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Some Inequalities of Hermite-Hadamard Type for Convex Functions of Commuting Selfadjoint Operators in Hilbert Spaces

S. S. Dragomir

Abstract. – *Some operator inequalities for convex functions of commuting selfadjoint operators that are related to the Hermite-Hadamard inequality are given. Natural examples for some fundamental convex functions are presented as well.*

1. – Introduction

If $f : I \to \mathbb{R}$ is a convex function on the interval I, then for any $a, b \in I$ with $a \neq b$ we have the following double inequality

$$\text{(HH)} \qquad f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}.$$

This remarkable result is well known in the literature as the *Hermite-Hadamard inequality* [25].

For various generalizations, extensions, reverses and related inequalities, see [1], [2], [3], [13], [16], [18], [19], [20], [21], [25] the monograph [11] and the references therein.

Let A be a selfadjoint linear operator on a complex Hilbert space $(H; \langle .,. \rangle)$. The *Gelfand map* establishes a $*$-isometrically isomorphism Φ between the set $C(Sp(A))$ of all *continuous functions* defined on the *spectrum* of A, denoted $Sp(A)$, an the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows (see for instance [14, p. 3]):

For any $f, g \in C(Sp(A))$ and any $\alpha, \beta \in \mathbb{C}$ we have

(i) $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;
(ii) $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;
(iii) $\|\Phi(f)\| = \|f\| := \sup_{t \in Sp(A)} |f(t)|$;
(iv) $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in Sp(A)$.

With this notation we define

$$f(A) := \Phi(f) \text{ for all } f \in C(Sp(A))$$

and we call it the *continuous functional calculus* for a selfadjoint operator A.

If A is a selfadjoint operator and f is a real valued continuous function on $Sp(A)$, then $f(t) \geq 0$ for any $t \in Sp(A)$ implies that $f(A) \geq 0$, *i.e.* $f(A)$ is a positive operator on H. Moreover, if both f and g are real valued functions on $Sp(A)$ then the following important property holds:

$$\text{(P)} \qquad f(t) \geq g(t) \text{ for any } t \in Sp(A) \text{ implies that } f(A) \geq g(A)$$

in the operator order of $B(H)$.

For a recent monograph devoted to various inequalities for functions of selfadjoint operators, see [14] and the references therein.

The following result provides an operator version for the Jensen inequality (see [14, p. 5]):

THEOREM 1 (Jensen). – *Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A) \subseteq [m, M]$ for some scalars m, M with $m<M$. If f is a convex function on $[m, M]$, then*

$$\text{(MP)} \qquad f(\langle Ax, x\rangle) \leq \langle f(A)x, x\rangle$$

for each $x \in H$ with $\|x\| = 1$.

The Hermite-Hadamard inequality (HH) can be extended to selfadjoint operators as follows, see [7].

Let A and B selfadjoint operators on the Hilbert space H and assume that $Sp(A), Sp(B) \subseteq [m, M]$ for some scalars m, M with $m<M$. If f is a convex function on $[m, M]$, then

$$\begin{aligned} f\left(\frac{\langle Ax, x\rangle + \langle By, y\rangle}{2}\right) &\leq \int_0^1 f((1-t)\langle Ax, x\rangle + t\langle By, y\rangle)dt \\ &\leq \left\langle \left[\int_0^1 f((1-t)A + t\langle By, y\rangle 1_H)dt\right] x, x\right\rangle \\ &\leq \frac{1}{2}[\langle f(A)x, x\rangle + f(\langle By, y\rangle)] \\ &\leq \frac{1}{2}[\langle f(A)x, x\rangle + \langle f(B)y, y\rangle] \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} f\left(\frac{\langle Ax, x\rangle + \langle By, y\rangle}{2}\right) &\leq \left\langle f\left(\frac{A + \langle By, y\rangle 1_H}{2}\right) x, x\right\rangle \\ &\leq \left\langle \left[\int_0^1 f((1-t)A + t\langle By, y\rangle 1_H)dt\right] x, x\right\rangle \end{aligned} \tag{1.2}$$

for each $x, y \in H$ with $\|x\| = \|y\| = 1$.

It is important to remark that, from the inequalities (1.1) and (1.2) we have the following Hermite-Hadamard's type results in the operator order of $B(H)$ and for the convex function $f:[m,M]\to\mathbb{R}$

$$f\left(\frac{A+\langle By,y\rangle 1_H}{2}\right)\le\int_0^1 f((1-t)A+t\langle By,y\rangle 1_H)dt \le\frac{1}{2}[f(A)+f(\langle By,y\rangle)1_H] \tag{1.3}$$

for any $y\in H$ with $\|y\|=1$ and any selfadjoint operators A,B with spectra in $[m,M]$.

In particular, we have from (1.3)

$$f\left(\frac{A+\langle Ay,y\rangle 1_H}{2}\right)\le\int_0^1 f((1-t)A+t\langle Ay,y\rangle 1_H)dt \le\frac{1}{2}[f(A)+f(\langle Ay,y\rangle)1_H] \tag{1.4}$$

for any $y\in H$ with $\|y\|=1$ and

$$f\left(\frac{A+s1_H}{2}\right)\le\int_0^1 f((1-t)A+ts1_H)dt\le\frac{1}{2}[f(A)+f(s)1_H] \tag{1.5}$$

for any $s\in[m,M]$.

As a particular case of the above results we have the following refinement of the Jensen's inequality:

Let A be a selfadjoint operator on the Hilbert space H and assume that $Sp(A)\subseteq[m,M]$ for some scalars m,M with $m<M$. If f is a convex function on $[m,M]$, then

$$\begin{aligned} f(\langle Ax,x\rangle) &\le\left\langle f\left(\frac{A+\langle Ax,x\rangle 1_H}{2}\right)x,x\right\rangle \\ &\le\left\langle\left[\int_0^1 f((1-t)A+t\langle Ax,x\rangle 1_H)dt\right]x,x\right\rangle \\ &\le\frac{1}{2}[\langle f(A)x,x\rangle+f(\langle Ax,x\rangle)]\le\langle f(A)x,x\rangle. \end{aligned} \tag{1.6}$$

A real valued continuous function f on an interval I is said to be *operator convex (operator concave)* if

$$f((1-\lambda)A+\lambda B)\le(\ge)(1-\lambda)f(A)+\lambda f(B) \tag{OC}$$

in the operator order, for all $\lambda\in[0,1]$ and for every selfadjoint operator A and B

on a Hilbert space H whose spectra are contained in I. Notice that a function f is operator concave if $-f$ is operator convex.

A real valued continuous function f on an interval I is said to be *operator monotone* if it is monotone with respect to the operator order, i.e., $A \leq B$ with $Sp(A), Sp(B) \subset I$ imply $f(A) \leq f(B)$.

For some fundamental results on operator convex (operator concave) and operator monotone functions, see [14] and the references therein.

As examples of such functions, we note that $f(t) = t^r$ is operator monotone on $[0, \infty)$ if and only if $0 \leq r \leq 1$. The function $f(t) = t^r$ is operator convex on $(0, \infty)$ if either $1 \leq r \leq 2$ or $-1 \leq r \leq 0$ and is operator concave on $(0, \infty)$ if $0 \leq r \leq 1$. The logarithmic function $f(t) = \ln t$ is operator monotone and operator concave on $(0, \infty)$. The entropy function $f(t) = -t \ln t$ is operator concave on $(0, \infty)$. The exponential function $f(t) = e^t$ is neither operator convex nor operator monotone.

Let $f : I \to \mathbb{R}$ be an operator convex function on the interval I. Then for any selfadjoint operators A and B with spectra in I we have the inequality in the operator order [8]

$$f\left(\frac{A+B}{2}\right) \leq \int_0^1 f((1-t)A + tB)dt \leq \frac{f(A)+f(B)}{2}. \tag{1.7}$$

With the above assumptions for f, A and B we also have the inequality [8]

$$\begin{aligned}(0 \leq) \int_0^1 f((1-t)A + tB)dt - f\left(\frac{A+B}{2}\right) \\ \leq \frac{f(A)+f(B)}{2} - \int_0^1 f((1-t)A + tB)dt\end{aligned} \tag{1.8}$$

in the operator order.

For other inequalities of Hermite-Hadamard type for operator convex functions, see [8].

Motivated by the above results, we investigate in this paper the corresponding Hermite-Hadamard inequality for two commuting operators and show that, in fact, the inequality (1.7) remains valid in this case for convex functions. Other related results, refinements, reverse inequalities and some applications for particular functions of interest are provided as well.

2. – Some General Results

Let $(H, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space. Recall that a bounded linear operator A is *commuting* with the operator B if $AB = BA$. Moreover, if A and B are two commuting bounded selfadjoint operators, then for any continuous function $f : \mathbb{R} \to \mathbb{R}, f(A)$ is also commuting with B. Examples of selfadjoint commutative operators can be taken from Matrices Theory. Commutative matrices have been studied since late in the last century. They are not often the topic of an independent study like [31], but usually the reader can find a chapter on commutative matrices in monographs on linear algebra (e.g. [16] and [17]).

It is known, see for instance [30, p. 356-358], that if A and B are two *commuting bounded selfadjoint operators* on the complex Hilbert space H, then there exists a bounded selfadjoint operator S on H and two bounded functions φ and ψ such that $A = \varphi(S)$ and $B = \psi(S)$. Moreover, if $\{E_t\}$ is the spectral family over the closed interval $[0, 1]$ for the selfadjoint operator S, then $S = \int_{0-}^{1} t dE_t$, where the integral is taken in the Riemann-Stieltjes sense, the functions φ and ψ are summable with respect with $\{E_t\}$ on $[0, 1]$ and

$$A = \varphi(S) = \int_{0-}^{1} \varphi(t) dE_t \text{ and } B = \psi(S) = \int_{0-}^{1} \psi(t) dE_t. \tag{2.1}$$

Now, if A and B are as above with $Sp(A), Sp(B) \subseteq J$ an interval of real numbers, then for any continuous functions $f, g : J \to \mathbb{C}$ we have the representations

$$f(A) = \int_{0-}^{1} (f \circ \varphi)(t) dE_t \text{ and } g(B) = \int_{0-}^{1} (g \circ \psi)(t) dE_t. \tag{2.2}$$

For some applications of these facts to synchronous functions and Čebyšev type inequalities, see [12].

Now, if the function $f : J \to \mathbb{R}$ is continuous convex and if A and B are two commuting bounded selfadjoint operators on the complex Hilbert space H with $Sp(A), Sp(B) \subseteq J$, then utilizing the representations (2.1) we have in the operator order

$$\begin{aligned} f((1-\lambda)A + \lambda B) &= \int_{0-}^{1} f[(1-\lambda)\varphi(t) + \lambda\psi(t)] dE_t \\ &\leq (1-\lambda) \int_{0-}^{1} f(\varphi(t)) dE_t + \lambda \int_{0-}^{1} f(\psi(t)) dE_t \\ &= (1-\lambda) f(A) + \lambda f(B) \end{aligned} \tag{2.3}$$

for any $\lambda \in [0, 1]$.

This shows that the usual convexity is preserved for the operator order when commutativity of the operators A and B is assumed.

Therefore we can state the following particular inequalities of interest:

Let A and B be two commuting positive operators and $p \geq 1 (p \in (0,1))$, then

$$((1-\lambda)A+\lambda B)^p \leq (\geq)(1-\lambda)A^p+\lambda B^p, \lambda \in [0,1]. \tag{2.4}$$

Moreover, if A and B are positive definite then the inequality with "$\leq$" also holds for $p<0$.

If A and B are two commuting selfadjoint operators and $p \geq 1$ then we have the modulus inequality

$$|(1-\lambda)A+\lambda B|^p \leq (1-\lambda)|A|^p+\lambda|B|^p, \lambda \in [0,1]. \tag{2.5}$$

In particular, we have the triangle inequality

$$|A+B| \leq |A|+|B|. \tag{2.6}$$

For A and B positive definite and commuting, we also have the logarithmic inequalities

$$\ln((1-\lambda)A+\lambda B) \geq (1-\lambda)\ln A+\lambda \ln B, \lambda \in [0,1] \tag{2.7}$$

and

$$\begin{aligned} &((1-\lambda)A+\lambda B)\ln((1-\lambda)A+\lambda B) \\ &\leq (1-\lambda)A\ln A+\lambda B\ln B, \lambda \in [0,1]. \end{aligned} \tag{2.8}$$

THEOREM 2. – *If the function $f: J \to \mathbb{R}$ is continuous convex on the interval J and if A and B are two commuting bounded selfadjoint operators on the complex Hilbert space H with $Sp(A), Sp(B) \subseteq J$, then*

$$f\left(\frac{A+B}{2}\right) \leq \int_0^1 f((1-\lambda)A+\lambda B)d\lambda \leq \frac{f(A)+f(B)}{2}. \tag{2.9}$$

PROOF. – Since for any $\lambda \in [0,1]$ the operators $(1-\lambda)A+\lambda B$ and $\lambda A+(1-\lambda)B$ are commutative and $Sp((1-\lambda)A+\lambda B), Sp(\lambda A+(1-\lambda)B) \subseteq J$, then by (2.3) we have

$$f\left(\frac{A+B}{2}\right) \leq \frac{1}{2}[f((1-\lambda)A+\lambda B)+f(\lambda A+(1-\lambda)B)]$$

for any $\lambda \in [0,1]$.

Integrating this inequality on $[0,1]$ and taking into account that

$$\int_0^1 f((1-\lambda)A+\lambda B)d\lambda = \int_0^1 f(\lambda A+(1-\lambda)B)d\lambda$$

we obtain the first inequality in (2.9).

By (2.3) we also have

$$\frac{1}{2}[f((1-\lambda)A+\lambda B)+f(\lambda A+(1-\lambda)B)]\leq\frac{f(A)+f(B)}{2}$$

for any $\lambda \in [0,1]$, which by integration produces the second part of (2.9). □

COROLLARY 1. – *With the assumptions of Theorem* 2 *we have*

$$\begin{aligned}&\frac{1}{2}\left[f\left(\frac{3A+B}{4}\right)+f\left(\frac{A+3B}{4}\right)\right]\\ &\leq\int_0^1 f((1-\lambda)A+\lambda B)d\lambda\\ &\leq\frac{1}{2}\left[f\left(\frac{A+B}{2}\right)+\frac{f(A)+f(B)}{2}\right].\end{aligned}\tag{2.10}$$

PROOF. – On making use of the change of variable $u=2\lambda$ we have

$$\int_0^{1/2} f((1-\lambda)A+\lambda B)d\lambda=\frac{1}{2}\int_0^1 f\left((1-u)A+u\frac{A+B}{2}\right)du$$

and by the change of variable $u=2\lambda-1$ we have

$$\int_{1/2}^1 f((1-\lambda)A+\lambda B)d\lambda=\frac{1}{2}\int_0^1 f\left((1-u)\frac{A+B}{2}+uB\right)du.$$

Utilising the Hermite-Hadamard inequality (2.9) we can write

$$f\left(\frac{3A+B}{4}\right)\leq\int_0^1 f\left((1-u)A+u\frac{A+B}{2}\right)du\leq\frac{1}{2}\left[f(A)+f\left(\frac{A+B}{2}\right)\right]$$

and

$$f\left(\frac{A+3B}{4}\right)\leq\int_0^1 f\left((1-u)\frac{A+B}{2}+uB\right)du\leq\frac{1}{2}\left[f(A)+f\left(\frac{A+B}{2}\right)\right],$$

which by summation and division by two produces the desired result (2.10). □

REMARK 1. – The second inequality in (2.10) is equivalent with

$$\begin{aligned}0&\leq\int_0^1 f((1-\lambda)A+\lambda B)d\lambda-f\left(\frac{A+B}{2}\right)\\ &\leq\frac{f(A)+f(B)}{2}-\int_0^1 f((1-\lambda)A+\lambda B)d\lambda,\end{aligned}\tag{2.11}$$

which shows that $\int_0^1 f((1-\lambda)A+\lambda B)d\lambda$ is nearer in the operator order to $f\left(\frac{A+B}{2}\right)$ than to $\frac{f(A)+f(B)}{2}$.

The interested reader may continue to apply the Hermite-Hadamard inequality on other subintervals such as $\left[A, \frac{3A+B}{4}\right]$, $\left[\frac{3A+B}{4}, \frac{A+B}{2}\right]$, $\left[\frac{A+B}{2}, \frac{A+3B}{4}\right]$ and $\left[\frac{A+3B}{4}, B\right]$ etc., however the resulting inequalities are complicate and therefore are omitted here.

REMARK 2. – Assume that A and B are two commuting positive operators and such that $A-B$ is invertible. Then for $p>0$, by utilizing the representation (2.1) and Fubini's theorem we have

$$\begin{aligned}\int_0^1 ((1-\lambda)A+\lambda B)^p d\lambda &= \int_0^1 \left(\int_{0-}^1 [(1-\lambda)\varphi(t)+\lambda\psi(t)]^p dE_t\right) d\lambda \\ &= \int_{0-}^1 \left(\int_0^1 [(1-\lambda)\varphi(t)+\lambda\psi(t)]^p d\lambda\right) dE_t \\ &= \frac{1}{p+1}\int_{0-}^1 \frac{\varphi^{p+1}(t)-\psi^{p+1}(t)}{\varphi(t)-\psi(t)} dE_t \\ &= \frac{1}{p+1}(A-B)^{-1}\left(A^{p+1}-B^{p+1}\right).\end{aligned}$$

Similarly, if A and B are two commuting positive definite operators and such that $A-B$ is invertible, then for $p \in (-\infty, 0)\backslash\{-1\}$ we also have

$$\int_0^1 ((1-\lambda)A+\lambda B)^p d\lambda = \frac{1}{p+1}(A-B)^{-1}\left(A^{p+1}-B^{p+1}\right).$$

Also, if A and B are two commuting positive definite operators and such that $A-B$ is invertible, then

$$\int_0^1 ((1-\lambda)A+\lambda B)^{-1} d\lambda = (A-B)^{-1}(\ln A - \ln B).$$

Utilising the Hermite-Hadamard inequality (2.9) we then have for $p \geq 1 (p \in (0,1))$

$$\left(\frac{A+B}{2}\right)^p \leq (\geq) \frac{1}{p+1}(A-B)^{-1}\left(A^{p+1}-B^{p+1}\right) \leq (\geq) \frac{A^p+B^p}{2}, \tag{2.12}$$

where A and B are two commuting positive operators and such that $A-B$ is invertible.

If $p \in (-\infty, 0)\backslash\{-1\}$ the inequality (2.12) also holds for "$\geq$" provided A and B are two commuting positive definite operators and such that $A - B$ is invertible.

With the last assumptions for A and B we also have

$$\left(\frac{A+B}{2}\right)^{-1} \leq (A-B)^{-1}(\ln A - \ln B) \leq \frac{A^{-1}+B^{-1}}{2}. \tag{2.13}$$

We also have that

THEOREM 3. – *If the function $f : J \to \mathbb{R}$ is continuous convex on the interval J and if A and B are two commuting bounded selfadjoint operators on the complex Hilbert space H with $Sp(A), Sp(B) \subseteq J$, then*

$$\begin{aligned} &2\min\{t, 1-t\}\left[\frac{f(A)+f(B)}{2} - f\left(\frac{A+B}{2}\right)\right] \\ &\leq (1-\lambda)f(A) + \lambda f(B) - f((1-\lambda)A + \lambda B) \\ &\leq 2\max\{t, 1-t\}\left[\frac{f(A)+f(B)}{2} - f\left(\frac{A+B}{2}\right)\right] \end{aligned} \tag{2.14}$$

in the operator order.

PROOF. – First of all, we recall the following result obtained by the author in [6] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} &n \min_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right] \\ &\leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i) - f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \\ &\leq n \max_{i\in\{1,\dots,n\}}\{p_i\}\left[\frac{1}{n}\sum_{i=1}^{n} f(x_i) - f\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right], \end{aligned} \tag{2.15}$$

where $f : C \to \mathbb{R}$ is a convex function defined on the convex subset C of the linear space X, $\{x_i\}_{i\in\{1,\dots,n\}}$ are vectors in C and $\{p_i\}_{i\in\{1,\dots,n\}}$ are nonnegative numbers with $P_n := \sum_{i=1}^{n} p_i > 0$.

For $n = 2$ we deduce from (2.15) that

$$\begin{aligned} &2\min\{\lambda, 1-\lambda\}\left[\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right] \\ &\leq \lambda f(x) + (1-\lambda)f(y) - f(\lambda x + (1-\lambda)y) \\ &\leq 2\max\{\lambda, 1-\lambda\}\left[\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right] \end{aligned} \tag{2.16}$$

for any $x, y \in C$ and $\lambda \in [0, 1]$.

Now, let φ and ψ the functions that represent the commuting operators A and B as in the equation (2.1). Then by (2.16) we have

$$\begin{aligned} &2\min\{\lambda,1-\lambda\}\left[\frac{f(\varphi(t))+f(\psi(t))}{2}-f\left(\frac{\varphi(t)+\psi(t)}{2}\right)\right]\\ &\le \lambda f(\varphi(t))+(1-\lambda)f(\psi(t))-f(\lambda\varphi(t)+(1-\lambda)\psi(t))\\ &\le 2\max\{\lambda,1-\lambda\}\left[\frac{f(\varphi(t))+f(\psi(t))}{2}-f\left(\frac{\varphi(t)+\psi(t)}{2}\right)\right] \end{aligned} \tag{2.17}$$

for any $\lambda \in [0,1]$ and $t \in [0,1]$.

Integrating over dE_t we deduce from (2.17) the following inequality in the operator order

$$\begin{aligned} &2\min\{\lambda,1-\lambda\}\\ &\times\left[\frac{\int_{0-}^{1} f(\varphi(t))dE_t+\int_{0-}^{1} f(\psi(t))dE_t}{2}-\int_{0-}^{1} f\left(\frac{\varphi(t)+\psi(t)}{2}\right)dE_t\right]\\ &\le \lambda\int_{0-}^{1} f(\varphi(t))dE_t+(1-\lambda)\int_{0-}^{1} f(\psi(t))dE_t\\ &-\int_{0-}^{1} f(\lambda\varphi(t)+(1-\lambda)\psi(t))dE_t\\ &\le 2\max\{\lambda,1-\lambda\}\\ &\times\left[\frac{\int_{0-}^{1} f(\varphi(t))dE_t+\int_{0-}^{1} f(\psi(t))dE_t}{2}-\int_{0-}^{1} f\left(\frac{\varphi(t)+\psi(t)}{2}\right)dE_t\right]. \end{aligned} \tag{2.18}$$

Now, taking into account the integral representation of continuous functions of selfadjoint operators (2.2) we deduce the desired result (2.14). □

REMARK 3. – If A and B are two commuting selfadjoint operators and $p \ge 1$ then we have the modulus inequality

$$\begin{aligned} &2\min\{\lambda,1-\lambda\}\left[\frac{|A|^p+|B|^p}{2}-\left|\frac{A+B}{2}\right|^p\right]\\ &\le (1-\lambda)|A|^p+\lambda|B|^p-|(1-\lambda)A+\lambda B|^p\\ &\le 2\max\{\lambda,1-\lambda\}\left[\frac{|A|^p+|B|^p}{2}-\left|\frac{A+B}{2}\right|^p\right] \end{aligned} \tag{2.19}$$

where $\lambda \in [0,1]$.

In particular, we have

$$\begin{aligned}&\min\{\lambda, 1-\lambda\}[|A|+|B|-|A+B|]\\ &\le (1-\lambda)|A|+\lambda|B|-|(1-\lambda)A+\lambda B|\\ &\le 2\max\{\lambda,1-\lambda\}[|A|+|B|-|A+B|]\end{aligned} \tag{2.20}$$

for any $\lambda \in [0,1]$.

If A and B are two commuting positive definite operators, then we have the logarithmic inequalities

$$\begin{aligned}&2\min\{\lambda,1-\lambda\}\left[\ln\left(\frac{A+B}{2}\right)-\frac{\ln A+\ln B}{2}\right]\\ &\le \ln((1-\lambda)A+\lambda B)-(1-\lambda)\ln A-\lambda\ln B\\ &\le 2\max\{\lambda,1-\lambda\}\left[\left[\ln\left(\frac{A+B}{2}\right)-\frac{\ln A+\ln B}{2}\right]\right]\end{aligned} \tag{2.21}$$

for any $\lambda \in [0,1]$.

3. – Some Results for Differentiable Functions

The following reverse of the inequality (2.3) for differentiable functions holds:

THEOREM 4. – *Let $f : J \to \mathbb{R}$ be a convex continuously differentiable function on the interior $\mathring{J}$ of J and A and B two commuting selfadjoint operators with $Sp(A), Sp(A) \subset \mathring{J}$. Then we have in the operator order*

$$\begin{aligned}0 &\le (1-\lambda)f(A)+\lambda f(B)-f((1-\lambda)A+\lambda B)\\ &\le (1-\lambda)\lambda(A-B)[f'(A)-f'(B)],\end{aligned} \tag{3.1}$$

for any $\lambda \in [0,1]$.

In particular, we have

$$0 \le \frac{f(A)+f(B)}{2}-f\left(\frac{A+B}{2}\right) \le \frac{1}{4}(A-B)[f'(A)-f'(B)]. \tag{3.2}$$

PROOF. – We have the gradient inequality

$$f(u)-f(v) \ge f'(v)(u-v) \tag{3.3}$$

for any $u, v \in \mathring{J}$.

Utilising the gradient inequality (3.3) we have

$$f(\lambda x+(1-\lambda)y)-f(x) \ge (1-\lambda)f'(x)(y-x) \tag{3.4}$$

and

$$f(\lambda x+(1-\lambda)y)-f(y)\geq -\lambda f^{'}(y)(y-x), \tag{3.5}$$

for any $\lambda \in [0,1]$ and $x,y\in \mathring{J}$.

If we multiply (3.4) with λ and (3.5) with $1-\lambda$ and add the resultant inequalities we obtain

$$\begin{aligned}&f(\lambda x+(1-\lambda)y)-\lambda f(x)-(1-\lambda)f(y)\\&\geq (1-\lambda)\lambda f'(x)(y-x)-\lambda(1-\lambda)f'(y)(y-x)\end{aligned}$$

which is clearly equivalent with

$$\begin{aligned}0&\leq \lambda f(x)(1-\lambda)f(y)-f(\lambda x+(1-\lambda)y)\\&\leq (1-\lambda)\lambda(x-y)[f'(x)-f'(y)]\end{aligned} \tag{3.6}$$

for any $\lambda \in [0,1]$ and $x,y\in \mathring{J}$.

Now, let φ and ψ the functions that represent the commuting operators A and B as in the equation (2.1). Then by (3.6) we have

$$\begin{aligned}0&\leq \lambda f(\varphi(t))+(1-\lambda)f(\psi(t))-f(\lambda\varphi(t)+(1-\lambda)\psi(t))\\&\leq (1-\lambda)\lambda(\varphi(t)-\psi(t))[f'(\varphi(t))-f'(\psi(t))]\end{aligned} \tag{3.7}$$

for any $\lambda \in [0,1]$ and $t\in[0,1]$.

Integrating over dE_t we deduce from (3.7) the following inequality in the operator order

$$\begin{aligned}0&\leq \lambda\int_{0-}^{1} f(\varphi(t))dE_t+(1-\lambda)\int_{0-}^{1} f(\psi(t))dE_t\\&\quad-\int_{0-}^{1} f(\lambda\varphi(t)+(1-\lambda)\psi(t))dE_t\\&\leq (1-\lambda)\lambda\int_{0-}^{1}(\varphi(t)-\psi(t))[f'(\varphi(t))-f'(\psi(t))]dE_t\end{aligned} \tag{3.8}$$

for any $\lambda \in [0,1]$.

Utilising the representation (2.2) for continuous functions of selfadjoint operators, we deduce the desired result (3.1). □

REMARK 4. – For A and B positive definite and commuting operators, we have the following reverse of (2.7)

$$\begin{aligned}0&\leq \ln((1-\lambda)A+\lambda B)-(1-\lambda)\ln A-\lambda\ln B\\&\leq (1-\lambda)\lambda(A-B)\left(B^{-1}-A^{-1}\right)\end{aligned} \tag{3.9}$$

and the following reverse of (2.8)

$$(1-\lambda)A\ln A+\lambda B\ln B-((1-\lambda)A+\lambda B)\ln((1-\lambda)A+\lambda B) \le (1-\lambda)\lambda(A-B)(\ln A-\ln B), \tag{3.10}$$

where $\lambda\in[0,1]$.

We use now the following result whose proof in a slightly more general form can be found in [4] and [5].

LEMMA 1. – *Let $f: J\to\mathbb{R}$ be a convex differentiable function on the interior $\mathring{J}$. Then for any $a,b\in\mathring{J}$ we have the inequalities*

$$0\le\frac{f(a)+f(b)}{2}-\int_0^1 f((1-t)a+tb)dt\le\frac{1}{8}(a-b)(f'(a)-f'(b)) \tag{3.11}$$

and

$$0\le\int_0^1 f((1-t)a+tb)dt-f\left(\frac{a+b}{2}\right)\le\frac{1}{8}(a-b)(f'(a)-f'(b)). \tag{3.12}$$

The constant $\frac{1}{8}$ is best possible in both inequalities.

For extensions of these inequalities for functions defined on segments of vectors in linear spaces and applications for semi-inner products, see [4] and [5].

THEOREM 5. – *Let $f: J\to\mathbb{R}$ be a convex continuously differentiable function on the interior $\mathring{J}$ of J and A and B two commuting selfadjoint operators with $Sp(A), Sp(A)\subset\mathring{J}$. Then we have in the operator order*

$$0\le\frac{f(A)+f(B)}{2}-\int_0^1 f((1-\lambda)A+\lambda B)d\lambda \le\frac{1}{8}(A-B)(f'(A)-f'(B)) \tag{3.13}$$

and

$$0\le\int_0^1 f((1-\lambda)A+\lambda B)d\lambda-f\left(\frac{A+B}{2}\right) \le\frac{1}{8}(A-B)(f'(A)-f'(B)). \tag{3.14}$$

PROOF. – Let φ and ψ the functions that represent the commuting operators A and B as in the equation (2.1). Then by (3.11) we have

$$\begin{aligned} 0 &\leq \frac{f(\varphi(t)) + f(\psi(t))}{2} - \int_0^1 f((1-\lambda)\varphi(t) + \lambda\psi(t))d\lambda \\ &\leq \frac{1}{8}(\varphi(t) - \psi(t))(f'(\varphi(t)) - f'(\psi(t))) \end{aligned} \tag{3.15}$$

for any $t \in [0, 1]$.

Integrating over dE_t we deduce from (3.15) the following inequality in the operator order

$$\begin{aligned} 0 &\leq \frac{\int_{0-}^1 f(\varphi(t))dE_t + \int_{0-}^1 f(\psi(t))dE_t}{2} \\ &- \int_{0-}^1 \left(\int_0^1 f((1-\lambda)\varphi(t) + \lambda\psi(t))d\lambda \right) dE_t \\ &\leq \frac{1}{8} \int_{0-}^1 (\varphi(t) - \psi(t))(f'(\varphi(t)) - f'(\psi(t)))dE_t. \end{aligned} \tag{3.16}$$

By Fubini's theorem we have

$$\begin{aligned} &\int_{0-}^1 \left(\int_0^1 f((1-\lambda)\varphi(t) + \lambda\psi(t))d\lambda \right) dE_t \\ &= \int_0^1 \left(\int_{0-}^1 f((1-\lambda)\varphi(t) + \lambda\psi(t))dE_t \right) d\lambda \end{aligned}$$

and employing the representation of continuous functions of commuting operators (2.2) we deduce the desired result (3.13).

The inequality (3.14) can be obtained in a similar manner and the details are omitted. □

REMARK 5. – If A and B are two commuting positive operators and such that $A - B$ is invertible, then for $p \geq 1$ we have the inequalities

$$\begin{aligned} 0 &\leq \frac{A^p + B^p}{2} - \frac{1}{p+1}(A - B)^{-1}\left(A^{p+1} - B^{p+1}\right) \\ &\leq \frac{1}{8} p(A - B)\left(A^{p-1} - B^{p-1}\right) \end{aligned} \tag{3.17}$$

and

$$0 \le \frac{1}{p+1}(A-B)^{-1}\left(A^{p+1}-B^{p+1}\right) - \left(\frac{A+B}{2}\right)^{p} \le \frac{1}{8}\,p(A-B)\left(A^{p-1}-B^{p-1}\right). \tag{3.18}$$

If A and B are two commuting positive definite operators and such that $A-B$ is invertible, then these two inequalities also hold for $p \in (-\infty, 0)\setminus\{-1\}$.

If $p \in (0,1)$ and A and B are two commuting positive definite operators and such that $A-B$ is invertible, then we have

$$0 \le \frac{1}{p+1}(A-B)^{-1}\left(A^{p+1}-B^{p+1}\right) - \frac{A^{p}+B^{p}}{2} \le \frac{1}{8}\,p(A-B)\left(B^{p-1}-A^{p-1}\right) \tag{3.19}$$

and

$$0 \le \left(\frac{A+B}{2}\right)^{p} - \frac{1}{p+1}(A-B)^{-1}\left(A^{p+1}-B^{p+1}\right) \le \frac{1}{8}\,p(A-B)\left(B^{p-1}-A^{p-1}\right). \tag{3.20}$$

Moreover, if A and B are two commuting positive definite operators and such that $A-B$ is invertible, then we also have

$$0 \le (A-B)^{-1}(\ln A - \ln B) - \left(\frac{A+B}{2}\right)^{-1} \le \frac{1}{8}(A-B)\left(B^{-2}-A^{-2}\right) \tag{3.21}$$

and

$$0 \le \frac{A^{-1}+B^{-1}}{2} - (A-B)^{-1}(\ln A - \ln B) \le \frac{1}{8}(A-B)\left(B^{-2}-A^{-2}\right). \tag{3.22}$$

4. – More Results for Twice Differentiable Functions

The following result for twice differentiable functions holds.

THEOREM 6. – *Let $f : J \to \mathbb{R}$ be a twice differentiable function on the interior $\mathring{J}$ and k, K two real numbers such that*

$$k \le f''(\lambda) \le K \text{ for any } \lambda \in \mathring{J}. \tag{4.1}$$

If A and B are two commuting selfadjoint operators with $Sp(A), Sp(B) \subset \mathring{J}$, then we have in the operator order

$$\begin{aligned} &\frac{1}{2} k\lambda(1-\lambda)(A-B)^2 \\ &\leq (1-\lambda)f(A) + \lambda f(B) - f((1-\lambda)A + \lambda B) \\ &\leq \frac{1}{2} K\lambda(1-\lambda)(A-B)^2 \end{aligned} \tag{4.2}$$

and

$$\begin{aligned} &\frac{1}{2}k\left(\lambda - \frac{1}{2}\right)^2 (A-B)^2 \\ &\leq \frac{f((1-\lambda)A + \lambda B) + f((1-\lambda)B + \lambda A)}{2} - f\left(\frac{A+B}{2}\right) \\ &\leq \frac{1}{2}K\left(\lambda - \frac{1}{2}\right)^2 (A-B)^2 \end{aligned} \tag{4.3}$$

for any $\lambda \in [0,1]$.

PROOF. – Consider the auxiliary function $g_k : J \to \mathbb{R}$ given by $g_k(x) = f(x) - \frac{1}{2}kx^2$. This function is twice differentiable on $\mathring{J}$ and $g_k''(x) = f''(x) - k \geq 0$ which shows that g_k is convex on J.

By the definition of convexity we have

$$\begin{aligned} 0 &\leq (1-\lambda)g_k(a) + \lambda g_k(b) - g_k((1-\lambda)a + \lambda b) \\ &= (1-\lambda)f(a) + \lambda f(b) - f((1-\lambda)a + \lambda b) \\ &\quad - \frac{1}{2}k\left[(1-\lambda)a^2 + \lambda b^2 - ((1-\lambda)a + \lambda b)^2\right] \\ &= (1-\lambda)f(a) + \lambda f(b) - f((1-\lambda)a + \lambda b) \\ &\quad - \frac{1}{2}k\lambda(1-\lambda)(a-b)^2 \end{aligned}$$

for any $a, b \in \mathring{J}$ for any $\lambda \in [0,1]$.

This implies the inequality

$$\begin{aligned} &\frac{1}{2}k\lambda(1-\lambda)(a-b)^2 \\ &\leq (1-\lambda)f(a) + \lambda f(b) - f((1-\lambda)a + \lambda b) \end{aligned} \tag{4.4}$$

for any $a, b \in \mathring{J}$ and for any $\lambda \in [0,1]$.

If we consider the auxiliary function $g_K : J \to \mathbb{R}$ given by $g_K(x) = \frac{1}{2}Kx^2 - f(x)$ and apply a similar argument, we deduce the opposite inequality

$$(1-\lambda)f(a) + \lambda f(b) - f((1-\lambda)a + \lambda b) \le \frac{1}{2}K\lambda(1-\lambda)(a-b)^2 \tag{4.5}$$

for any $a, b \in \mathring{J}$ and for any $\lambda \in [0,1]$.

Now, utilizing the spectral representation (2.1) for the commuting selfadjoint operators A and B, the scalar inequalities (4.4), (4.5) and a similar argument to that in the proof of Theorem 3, we deduce the desired result (4.2).

From (4.2) for $\lambda = \frac{1}{2}$ we get in the operator order

$$\frac{1}{8}k(C-D)^2 \le \frac{f(C)+f(D)}{2} - f\left(\frac{C+D}{2}\right) \le \frac{1}{8}K(C-D)^2, \tag{4.6}$$

for any commuting selfadjoint operators C and D with $Sp(C), Sp(D) \subset \mathring{J}$.

If we take $C = (1-\lambda)A + \lambda B$ and $D = (1-\lambda)B + \lambda A$, then C and D are commuting operators with $Sp(C), Sp(D) \subset \mathring{J}$ and by (4.6) we deduce the desired result (4.3). □

REMARK 6. – The above operator inequalities have some particular cases of interest. For instance, for $p \ge 2$, if A and B are two commuting selfadjoint operators such that $Sp(A), Sp(A) \subset [m, M] \subset (0, \infty)$, then we have in the operator order

$$\begin{aligned} &\frac{1}{2}p(p-1)m^{p-2}\lambda(1-\lambda)(A-B)^2 \\ &\le (1-\lambda)A^p + \lambda B^p - ((1-\lambda)A + \lambda B)^p \\ &\le \frac{1}{2}p(p-1)M^{p-2}\lambda(1-\lambda)(A-B)^2 \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} &\frac{1}{2}\,p(p-1)m^{p-2}\left(\lambda - \frac{1}{2}\right)^2 (A-B)^2 \\ &\le \frac{((1-\lambda)A + \lambda B)^p + ((1-\lambda)B + \lambda A)^p}{2} - \left(\frac{A+B}{2}\right)^p \\ &\le \frac{1}{2}\,p(p-1)M^{p-2}\left(\lambda - \frac{1}{2}\right)^2 (A-B)^2 \end{aligned} \tag{4.8}$$

for any $\lambda \in [0,1]$.

We have also the logarithmic inequalities

$$\begin{aligned}&\frac{1}{2}M^{-2}\lambda(1-\lambda)(A-B)^2\\ &\le \ln\left((1-\lambda)A+\lambda B\right)-(1-\lambda)\ln(A)-\lambda\ln(B)\\ &\le \frac{1}{2}m^{-2}\lambda(1-\lambda)(A-B)^2\end{aligned} \tag{4.9}$$

and

$$\begin{aligned}&\frac{1}{2}M^{-2}\left(\lambda-\frac{1}{2}\right)^2(A-B)^2\\ &\le \frac{\ln\left((1-\lambda)A+\lambda B\right)+\ln\left((1-\lambda)B+\lambda A\right)}{2}-\ln\left(\frac{A+B}{2}\right)\\ &\le \frac{1}{2}m^{-2}\left(\lambda-\frac{1}{2}\right)^2(A-B)^2\end{aligned} \tag{4.10}$$

for any $\lambda\in[0,1]$.

We use the following inequalities for twice differentiable functions that are related to the Hermite-Hadamard inequality:

LEMMA 2. – *Let $f:J\to\mathbb{R}$ be a twice differentiable function on the interior $\mathring{J}$ and k,K two real numbers such that* (4.1) *is true. Then for any $a,b\in\mathring{J}$ we have the inequalities*

$$\frac{1}{24}k(a-b)^2\le\int_0^1 f((1-\lambda)a+\lambda b)d\lambda-f\left(\frac{a+b}{2}\right)\le\frac{1}{24}K(a-b)^2, \tag{4.11}$$

$$\frac{1}{12}k(a-b)^2\le\frac{f(a)+f(b)}{2}-\int_0^1 f((1-\lambda)a+\lambda b)d\lambda\le\frac{1}{12}K(a-b)^2 \tag{4.12}$$

and

$$\begin{aligned}\frac{1}{48}k(a-b)^2&\le\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\int_0^1 f((1-\lambda)a+\lambda b)d\lambda\\ &\le\frac{1}{48}K(a-b)^2.\end{aligned} \tag{4.13}$$

PROOF. – The inequality (4.11) was obtained in [9] while (4.12) was obtained in [10]. The inequality (4.13) was established in [11, p. 44]. □

THEOREM 7. – *Let $f : J \to \mathbb{R}$ be a twice differentiable function on the interior $\mathring{J}$ and satisfying the condition* (4.1). *If A and B are two commuting selfadjoint operators with $Sp(A), Sp(A) \subset \mathring{J}$, then we have in the operator order*

$$\frac{1}{24}k(A-B)^2 \leq \int_0^1 f((1-\lambda)A+\lambda B)d\lambda - f\left(\frac{A+B}{2}\right) \leq \frac{1}{24}K(A-B)^2, \tag{4.14}$$

$$\frac{1}{12}k(A-B)^2 \leq \frac{f(A)+f(B)}{2} - \int_0^1 f((1-\lambda)A+\lambda B)d\lambda \tag{4.15}$$

$$\leq \frac{1}{12}K(A-B)^2 \tag{4.16}$$

and

$$\frac{1}{48}k(A-B)^2 \leq \frac{1}{2}\left[\frac{f(A)+f(B)}{2} + f\left(\frac{A+B}{2}\right)\right] - \int_0^1 f((1-\lambda)A+\lambda B)d\lambda \leq \frac{1}{48}K(A-B)^2. \tag{4.17}$$

The argument follows in a similar way to that in the proof of Theorem 5 by utilizing Lemma 2 and the details are omitted.

REMARK 7. – The above operator inequalities have some particular cases of interest. For instance, for $p \geq 2$, if A and B are two commuting selfadjoint operators such that $A-B$ is invertible and $Sp(A), Sp(A) \subset [m, M] \subset (0, \infty)$, then we have in the operator order

$$\frac{p(p-1)}{24}m^{p-2}(A-B)^2 \leq \frac{1}{p+1}(A-B)^{-1}\left(A^{p+1}-B^{p+1}\right) - \left(\frac{A+B}{2}\right)^p \leq \frac{p(p-1)}{24}M^{p-2}(A-B)^2, \tag{4.18}$$

$$\frac{p(p-1)}{12}m^{p-2}(A-B)^2 \leq \frac{A^p+B^p}{2} - \frac{1}{p+1}(A-B)^{-1}\left(A^{p+1}-B^{p+1}\right) \leq \frac{p(p-1)}{12}M^{p-2}(A-B)^2 \tag{4.19}$$

and

$$\frac{p(p-1)}{48}m^{p-2}(A-B)^2 \leq \frac{1}{2}\left[\frac{A^p+B^p}{2} + \left(\frac{A+B}{2}\right)^p\right] - \frac{1}{p+1}(A-B)^{-1}\left(A^{p+1}-B^{p+1}\right) \leq \frac{p(p-1)}{48}M^{p-2}(A-B)^2. \tag{4.20}$$

If $p \in [1,2)$, then the above inequalities (4.18)-(4.20) also hold by replacing M^{p-2} with m^{p-2}.

Similar results may be stated for $p \in (0,1), p \in (\infty, 0) \setminus \{-1\}$ and $p = -1$. However the details are not presented here.

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REFERENCES

[1] G. ALLASIA - C. GIORDANO - J. PEČARIĆ, *Hadamard-type inequalities for (2r)-convex functions with applications*, Atti Acad. Sci. Torino-Cl. Sc. Fis., **133** (1999), 1-14.
[2] A. G. AZPEITIA, *Convex functions and the Hadamard inequality*, Rev.-Colombiana-Mat., **28** (1) (1994), 7-12.
[3] E. F. BECKENBACH - R. BELLMAN, *Inequalities*, 4th Edition, Springer-Verlag, Berlin, 1983.
[4] S. S. DRAGOMIR, *An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products*. J. Inequal. Pure Appl. Math. **3**, no. 2 (2002), 8.
[5] S. S. DRAGOMIR, *An inequality improving the second Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products*. J. Inequal. Pure Appl. Math. **3**, no. 3 (2002), 8.
[6] S. S. DRAGOMIR, *Bounds for the normalized Jensen functional*, Bull. Austral. Math. Soc. **74** (3) (2006), 471-476.
[7] S. S. DRAGOMIR, *Hermite-Hadamard's type inequalities for convex functions of selfadjoint operators in Hilbert spaces*, Linear Algebra and its Applications, **436**, Issue 5, Pages 1503-1515.
[8] S. S. DRAGOMIR, *Hermite-Hadamard's type inequalities for operator convex functions*, Applied Mathematics and Computation, **218**, Issue 3, Pages 766-772.
[9] S. S. DRAGOMIR - P. CERONE - A. SOFO, *Some remarks on the midpoint rule in numerical integration*, Studia Univ. Babes - Bolyai Math. **45**, no. 1 (2000), 63-74. Preprint *RGMIA Res. Rep. Coll.*, **1**, no. 2 (1998), 25-33.
[10] S. S. DRAGOMIR - P. CERONE - A. SOFO, *Some remarks on the trapezoid rule in numerical integration*. Indian J. Pure Appl. Math. **31**, no. 5 (2000), 475-494. Preprint *RGMIA Res. Rep. Coll.* , **2**, no. 5 (1998), Art. 1.
[11] S. S. DRAGOMIR - C. E. M. PEARCE, *Selected Topics on Hermite-Hadamard Type Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000. [http://www.staff.vu.edu.au/RGMIA/monographs/hermite_hadamard.html].
[12] S. S. DRAGOMIR, *Some inequalities of Čeby šev type for functions of operators in Hilbert spaces*, Preprint *RGMIA Res. Rep. Coll.*, **15** (2012), Article 41, 12 pp.
[13] A. M. FINK, *Toward a theory of best possible inequalities*, Nieuw Archief von Wiskunde, **12** (1994), 19-29.
[14] T. FURUTA - J. MIĆIĆ HOT - J. PEČARIĆ - Y. SEO, *Mond-Pečarić Method in Operator Inequalities. Inequalities for Bounded Selfadjoint Operators on a Hilbert Space*, Element, Zagreb, 2005.
[15] F. R. GANTMACHER, *The Theory of Matrices*, Chelsea, 1959.

[16] B. GAVREA, *On Hadamard's inequality for the convex mappings defined on a convex domain in the space*, Journal of Ineq. in Pure 1, no. 1(2000), Article 9, http://jipam.vu.edu.au/.

[17] I. GOHBERG - P. LANCASTER - L. RODMAN, *Invariant Subspaces of Matrices with Application,* Wiley-Interscience, 1986.

[18] P. M. GILL - C. E. M. PEARCE - J. PEČARIĆ, *Hadamard's inequality for* $r-$*convex functions*, J. of Math. Anal. and Appl., **215** (1997), 461-470.

[19] K.-C. LEE - K.-L. TSENG, *On a weighted generalisation of Hadamard's inequality for G-convex functions*, Tamsui Oxford Journal of Math. Sci., **16** (1) (2000), 91-104.

[20] A. LUPAS, *A generalisation of Hadamard's inequality for convex functions*, Univ. Beograd. Publ. Elek. Fak. Ser. Mat. Fiz., **544-576,** (1976), 115-121.

[21] D. M. MAKSIMOVIĆ, *A short proof of generalized Hadamard's inequalities*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz., no. 634-677 (1979), 126-128.

[22] A. MATKOVIĆ - J. PEČARIĆ - I. PERIĆ, *A variant of Jensen's inequality of Mercer's type for operators with applications*. Linear Algebra Appl., **418**, no. 2-3 (2006), 551-564.

[23] C. A. MCCARTHY, c_p, Israel J. Math., **5** (1967), 249-271.

[24] J. MIĆIĆ - Y. SEO - S.-E. TAKAHASI - M. TOMINAGA, *Inequalities of Furuta and Mond-Pečarić*, Math. Ineq. Appl., **2** (1999), 83-111.

[25] D. S. MITRINOVIĆ - I. LACKOVIĆ, *Hermite and convexity*, Aequat. Math., **28** (1985), 229-232.

[26] D. S. MITRINOVIĆ - J. PEČARIĆ - A. M. FINK, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht, 1993.

[27] B. MOND - J. PEČARIĆ, *Convex inequalities in Hilbert space*, Houston J. Math., **19** (1993), 405-420.

[28] B. MOND - J. PEČARIĆ, *On some operator inequalities*, Indian J. Math., **35** (1993), 221-232.

[29] B. MOND - J. PEČARIĆ, *Classical inequalities for matrix functions*, Utilitas Math., **46** (1994), 155-166.

[30] F. RIESZ - B. SZ-NAGY, *Functional Analysis*, New York, Dover Publications, 1990.

[31] D. A. SUPRUNENKO - R. I. TYSHKEVICH, *Commutative Matrices*, Academic, 1968.

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