

yields

$$1)\begin{cases} t_j^{12}t_k^{23} - t_k^{12}t_j^{23} &= t_k^{13}t_j^{23} - t_j^{13}t_k^{23} &= t_k^{12}t_j^{13} - t_j^{12}t_k^{13} \\ t_j^{12}t_k^{24} - t_k^{12}t_j^{24} &= t_k^{14}t_j^{34} - t_j^{14}t_k^{34} &= t_k^{12}t_j^{14} - t_j^{12}t_k^{14} \\ t_j^{13}t_k^{34} - t_k^{13}t_j^{34} &= t_k^{14}t_j^{24} - t_j^{14}t_k^{24} &= t_k^{13}t_j^{14} - t_j^{13}t_k^{14} \\ t_j^{23}t_k^{24} - t_k^{23}t_j^{24} &= t_k^{34}t_j^{24} - t_j^{34}t_k^{24} &= t_k^{23}t_j^{34} - t_j^{23}t_k^{34} \end{cases}$$

for $(j,k) \in \{(1,2),(2,3),(1,5),(3,4)\}$, and

$$2)\begin{cases} t_j^{12}t_k^{34} - t_k^{12}t_j^{34} &= 0 \\ t_j^{13}t_k^{24} - t_k^{13}t_j^{24} &= 0 \\ t_j^{23}t_k^{14} - t_k^{23}t_j^{14} &= 0 \end{cases}$$

where $(j,k) \in \{(1,2),(2,3),(1,5),(3,4)\}$, whilst for $k = 6$ the coefficients are arbitrary.

Now, the main point is that independence of Arnol'd's relations allows us to resort to the above geometric interpretation "locally" (one has a sort of complete "decoupling"), abutting in general at a homogeneous *linear* system of 20 equations in 30 unknowns (keeping all forms v_i, $i = 1, 2, \ldots, 5$ and discarding v_6, which does not contribute to the zero curvature condition). Since its rank is full, one finds a 10-parameter solution (two parameters for each form v_i, $i = 1, 2, \ldots, 5$). Actually, it is readily verified that all solutions above fulfil the remaining 12 quadratic equations *automatically*. Thus, in more detail, instead of solving system 1) we solve the homogeneous linear system

$$\begin{cases} t_k^{12} + t_k^{13} + t_k^{23} &= 0 \\ t_k^{12} + t_k^{14} + t_k^{24} &= 0 \\ t_k^{13} + t_k^{34} + t_k^{41} &= 0 \\ t_k^{23} + t_k^{34} + t_k^{42} &= 0 \end{cases}$$

($k = 1, \dots, 5$) whose solutions automatically fulfil system 2). Complex parametric homogeneous solutions are

$$\begin{array}{rcl}
v_1 &=& \alpha\,(\omega_{12} - \omega_{13} - \omega_{24} + \omega_{34}) + \beta\,(\omega_{12} - \omega_{14} - \omega_{23} + \omega_{34}) \\
v_2 &=& \gamma\,(\omega_{12} - \omega_{13} - \omega_{24} + \omega_{34}) + \delta\,(\omega_{12} - \omega_{14} - \omega_{23} + \omega_{34}) \\
v_3 &=& \xi\,(\omega_{12} - \omega_{13} - \omega_{24} + \omega_{34}) + \eta\,(\omega_{12} - \omega_{14} - \omega_{23} + \omega_{34}) \\
v_4 &=& \sigma\,(\omega_{12} - \omega_{13} - \omega_{24} + \omega_{34}) + \tau\,(\omega_{12} - \omega_{14} - \omega_{23} + \omega_{34}) \\
v_5 &=& \zeta\,(\omega_{12} - \omega_{13} - \omega_{24} + \omega_{34}) + \lambda\,(\omega_{12} - \omega_{14} - \omega_{23} + \omega_{34}) \\
v_6 &=& x_{12}\,\omega_{12} + x_{13}\,\omega_{13} + x_{14}\,\omega_{14} + x_{23}\,\omega_{23} + x_{24}\,\omega_{24} + x_{34}\,\omega_{34}
\end{array}$$

or

$$\begin{array}{rcl}
v_1 &=& (\alpha + \beta)\,\omega_{12} - \alpha\,\omega_{13} - \beta\,\omega_{14} - \beta\,\omega_{23} - \alpha\,\omega_{24} + (\alpha + \beta)\,\omega_{34} \\
v_2 &=& (\gamma + \delta)\,\omega_{12} - \gamma\,\omega_{13} - \delta\,\omega_{14} - \delta\,\omega_{23} - \gamma\,\omega_{24} + (\gamma + \delta)\,\omega_{34} \\
v_3 &=& (\xi + \eta)\,\omega_{12} - \xi\,\omega_{13} - \eta\,\omega_{14} - \eta\,\omega_{23} - \xi\,\omega_{24} + (\xi + \eta)\,\omega_{34} \\
v_4 &=& (\sigma + \tau)\,\omega_{12} - \sigma\,\omega_{13} - \tau\,\omega_{14} - \tau\,\omega_{23} - \sigma\,\omega_{24} + (\sigma + \tau)\,\omega_{34} \\
v_5 &=& (\zeta + \lambda)\,\omega_{12} - \zeta\,\omega_{13} - \lambda\,\omega_{14} - \lambda\,\omega_{23} - \zeta\,\omega_{24} + (\zeta + \lambda)\,\omega_{34} \\
v_6 &=& x_{12}\,\omega_{12} + x_{13}\,\omega_{13} + x_{14}\,\omega_{14} + x_{23}\,\omega_{23} + x_{24}\,\omega_{24} + x_{34}\,\omega_{34}
\end{array}$$

Thus we find, for the terms entering the curvature form

$$\begin{array}{rcl}
v_1 \wedge v_2 &=& (\alpha\delta - \beta\gamma)(-\mathbb{I}_1 + \mathbb{I}_2 - \mathbb{I}_3 + \mathbb{I}_4) \\
v_2 \wedge v_3 &=& (\gamma\eta - \delta\xi)(-\mathbb{I}_1 + \mathbb{I}_2 - \mathbb{I}_3 + \mathbb{I}_4) \\
v_1 \wedge v_5 + v_4 \wedge v_3 &=& (\alpha\lambda - \beta\zeta + \eta\sigma - \xi\tau)(-\mathbb{I}_1 + \mathbb{I}_2 - \mathbb{I}_3 + \mathbb{I}_4)
\end{array}$$

Therefore the latter is indeed zero by virtue of Arnol'd's relations. In order to avoid trivialities we require that at least one of the coefficients in the above formula does not vanish.

The holonomy matrix reads:

$$\varrho_4(b) = \begin{pmatrix} 1 & \int_b v_1 & \int_b v_1 v_2 + v_4 & \int_b v_1 v_2 v_3 + v_1 v_5 + v_4 v_3 + v_6 \\ 0 & 1 & \int_b v_2 & \int_b v_2 v_3 + v_5 \\ 0 & 0 & 1 & \int_b v_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

for $b \in P_4$, depending on the sixteen complex parameters $\alpha, \beta, \dots, x_{34}$.

REMARK. – We observe that the *double* iterated integral $\int v_1 v_5 + v_4 v_3$ and the *triple* iterated integral $\int v_1 v_2 v_3$ appear only in the entry $(1, 4)$ of the holonomy matrix (as before, it will be enough to calculate it for the Artin generators);

however, such an entry also depends on the arbitrary 1-form v_6, thus it is *unnecessary* to evaluate these integrals, since the term $(1,4)$ can accommodate *any* complex value.

In order to compute holonomies around generators, we need further double iterated integrals; the relevant formulae are given below.

Moving "1" around "2" *clockwise*:

$$\int_{A_{12}} \omega_{12}\omega_{14} = +\frac{\sqrt{-1}}{2\pi}\log\frac{3}{2} \qquad \int_{A_{12}} \omega_{14}\omega_{12} = -\frac{\sqrt{-1}}{2\pi}\log\frac{3}{2}$$

Moving "1" around "3" *clockwise*:

$$\int_{A_{13}} \omega_{13}\omega_{14} = +\frac{\sqrt{-1}}{2\pi}\log 3 \qquad \int_{A_{13}} \omega_{14}\omega_{13} = -\frac{\sqrt{-1}}{2\pi}\log 3$$

Moving "1" around "4" *clockwise*:

$$\int_{A_{14}} \omega_{14} = -1 \qquad \int_{A_{14}} \omega_{14}\omega_{14} = +\frac{1}{2}$$

$$\int_{A_{14}} \omega_{14}\omega_{12} = -\frac{1}{2}-\frac{\sqrt{-1}}{2\pi}\log 2 \qquad \int_{A_{14}} \omega_{14}\omega_{12} = +\frac{1}{2}+\frac{\sqrt{-1}}{2\pi}\log 2$$

$$\int_{A_{14}} \omega_{14}\omega_{13} = -\frac{1}{2}+\frac{\sqrt{-1}}{2\pi}\log 2 \qquad \int_{A_{14}} \omega_{13}\omega_{14} = +\frac{1}{2}-\frac{\sqrt{-1}}{2\pi}\log 2$$

Moving "2" around "4" *clockwise*:

$$\int_{A_{24}} \omega_{24} = -1 \qquad \int_{A_{24}} \omega_{24}\omega_{24} = +\frac{1}{2}$$

$$\int_{A_{24}} \omega_{24}\omega_{21} = -\frac{\sqrt{-1}}{2\pi}\log 3 \qquad \int_{A_{24}} \omega_{21}\omega_{24} = +\frac{\sqrt{-1}}{2\pi}\log 3$$

$$\int_{A_{24}} \omega_{24}\omega_{23} = -\frac{1}{2} \qquad \int_{A_{24}} \omega_{23}\omega_{24} = +\frac{1}{2}$$

Moving "3" around "4" *clockwise*:

$$\int_{A_{34}} \omega_{34} = -1 \qquad \int_{A_{34}} \omega_{34}\omega_{34} = +\frac{1}{2}$$

$$\int_{A_{34}} \omega_{34}\omega_{31} = -\frac{\sqrt{-1}}{2\pi}\log\frac{3}{2} \qquad \int_{A_{34}} \omega_{31}\omega_{34} = +\frac{\sqrt{-1}}{2\pi}\log\frac{3}{2}$$

$$\int_{A_{34}} \omega_{32}\omega_{34} = +\frac{\sqrt{-1}}{2\pi}\log 2 \qquad \int_{A_{34}} \omega_{34}\omega_{32} = -\frac{\sqrt{-1}}{2\pi}\log 2$$

Notice that, as in the previous cases, the single entries of the holonomy matrix are homotopy invariant, but the explicit computation of the various summands requires a consistent choice of the mutual winding of the strings.

Eventually, we have

THEOREM 3. – (i) *There exists a 16-complex parameter family of* 4×4 *nilpotent representations* ϱ_4 *of* P_4 *reading, on Artin's generators:*

$$\varrho_4(A_{12}) = \begin{pmatrix} 1 & -\alpha-\beta & \frac{1}{2}(\alpha+\beta)(\gamma+\delta)+(\alpha\delta-\beta\gamma)(\frac{\sqrt{-1}}{2\pi}\log\frac{4}{3})-(\sigma+\tau) & X_{12} \\ 0 & 1 & -\gamma-\delta & \frac{1}{2}(\gamma+\delta)(\eta+\xi)+(\gamma\eta-\delta\xi)(\frac{\sqrt{-1}}{2\pi}\log\frac{4}{3})-(\zeta+\lambda) \\ 0 & 0 & 1 & -\eta-\xi \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\varrho_4(A_{13}) = \begin{pmatrix} 1 & \alpha & \frac{\alpha\gamma}{2}(\alpha\delta-\beta\gamma)\left(+\frac{1}{2}+\frac{\sqrt{-1}}{2\pi}\log 3\right)+\sigma & X_{13} \\ 0 & 1 & \gamma & \frac{\gamma\xi}{2}+(\gamma\eta-\delta\xi)\left(+\frac{1}{2}+\frac{\sqrt{-1}}{2\pi}\log 3\right)+\zeta \\ 0 & 0 & 1 & \xi \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\varrho_4(A_{14}) = \begin{pmatrix} 1 & \beta & \frac{\beta\delta}{2}+(\alpha\delta-\beta\gamma)\left(-\frac{\sqrt{-1}}{\pi}\log 2\right)+\tau & X_{14} \\ 0 & 1 & \delta & \frac{\delta\eta}{2}+(\gamma\eta-\delta\xi)\left(-\frac{\sqrt{-1}}{\pi}\log 2\right)+\lambda \\ 0 & 0 & 1 & \eta \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\varrho_4(A_{23}) = \begin{pmatrix} 1 & \beta & \frac{\beta\delta}{2}+(\alpha\delta-\beta\gamma)\left(-\frac{\sqrt{-1}}{\pi}\log 2\right)+\tau & X_{23} \\ 0 & 1 & \delta & \frac{\delta\eta}{2}+(\gamma\eta-\delta\xi)\left(-\frac{\sqrt{-1}}{\pi}\log 2\right)+\lambda \\ 0 & 0 & 1 & \eta \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\varrho_4(A_{24}) = \begin{pmatrix} 1 & \alpha & \frac{\alpha\gamma}{2}+(\alpha\delta-\beta\gamma)\left(-\frac{1}{2}+\frac{\sqrt{-1}}{2\pi}\log 3\right)+\sigma & X_{24} \\ 0 & 1 & \gamma & \frac{\gamma\xi}{2}+(\gamma\eta-\delta\xi)\left(-\frac{1}{2}+\frac{\sqrt{-1}}{2\pi}\log 3\right)+\zeta \\ 0 & 0 & 1 & \xi \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\varrho_4(A_{34}) = \begin{pmatrix} 1 & -\alpha-\beta & \frac{1}{2}(\alpha+\beta)(\gamma+\delta)+(\alpha\delta-\beta\gamma)\left(\frac{\sqrt{-1}}{2\pi}\log\frac{4}{3}\right)-(\sigma+\tau) & X_{34} \\ 0 & 1 & -\gamma-\delta & \frac{1}{2}(\gamma+\delta)(\eta+\xi)+(\gamma\eta-\delta\xi)\left(\frac{\sqrt{-1}}{2\pi}\log\frac{4}{3}\right)-(\zeta+\lambda) \\ 0 & 0 & 1 & -\eta-\xi \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(ii) *The central element reads, in turn*

$$\rho_4(\varDelta_4^2) = \begin{pmatrix} 1 & 0 & 0 & W \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$W = -(\alpha\lambda+\beta\zeta+\xi\tau+\eta\sigma)-2(\alpha\zeta+\beta\lambda+\xi\sigma+\eta\tau)+\frac{1}{2}\gamma\eta(\alpha+\beta)+\frac{1}{2}\beta\gamma(\eta+\xi)- \\ -\frac{\sqrt{-1}}{2\pi}(\alpha\eta-\beta\xi)\left(\gamma\log\frac{9}{4}-\delta\log\frac{16}{3}\right)+X_{12}+X_{13}+X_{14}+X_{23}+X_{24}+X_{34}$$

REMARKS – 1. Matrices are again written in the reverse order; fulfilment of the Artin relations can be again readily checked.

2. Notice that, upon restriction to $P_3 \hookrightarrow P_4$ (obvious inclusion) one has

$$\varrho_4|_{P_3} \neq \rho_3$$

6. – Brunnian type braids

In this section we wish to compute our representations on Brunnian type pure braids, showing that they are indeed able to detect this kind of phenomenon, in the sense that, in general, evaluating the monodromy matrix on such braids yields a non trivial result and these kinds of braids can be (partially) distinguished among them via our invariants. We do not attempt to give a systematic classification but provide specific significant examples.

In analogy to the link case, a pure braid is called Brunnian if upon removing any strand therefrom, it becomes trivial. One may also think of stratified Brunnian braids B_k^n, $k = 0, 1, \ldots, n-2$, i.e. those n-strand braids which become trivial after (and only after) arbitrarily removing k strands therefrom (so Brunnian braids yield B_1^n, and the trivial braid is the only element of B_0^n). Removal of a strand, the j-th, say, of a braid b, is obtained by erasing the generators containing the index j in any word representing b ("forgetting homomorphism").

An immediate example of Brunnian braid is given below:

$$b = [A_{12}, [A_{13}, [\dots, [A_{1,n-1}, A_{1n}]\dots]]]$$

generalizing the *pigtail braid* $b = [A_{12}, A_{13}] = A_{12}A_{13}A_{12}^{-1}A_{13}^{-1}$ which, upon closure, provides a realization of the *Borromean rings*. More generally, the Brunnian braid $[A_{12}^n, A_{13}^m]$, $n, m \in \mathbb{Z}$ can be represented, via ρ_3, as follows:

$$\rho_3\big([A_{12}^n, A_{13}^m]\big) = \begin{pmatrix} 1 & 0 & mn(\alpha\delta - \beta\gamma) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Other Brunnian type words can be given as $a\,b\,c\,a^{-1}b^{-1}c^{-1}$ with a,b,c distinct generators; consider, for example

$$b' = A_{12}A_{13}A_{23}A_{12}^{-1}A_{13}^{-1}A_{23}^{-1}$$

Its 3×3 representation is

$$\rho_3(b') = \begin{pmatrix} 1 & 0 & \beta\gamma - \alpha\delta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

whereas its 4×4 representation ρ_4 reads, in turn,

$$\rho_4(b') = \begin{pmatrix} 1 & 0 & \beta\gamma - \alpha\delta & X \\ 0 & 1 & 0 & \delta\xi - \gamma\eta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with

$$X = -(\alpha\delta - \beta\gamma)\Big(\frac{\xi}{2} + \eta\Big) + (\gamma\eta - \delta\xi)\Big(\frac{\alpha}{2} + \beta\Big) - (\alpha\lambda - \beta\zeta + \eta\sigma - \xi\tau) +$$
$$+ \frac{\sqrt{-1}}{2\pi}\big[-(\alpha\delta - \beta\gamma)\xi\,\log 2 + (\gamma\eta - \delta\xi)\alpha\,\log 2\big]$$

This is to be compared with the ϱ_4-representation

$$\varrho_4(b') = \begin{pmatrix} 1 & 0 & \beta\gamma - \alpha\delta & Y \\ 0 & 1 & 0 & \delta\xi - \gamma\eta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$Y = -(\alpha\delta - \beta\gamma)\Big(\frac{\xi}{2} + \eta\Big) + (\gamma\eta - \delta\xi)\Big(\frac{\alpha}{2} + \beta\Big) - (\alpha\lambda - \beta\zeta + \eta\sigma - \xi\tau) + \\ + \frac{\sqrt{-1}}{2\pi}\big[-(\alpha\delta - \beta\gamma)(\xi\,\log 4 + \eta\,\log 3) + (\gamma\eta - \delta\xi)(\alpha\,\log 4 + \beta\,\log 3)\big]$$

We record the ϱ_4-representation of the Brunnian braid $b = [A_{12}, [A_{13}, A_{14}]]$:

$$\varrho_4(b) = \begin{pmatrix} 1 & 0 & 0 & Z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with

$$Z = (\alpha\delta - \beta\gamma)(\eta + \xi) - (\gamma\eta - \delta\xi)(\alpha + \beta)$$

The same "mirror-inverse" $a\,b\,c\,d\ldots(\ldots d\,c\,b\,a)^{-1}$ words, using the six generators of P_4 (and any permutation thereof) give rise to braids of type B^4_2: indeed, upon deleting a strand, we easily see that the remaining braid is of type B^3_1. For instance, take

$$b'' = A_{12}A_{13}A_{14}A_{23}A_{24}A_{34}A_{12}^{-1}A_{13}^{-1}A_{14}^{-1}A_{23}^{-1}A_{24}^{-1}A_{34}^{-1}$$

with ϱ_4-representation

$$\varrho_4(b'') = \begin{pmatrix} 1 & 0 & 0 & -(\alpha\delta-\beta\gamma)(2\eta+\xi)+(\alpha+2\beta)(\gamma\eta-\delta\xi) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This does not exhaust all possibilities since, for instance, the "shorter" braid $b''' = A_{12}A_{24}A_{13}A_{34}A_{12}^{-1}A_{24}^{-1}A_{13}^{-1}A_{34}^{-1}$ is also of type B^4_2, and differs from b'':

$$\varrho_4(b''') = \begin{pmatrix} 1 & 0 & 0 & \xi(\alpha\delta-\beta\gamma)+\alpha(\delta\xi-\gamma\eta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Notice that, in dealing with Brunnian phenomena, the parameters σ,τ,ζ,λ and X_{ij} play no role. An inductive argument for P_n gives that symmetric type words containing all generators of P_n give B^n_{n-2} Brunnian braids (and the argument can be generalized by using higher order powers of the generators throughout).

Examination of multiple (or nested) commutators such as $[A_{ij}, A_{kh}]$ or $[A_{ij}, [A_{rs}, A_{kh}]]$ yields interesting topological consequences: for example by the 3×3 representation of P_3 we detect only two types of Brunnian braids of the kind $[a, b]$ where a and b represent some generator or its inverse, but we distinguish 12 braid types in the same set using the 4×4 representation (the Burau representation yields, in turn, 13 different braid types).

For type $[a, [b, c]]$ braids we always get 3×3 trivial representations, but 6 distinct braid types via the 4×4 representation (Burau distinguishes 60 braid types).

7. – Heisenberg group representations

Recall that the Heisenberg group $H_n(\mathbb{C})$ is the (multiplicative) group of $n \times n$ upper triangular (complex) matrices of the form

$$\begin{pmatrix} 1 & \mathbf{a}^T & c \\ 0 & \mathbf{1}_{n-2} & \mathbf{b} \\ 0 & 0 & 1 \end{pmatrix}$$

for $\mathbf{a}$, $\mathbf{b} \in \mathbb{C}^{n-2}$ (column vectors) and $c \in \mathbb{C}$.

We show below how to construct various kinds of Heisenberg group representations for P_n, $n = 3, 4$. The role of the Heisenberg group in braid theory has also been stressed by Adem et al. ([1]).

7.1 – 4×4 *Heisenberg representations of* P_4

They can be obtained simply by setting $\delta = \gamma = 0$ in the previously constructed family. The above construction matches with the following purely algebraic procedure: one tries to build up 4×4 nilpotent matrix representations of P_4 by setting, formally

$$A_{ij} := \exp t_{ij}$$

(i.e. $t_{ij} \in Nil_4$) and by attempting at enforcing Artin's relations by making use of the Baker-Campbell-Hausdorff series (which is truncated by nilpotency). One finds, successively,

$$\begin{aligned} e^X e^Y &= e^{X+Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]+\frac{1}{12}[Y,[Y,X]]+\dots} \\ &=: e^{\xi} \end{aligned}$$

and setting $\eta := -X$

$$\begin{aligned} e^X e^Y e^{-X} &= e^{\xi} e^{\eta} \\ &= e^{\xi+\eta+\frac{1}{2}[\xi,\eta]+\frac{1}{12}[\xi,[\xi,\eta]]+\frac{1}{12}[\eta,[\eta,\xi]]+\dots} \end{aligned}$$

Now, in a fourth order nilpotent algebra $[,[,[\ ,\ ]]]\equiv 0$, thus a short computation yields

$$\begin{array}{rrcl} 1) & \xi+\eta & = & Y+\frac{1}{2}[X,Y]+\frac{1}{12}[X,[X,Y]]+\frac{1}{12}[Y,[Y,X]] \\ 2) & [\xi,\eta] & = & [X,Y]+\frac{1}{2}[X,[X,Y]] \\ 3) & [\xi,[\xi,\eta]] & = & [X,[X,Y]]-[Y,[Y,X]] \\ 4) & [\eta,[\eta,\xi]] & = & [X,[X,Y]] \end{array}$$

and, finally, collecting the above results, the following

PROPOSITION 4. – *For any X, $Y \in Nil_4$ one has*

$$e^X e^Y e^{-X} = e^{Y+[X,Y]+\frac{1}{2}[X,[X,Y]]}$$

Now consider, in particular, the braid relation

$$A_{12}^{-1}A_{23}A_{12} = A_{13}A_{12}A_{13}^{-1}$$

We find

$$\begin{aligned} e^{-t_{12}}e^{t_{23}}e^{t_{12}} &= e^{t_{23}+[-t_{12},t_{23}]+\frac{1}{2}[-t_{12},[-t_{12},t_{23}]]} \\ &= e^{t_{23}+[t_{23},t_{12}]+\frac{1}{2}[t_{12},[t_{12},t_{23}]]} \end{aligned}$$

and

$$e^{t_{13}}e^{t_{23}}e^{-t_{13}} = e^{t_{23}+[t_{13},t_{23}]+\frac{1}{2}[t_{13},[t_{13},t_{23}]]}$$

and this first of all requires $[t_{23},t_{12}] = -[t_{23},t_{13}]$ i.e. the infinitesimal braid relations. But, from

$$[t_{13},[t_{13},t_{23}]] = -[t_{13},[t_{12},t_{23}]] = [t_{12},[t_{23},t_{13}]] + [t_{23},[t_{13},t_{12}]]$$

and

$$[t_{12},[t_{12},t_{23}]] = -[t_{12},[t_{13},t_{23}]] = [t_{12},[t_{23},t_{13}]]$$

we see that in general the Artin relation above is *not* fulfilled, the discrepancy being given by a central element in Nil_4. However, if we stick to the *Heisenberg group* $H_4(\mathbb{C})$, then all double commutators vanish and the identity is fulfilled. This persists for the other identities, giving no obstruction to the existence of Heisenberg group representations, provided the infinitesimal braid relations are fulfilled, and one abuts at the ones already found for $\delta=\gamma=0$. Such Heisenberg group representations are nevertheless *unable* to detect Brunnian phenomena (see the preceding sections).

In the sequel we construct some additional representations $P_3 \to H_n(\mathbb{C})$ for $n=4,5$, which can be manufactured via elementary geometric reasoning. They *do not* refine the basic 3×3 ones.

7.2 – 4×4 *Heisenberg representations of* P_3

Start from the connection matrix

$$\mathbf{v}=\begin{pmatrix}0 & v_1 & v_4 & 0\\ 0 & 0 & 0 & v_5\\ 0 & 0 & 0 & v_3\\ 0 & 0 & 0 & 0\end{pmatrix}$$

with curvature form

$$\Omega=\begin{pmatrix}0 & 0 & 0 & v_1\wedge v_5+v_4\wedge v_3\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$$

Its holonomy is

$$\rho(b)=\begin{pmatrix}1 & \int_b v_1 & \int_b v_4 & \int_b v_1v_5+v_4v_3\\ 0 & 1 & 0 & \int_b v_5\\ 0 & 0 & 1 & \int_b v_3\\ 0 & 0 & 0 & 1\end{pmatrix}$$

We look for solutions such that $v_1\wedge v_5+v_4\wedge v_3=0$ but $v_1\wedge v_5\neq 0$ and $v_4\wedge v_3\neq 0$. Setting

$$\begin{aligned}
v_1 &= t_1^{12}\omega_{12}+t_1^{13}\omega_{13}+t_1^{23}\omega_{23}\\
v_3 &= t_3^{12}\omega_{12}+t_3^{13}\omega_{13}+t_3^{23}\omega_{23}\\
v_4 &= t_4^{12}\omega_{12}+t_4^{13}\omega_{13}+t_4^{23}\omega_{23}\\
v_5 &= t_5^{12}\omega_{12}+t_5^{13}\omega_{13}+t_5^{23}\omega_{23}
\end{aligned}$$

we have

$$\begin{aligned}
(t_1^{12},t_1^{13},t_1^{23}) &= (\cos(\alpha),\sin(\alpha),0)\\
(t_3^{12},t_3^{13},t_3^{23}) &= (\sin(\beta+\varepsilon),-\sin(\beta+\varepsilon),\cos(\beta+\varepsilon))\\
(t_4^{12},t_4^{13},t_4^{23}) &= (\sin(\beta),-\sin(\beta),\cos(\beta))\\
(t_5^{12},t_5^{13},t_5^{23}) &= (\cos(\alpha+\varepsilon),\sin(\alpha+\varepsilon),0)
\end{aligned}$$

and the representations (ρ', say) read, on Artin's generators

$$\rho_4'(A_{12}) = \begin{pmatrix} 1 & \cos(\alpha) & \sin(\beta) & \frac{1}{2}(\cos(\alpha)\cos(\alpha+\varepsilon)+\sin(\beta)\sin(\beta+\varepsilon)) \\ 0 & 1 & 0 & \cos(\alpha+\varepsilon) \\ 0 & 0 & 1 & \sin(\beta+\varepsilon) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho_4'(A_{13}) = \begin{pmatrix} 1 & \sin(\alpha) & -\sin(\beta) & \frac{1}{2}(\sin(\alpha)\sin(\alpha+\varepsilon)+\sin(\beta)\sin(\beta+\varepsilon)) \\ 0 & 1 & 0 & \sin(\alpha+\varepsilon) \\ 0 & 0 & 1 & -\sin(\beta+\varepsilon) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho_4'(A_{23}) = \begin{pmatrix} 1 & 0 & \cos(\beta) & \frac{1}{2}\cos(\beta)\cos(\beta+\varepsilon) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \cos(\beta+\varepsilon) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

valid for generic angles.

We could get representations with more parameters upon application of successive rotations in the triples of t_{ij} coefficients such as

$$\begin{aligned} (t_1^{12}, t_1^{13}, t_1^{23}) &= R_v(\theta)R_u(\zeta)\,(\cos(\alpha), \sin(\alpha), 0) \\ (t_3^{12}, t_3^{13}, t_3^{23}) &= R_v(\theta)R_u(\gamma)\,(\sin(\beta+\varepsilon), -\sin(\beta+\varepsilon), \cos(\beta+\varepsilon)) \\ (t_4^{12}, t_4^{13}, t_4^{23}) &= R_v(\theta)R_u(\gamma)\,(\sin(\beta), -\sin(\beta), \cos(\beta)) \\ (t_5^{12}, t_5^{13}, t_5^{23}) &= R_v(\theta)R_u(\zeta)\,(\cos(\alpha+\varepsilon), \sin(\alpha+\varepsilon), 0) \end{aligned}$$

where $R_u(\zeta)$ is the counterclockwise rotation matrix around the direction $u = (1, -1, 0)$ given by

$$R_u(\zeta) = \begin{pmatrix} \cos^2\frac{\zeta}{2} & -\sin^2\frac{\zeta}{2} & -\dfrac{\sin\zeta}{\sqrt{2}} \\ -\sin^2\frac{\zeta}{2} & \cos^2\frac{\zeta}{2} & -\dfrac{\sin\zeta}{\sqrt{2}} \\ \dfrac{\sin\zeta}{\sqrt{2}} & \dfrac{\sin\zeta}{\sqrt{2}} & \cos\zeta \end{pmatrix}$$

where the "angles" γ and ζ are related by

$$\cos\zeta + \sqrt{2}\sin\gamma = \cos\gamma - \frac{1}{\sqrt{2}}\sin\zeta$$

and the rotation $R_v(\theta)$ around the vector $v=(1,1,1)$ given by

$$R_v(\theta)=\begin{pmatrix} \frac{1}{3}(2\cos(\theta)+1) & -\frac{4}{3}\cos\left(\frac{\theta}{2}+\frac{\pi}{6}\right)\sin\left(\frac{\theta}{2}\right) & \frac{4}{3}\cos\left(\frac{\pi}{6}-\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) \\ \frac{4}{3}\cos\left(\frac{\pi}{6}-\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) & \frac{1}{3}(2\cos(\theta)+1) & -\frac{4}{3}\cos\left(\frac{\theta}{2}+\frac{\pi}{6}\right)\sin\left(\frac{\theta}{2}\right) \\ -\frac{4}{3}\cos\left(\frac{\theta}{2}+\frac{\pi}{6}\right)\sin\left(\frac{\theta}{2}\right) & \frac{4}{3}\cos\left(\frac{\pi}{6}-\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right) & \frac{1}{3}(2\cos(\theta)+1) \end{pmatrix}$$

This relation easily stems again from the three-space geometric interpretation of 1-forms and their wedge products and comes from applying R_u. Geometrically, it forces the sum of two parametric vectors of fixed (but different) lengths, spanning the same plane, to be proportional to the Arnol'd form vector $\mathbb{I}_1$, namely

$$v_1\wedge v_5+v_4\wedge v_3 \quad = \quad \sin\varepsilon\Big(\cos\zeta+\sqrt{2}\,\sin\gamma\Big)\cdot\mathbb{I}_1=0$$

i.e. the connection is indeed flat. The above construction can be also easily carried out via ruler and compass.

The expression for the Artin generators becomes however too complicated in general to be effectively displayed here.

7.3 – 5×5 *Heisenberg representations of* P_3

Consider the following connection matrix:

$$\mathbf{v}=\begin{pmatrix} 0 & v_1 & v_2 & v_3 & 0 \\ 0 & 0 & 0 & 0 & v_4 \\ 0 & 0 & 0 & 0 & v_5 \\ 0 & 0 & 0 & 0 & v_6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

with curvature form

$$\Omega=\begin{pmatrix} 0 & 0 & 0 & 0 & v_1\wedge v_4+v_2\wedge v_5+v_3\wedge v_6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and holonomy

$$\rho(b) = \begin{pmatrix} 1 & \int_b v_1 & \int_b v_2 & \int_b v_3 & \int_b v_1 v_4 + v_2 v_5 + v_3 v_6 \\ 0 & 1 & 0 & 0 & \int_b v_4 \\ 0 & 0 & 1 & 0 & \int_b v_5 \\ 0 & 0 & 0 & 1 & \int_b v_6 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

One finds, again via the same kind of geometric reasoning, the solutions

$$\begin{aligned} (t_1^{12}, t_1^{13}, t_1^{23}) &= (0, \cos(\alpha), \sin(\alpha)) \\ (t_4^{12}, t_4^{13}, t_4^{23}) &= (0, \cos(\alpha+\varepsilon), \sin(\alpha+\varepsilon)) \\ (t_2^{12}, t_2^{13}, t_2^{23}) &= (\sin(\beta), 0, \cos(\beta)) \\ (t_5^{12}, t_5^{13}, t_5^{23}) &= (\sin(\beta+\varepsilon), 0, \cos(\beta+\varepsilon)) \\ (t_3^{12}, t_3^{13}, t_3^{23}) &= (\cos(\gamma), \sin(\gamma), 0) \\ (t_6^{12}, t_6^{13}, t_6^{23}) &= (\cos(\gamma+\varepsilon), \sin(\gamma+\varepsilon), 0) \end{aligned}$$

Indeed

$$\begin{aligned} v_1 \wedge v_4 &= -\sin(\varepsilon)\,\omega_{13} \wedge \omega_{23} \\ v_2 \wedge v_5 &= +\sin(\varepsilon)\,\omega_{12} \wedge \omega_{23} \\ v_3 \wedge v_6 &= -\sin(\varepsilon)\,\omega_{12} \wedge \omega_{13} \end{aligned}$$

therefore

$$v_1 \wedge v_4 + v_2 \wedge v_5 + v_3 \wedge v_6 = +\sin(\varepsilon) \cdot \mathbb{I}_1 = 0$$

and one has the following expressions for the generators:

$$\rho_5(A_{12}) = \begin{pmatrix} 1 & 0 & \sin(\beta) & \cos(\gamma) & \frac{1}{2}(\cos(\gamma)\cos(\gamma+\varepsilon) + \sin(\beta)\sin(\beta+\varepsilon)) \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \sin(\beta+\varepsilon) \\ 0 & 0 & 0 & 1 & \cos(\gamma+\varepsilon) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho_5(A_{13}) = \begin{pmatrix} 1 & \cos(\alpha) & 0 & \sin(\gamma) & \frac{1}{2}(\cos(\alpha)\cos(\alpha+\varepsilon)+\sin(\gamma)\sin(\gamma+\varepsilon)) \\ 0 & 1 & 0 & 0 & \cos(\alpha+\varepsilon) \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \sin(\gamma+\varepsilon) \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rho_5(A_{23}) = \begin{pmatrix} 1 & \sin(\alpha) & \cos(\beta) & 0 & \frac{1}{2}(\cos(\beta)\cos(\beta+\varepsilon)+\sin(\alpha)\sin(\alpha+\varepsilon)) \\ 0 & 1 & 0 & 0 & \sin(\alpha+\varepsilon) \\ 0 & 0 & 1 & 0 & \cos(\beta+\varepsilon) \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

also valid for generic angles.

8. – Conclusions and outlook

In this paper we have tackled the problem of constructing concrete representations of the pure braid group via parallel transport of nilpotent flat connections (nilpotency is required in order to have a terminating Chen series). This led to the discovery – via a simple linearization method – of possibly interesting families of connections, yielding representations which are able to distinguish, in particular, several classes of Brunnian type braids. *However, the proliferation of constraints dictated by the Arnol'd relations makes the linearization approach, as it stands, unavailable for $n > 4$.* Specifically, the argument goes as follows: in the general case one has $\binom{n}{2}$ basis forms, and $\binom{n}{3}$ Arnol'd's relations. The above linearization method (depending on the appropriate generalization of the crucial assumption ($***$) in Section 4) would give rise to a homogeneous system of nullity given by

$$[\binom{n}{2} - \binom{n}{3}] \cdot [\binom{n}{2} - 1]$$

(the second factor being the number of forms v_i entering the zero curvature condition), which is *non positive* for $n > 4$, so, a fortiori, one cannot fulfil the remaining quadratic conditions non trivially. Obviously, for $n = 3, 4$ we recover the preceding results.

We finally notice that, even if one finds a way out of the above *impasse*, one is still left with the computation of higher order hyperlogarithms, and this appears to be quite difficult at a first glance, in view of the intricate monodromy problems arising therein. The same problem crops up if one tried to build up higher dimensional representations of P_3 and P_4 via the methods of this note, in view of improving their efficacy in distinguishing braid types. Nevertheless this could in principle lead to possibly interesting new hyperlogarithmic identities.

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