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Abstract. – *Notion of complex derivatives is used to prove interpolation theorems mainly in weighted Banach spaces studied in* [5]. *A conjecture of* [4], *concerning the weights is solved and a characterization is given. Thus* [3], [4], [5], *are somewhat revisited.*

1. – Introduction

The paper extends to weighted Banach spaces studied in [5] an interpolation theorem of intermediate derivatives previously proved in [3] using Fourier transform and the notion of derivative of complex order defined in [30].

The property is generally the following:

"*If u is a function on R^N, p_0-intégrable with values in a Banach space A_0, such that all derivatives of order m are p_1-integrable with values in an other Banach space A_1, then the derivatives of order μ, $0<\mu<m$, are p_μ-integrable (p_μ given in section 4) with values in an intermediate Banach space "between" A_0 and A_1 which is obtained by interpolation*"

Several authors were interested in some properties of that type: see (by chronologic order) ([15], [14], [11], [26]) for "scalar" cases and ([17], [19], [6], [12], [1], [2], [3], [4],) for "vector" cases with weights or not.

In particular in ([6], [12]) $N = 1$ and weights of t^α type are considered and in [1], [2] results of [17] are extended in the framework of [20] to weighted Hilbert spaces with decreasing weights. (1)

For unweighted Banach spaces the result given in [19] is obtained by real interpolation while for Hilbert spaces the results generally related with complex interpolation method are slightly distincts, so that it was to obtain a satisfactory generalization for Banach spaces that [3] was done. Two proofs were given there, one with the help of Fourier Transform, the other with the notion of derivatives of complex order but, in both cases, the complex interpolation method gives the result.

(1) And also for some increasing weights by duality.

The disadventage to use Fourier transform for functions with vectorial values is that generally the Fourier transform is not locally intégrable [27], so that the class of Banach spaces must be "restricted" to those, B, for which the following condition holds:

$(Condition\ \mathcal{M})$: *the theorem of Michlin on multipliers is true in* $\mathcal{F}(L^p(B))$.

A real characterization of such spaces seems still unknown, however for applications the spaces A_i $(i = 0, 1)$ are mainly $L^q(R^N)$, $W^{m,q}(R^N)$ (Sobolev spaces),or $L^{p,q}(R^N)$ (Lorentz spaces), Besov spaces, ..., spaces for which the condition $\mathcal{M}$ is valid.

On the other hand the method is based on the properties of the basic derivative $D^{i\eta}$ of complex order $i\eta$, $\eta \in \boldsymbol{R}$, defined by convolution with the tempered distribution $Y_{-i\eta} = \dfrac{1}{\Gamma(-i\eta)} Pf. \dfrac{1}{x_+^{1+i\eta}}$, $\eta \in \boldsymbol{R}$ where the symbol $Pf.$ represents the *"finite part"* or *"the pseudo-function"* in the sense of Laurent Schwartz, $x_+ = max(0, x)$, Γ is the usual Gamma function ([30], [31]). (2)

Here, in a way, that is the "dual condition" which play the main part *i.e.*:

$$(\mathcal{C} - condition) : \text{“} Y_{-i\eta} \textit{ is a convolutor of } L^p(B)\text{”}$$

if B is the Banach space considered.

In fact from [3] if $p > 1$ every Banach space satisfies the $\mathcal{C}$-*condition* and from certain point of wiew the second method is more general.

A set of conditions on the weights is given in [4] and in particular here the $\mathcal{C} - condition$ reads "$Y_{-i\eta}$ *is a convolutor of* $L^p_w(B)$" where w is a weight. Such a condition seems both a condition on spaces and on weights. An other condition, on the weights, is also to be in a prominent position in [4]: the $(\mathcal{P} - condition)$:

$$\text{“}\textit{If } \phi \in L^1(\boldsymbol{R}) \textit{ and } \tilde{f} = Y_{i\eta} * f \textit{ then } \tilde{f} * \phi \textit{ is continuous from } L^p_w(B) \textit{ in itself.”}$$

Thus we can find sufficient conditions like: either w belongs to the class $A(p)$ introduced by B. Muckenhoupt (see [25]) for maximal Hardy functions, or the weigth w is non increasing.

If we introduce (see [4], [5]) the class of Hardy $\mathcal{H}(p)$, we know ([5]) that $A(p) \subset \mathcal{H}(p)$ on the line, and that every non increasing weight belongs to $\mathcal{H}(p)$ but may be not in $A(p)$ (like $w(t) = e^{-\lambda t}$, $\lambda > 0$, for example).

In such conditions it was natural to conjecture that the good class for the weights is the Hardy class. This is done and solved below.

(2) see also [29], where the comparison with différent approachs may be found.

The present paper is constructed as follows. Section2 contains the main notations and is devoted to remember some class of weighted spaces studied in [5] as well as the class of weights $A(p)$, $\mathcal{H}(p)$. General derivatives of complex order are introduced in Section 3 and the properties of $Y_{-i\eta} * f$, when f is a step function gives necessary and sufficient conditions on the weights to prove conditon $\mathcal{C}$ and condition $\mathcal{P}$.

Therefore, in Section 4, one uses complex interpolation method to obtain the main theorem of intermediate derivatives which improve a result of [5] using a method of interpolation relatd with the notion of space of traces. Results for fractional derivative spaces are also considered.

Finally, in Section 5, we return to the case of unweighted spaces to revisite the Fourier method and the condition $\mathcal{M}$ related with the notion of Banach spaces of p type introduced in [27].

2. – Notations and background

If $\mathcal{X}$, $\mathcal{Y}$ are vectorial topologic spaces, $\mathcal{X} \subset \mathcal{Y}$ means always algebric inclusion with continuous injective mapping and we denote $\mathcal{L}(\mathcal{X}, \mathcal{Y})$, or $\mathcal{L}(\mathcal{X})$ (if $\mathcal{Y} = \mathcal{X}$) the space of continuous linear mappings from $\mathcal{X}$ into $\mathcal{Y}$.

Let X be a normed space with norm $|.|_X$, $L^r(a, b.X)$ (*resp.* $L^r(X)$ *if* $(a, b) = [0, +\infty)$) means the space of strongly measurable functions $r-$ integrables on $(a, b) \subset \boldsymbol{R}$ with values in X. Provided with the norm $u \longrightarrow |u|_r = \left(\int_a^b |u(\sigma)|_X^r d\sigma \right)^{1/r}$, $L^r(a, b; X)$ is a Banach space. We take usual modification when $r = +\infty$.

2.1 – *Weighted L^p spaces*

Let B a real or complex Banach space equipped with the norm $|.|_B$, and let ω be a positive locally integrable function defined on $\Omega \subset \boldsymbol{R}^n$ and tacking values in $\boldsymbol{R}^+ =]0, +\infty[$. Define the measure ν, to be such that $d\nu = \omega(x)dx$, *where* $\omega > 0$, is a density with respect to the measure of Lebesgue in $\boldsymbol{R}^N$. Such a density ω will be called a *weight*.

One denotes by $L^p_\omega(B)$, $1 \leq p \leq +\infty$, the space of functions u strongly mesurables with values in B satisfying

$$\int_\Omega |u(x)|_B^p \, d\nu < +\infty$$

with usual modification for $p = +\infty$. The space $L^p_\omega(B)$ becomes a Banach space when equipped with the norm

$$u \longrightarrow |u|_{L^p_\omega(B)} = \left(\int_\Omega |u(x)|_B^p d\nu \right)^{1/p}.$$

In what follows, we shall take $N = 1$, $\Omega = \boldsymbol{R}^+$, and like in [5] it is also of interest to set $\omega = c^p$, where $c > 0$ satisfies

$$c \in L^p_{loc}(\boldsymbol{R}^+),\ \frac{1}{c} \in L^{p'}_{loc}(\boldsymbol{R}^+),\ \frac{1}{p} + \frac{1}{p'} = 1. \tag{2.1}$$

Indeed, when $\omega = c^p$, the condition $u \in L^p_\omega(B)$ is equivalent to $cu \in L^p(B)$, using Lebesque measure. Accordingly, we still refer to c as a weight. So, in what follows the space $L^p_c(B)$ always denotes *the space of function u, such that* $cu \in L^p(B)$. The letter ω is reserved exclusively for the density $\omega = c^p$, where c satisfies (2.1).

REMARK 2.1. – Suppose B reflexive and B' the dual (or the antidual) of B, then the dual space of $L^p_\omega(B)$ is $L^{p'}_{\omega'}(B')$ and $\omega' = \omega^{1-p'} = c^{-p'}$.

Obviously condition (2.1) is reasonable for c to have by Hölder inequality

$$\forall T > 0,\ L^p_\omega(0, T; B) \subset L^1(0, T; B),$$

with continuous injective mapping.

In what follows we are interested to say the least by two kinds of weights: the class $A(p)$ and the class $\mathcal{H}(p)$. (see [5]).

We recall that c (*or* ω) belongs to $\mathcal{H}(p)$, if c satisfies :

$$\sup_{t>0} \left(\int_t^{+\infty} \left[\frac{c(\sigma)}{\sigma}\right]^p d\sigma \right)^{1/p} \left(\int_0^t \frac{d\sigma}{(c'\sigma))^{p'}} \right)^{1/p'} < +\infty \tag{2.2}$$

while, c belongs to $A(p)$ if

$$\operatorname*{Sup}_{I \in R^+} \left(\frac{1}{|I|} \int_I (c(\sigma))^p d\sigma \right)^{1/p} \left(\frac{1}{|I|} \int_I \frac{d\sigma}{(c(\sigma))^{p'}} \right)^{1/p'}, \tag{2.3}$$

where I are any intervals $I \subset \boldsymbol{R}^+$.($|I| =$ *length of* I).

One recalls that (2.2) is a necessary and sufficient contition for the continuity of Hardy's operator $\mathcal{H} : f \longrightarrow \dfrac{1}{t}\int_0^t f(\sigma)d\sigma$ in $L^p_c(B)$), while (2.3) play a similar condition for the maximal Hardy-Littlewood operator on the line.

2.2 – *The spaces* $W^{(m)}_{c_0,c_1}$

We follows the notations of [5] to define the space $W^{(m)}_{c_0,c_1}$.

Let A_0, A_1 be two Banach spaces continuously embeded into a (real or complex) topologic vector space $\mathcal{A}$ with

(2.4) $\quad X = A_0 \cap A_1$ *is equipped with the norm* $|u|_X = max(|u|_{A_0}, |u|_{A_1})$

(2.5) $\quad Y = A_0 + A_1,$ *is equipped with the norm* $|u|_Y = \inf_{a=a_0+a_1} \left(|a_0|_{A_0} + |a_1|_{A_1} \right)$

thus X, Y, are banach spaces, and, of course, we have $X \subset A_i \subset Y,\ (i = 0, 1)$. We assume that

(2.6) $\quad A_i$ *is reflexive*

and

(2.7) $\quad X$ *is dense into* $A_i,\ i = 0, 1.$

For $i = 0, 1$, let $\omega_i = c_i^{p_i},\ p_i \in [1, +\infty]$ be weights satisfying (2.1), we consider the spaces

(2.8) $\quad X_i = L^{p_i}_{c_i}(\boldsymbol{R}^+; A_i) = L^{p_i}_{c_i}(A_i),$ *the norm being denoted* $N_i(.)$.

and one defines for $m \geq 1$,

$$W^{(m)}[p_0, c_0, A_0; p_1, c_1, A_1] = W^{(m)}$$

the space of functions u, locally integrable on R^+, with $u \in X_0$ and such that $D^m u \in X_1$.The last condition must be understood as follows: u is m-time differentiable at the sense of distributions on R^+ with values in Y and $D^m u$ is locally integrable, so that the product with c_1 makes a sense.

Indeed, since $D^m u$ is locally integrable, then $D^{m-1}u$ is absolutely continuous, hence continuous, then we can consider that u is $(m-1)$-time continuously differentiable on $R^+$with values in Y and $D^i u(t),\ 0 \leq i \leq m-1$, is defined for $t \in]0, +\infty[$. Therefore if $\lim_{t \longrightarrow 0} D^i u(t) = a$ in Y exists, we shall said that $D^i u$ has a trace $D^i u(0) = a$ at $t = 0$.

Equipped with the norm

(2.9) $\quad u \longrightarrow \|u\|_{W^m} = \ max(N_0(u),\ N_1(D^m u),$

$W^{(m)}$is a Banach space.

If we let $\mathcal{D}(\overline{\boldsymbol{R}}^+; X)$ the space of functions in $\mathcal{D}(\boldsymbol{R}; X)$ restricted to $\overline{\boldsymbol{R}}^+ = [0, +\infty[$, one has

Lemma 2.2. – *The space* $\mathcal{D}(\overline{\boldsymbol{R}}^+; X)$ *is dense in* $W^{(m)}_{c_0,c_1}$ *if* $c_1 \in \mathcal{H}(p_1)$

Proof. – see [5].

3. – Complex derivatives and main results

To define derivatives of complex order z for a distribution Φ, we consider first, following Laurent Schwartz ([30], tome II,V,&:VI), the convolution $Y_z * \Phi$, where

(3.1) $$Y_z = \frac{1}{\Gamma(z)} Pf.x_+^{z-1} \quad if\ z \notin -\boldsymbol{N}, \quad Y_{-k} = \delta^{(k)}\ if\ z = k \in -\boldsymbol{N}.$$

$Pf.$ meaning the *finite part* (3) *or Pseudo function*, defined by the function x_+^{z-1}, the symbol $Pf.$ being unavailing if $\mathcal{R}e(-z) > 0$.

Γ is the classical eulerian function and $x_+ = x\ if\ x > 0,\ = 0\ if\ x < 0$.

We recall, that one has $(Pf.f, \phi) = Pf.(f, \phi)$ at the sense of distributions, that is

$$(Pseudofunction\ f, \phi) = \ Finite\ part(^3)\ (f, \phi).$$

Like $\Gamma(z)$ has the same pôles that $Pf.x_+^{z-1}$ it follows that $z \longrightarrow Y_z$ is continuous with respect to z, and we can say that Y_z is an analytic (holomorphic) function of the z variable on $\boldsymbol{C}$ tacking its values in $\mathcal{S}'(\boldsymbol{R})$.

From eulerian integral properties, one deduces by analytic prolongation that

(3.2) $$\forall z, \zeta \in \boldsymbol{C},\ Y_z * Y_\zeta = Y_{z+\zeta}.$$

If $\mathcal{D}'_+$ is the space of distributions on $\boldsymbol{R}$, whose the support is bounded on the left, then $\mathcal{D}'_+$ is an algebra with the respect of convolution product and δ is the unity, so that from (3.2) the inverse of Y_z is Y_{-z} and like from (3.1) $Y_{-k} = D^k$ one defines the derivative of order z for a distribution Φ *by*

(3.3) $$D^z\phi = Y_{-z} * \Phi.$$

3.1 – *The operator* $D^{i\eta} = Y_{-i\eta}*,\ \eta \in \boldsymbol{R}$

For $\mathcal{R}e(z) = 0$ we have

$$Y_{-i\eta} = \frac{1}{\Gamma(-i\eta)} Pf.\left[\frac{1}{x_+^{1+i\eta}}\right]\ if\ \eta \neq 0, \quad Y_0 = \delta.$$

If $\phi \in \mathcal{D}(\boldsymbol{R}^+)$ then

(3.4) $$D^{i\eta}\phi = \frac{1}{\Gamma(-i\eta)} \underset{\varepsilon \longrightarrow +0}{Lim} \left(\int_\varepsilon^t \frac{\phi(t-x)}{x^{1+i\eta}} - \frac{\phi(t)\varepsilon^{-i\eta}}{-i\eta} \right),\ \eta \neq 0,$$

and obviousely $D^0\phi = \delta * \phi$.

(3) Partie finie in french.

It is known in the unweighted case that if either $B = \boldsymbol{R},\ or\ \boldsymbol{C},$ (see [24], [32], [3]) and $B = H$ (Hilbert space), or B is a Banach space subject to some conditions (see [3]), then one has

THEOREM 3.1. – *For* $p > 1$, *the mapping* $f \longrightarrow D^{i\eta}f = Y_{-i\eta} * f$, *is continuous from* $L^p(B)$ *into itself.*

REMARK 3.2. – It is easy to see that $D^{i\eta} \notin \mathcal{L}(L^1(B))$ if $\eta \neq 0$ except for $\eta = 0$. (cf. remark 3.4).

Quasi systematically the proof in [24] and [25] are based on the study of the kernel $K_\varepsilon(x) = \dfrac{1}{x^{1+i\eta.}}$, *if* $|x| > \varepsilon$, $= 0$ *if* $|x| < \varepsilon$ and its Fourier transform following the methods for singular integral improving the results of [9] in [32].

Like it was said in the Introduction, we have independantly studied the point in [3] for some Banach spaces using two methods. We will return on the case in section 5.

In what follows we give a very simple direct proof which avoid the methods of [32] on $\boldsymbol{R}^+$ for the space $L^p_c(B)$ now valid for $p \geq 1$.

3.2 – *Main results*

THEOREM 3.3. – *Assume* $1 \leq p \leq +\infty$, *and that* c *satisfies* (2.1), *then* $D^{i\eta} \in \mathcal{L}(L^p_c(B))$ *if and only if* $c \in \mathcal{H}(p)$.

PROOF. – In what follows, to simplify notations, we let $\tilde{f} = D^{i\eta}f$.

a) *Necessity of condition* (2.2).

χ being the caracteristic function of the interval $]a, b[,\ 0 < a < b$, and $\beta \in B$, we can choose $u(t) = \chi(t) \otimes \beta$ and (3.4) gives

$$\tilde{u}(t) = o,\ if\ 0 \leq t < b,\ =\ 1 \otimes \beta\ if\ a < t < b,$$

$$= \frac{i}{\eta\Gamma(-i\eta)}\left[\frac{1}{(t-a)^{i\eta}} - \frac{1}{(t-b)^{i\eta}}\right] \otimes \beta,\ if\ t > b$$

and we check that

$$t > b \Longrightarrow |\tilde{u}(t)|_B = \gamma(\eta)\frac{2}{|\eta|}\left|sin\left[\frac{\eta}{2}Log\left(1 + \frac{b-a}{t-b}\right)\right]\right||\beta|_B,$$

with

$$\gamma(\eta) = \frac{1}{|\Gamma(-i\eta)|} = \left(\frac{\eta sh\eta}{\pi}\right)^{1/2}$$

If we want $\tilde{u}$ in $L_c^p(B)$ then (a fortiori) $\int_t^{+\infty} |\tilde{u}(\tau)|_B^p c^p(\tau)dt \leq c_1(b-a)|\beta|_B \ t > b$, and like for large t, $|\tilde{u}(t)|_B^p \simeq c(\beta, B, \eta)t^{-p}$ one deduces à first necessary condition:

$$\forall t > 0, \quad \int_t^{+\infty} \frac{c^p(\tau)}{\tau^p} d\tau \ \textit{exists.} \tag{3.5}$$

Now we see that if the function c *is of order* -1 with respect to *logt* when $t \longrightarrow +0$ then there is a contradiction with the assumption (2.1). On the other part if c *is of order* $+\infty$ with respect to *logt* when $t \longrightarrow +\infty$, then it is (3.5) wich is not true.

Then from Bourbaki [8] the integral in (3.5) is equivalent to $c^p(t)t^{-p+1}$(up to a multiplicative constant) and $\int_0^t c^{-p'}(\tau)d\tau \simeq (\textit{constant})\ tc^{-p'}(t)$ when $t \longrightarrow +\infty$ or $t \longrightarrow +0$. One concludes that (2.2) is satisfied, so that

$$c \in \mathcal{H}(p) \ \textit{is a necessary condition for} \ D^{i\eta} \in \mathcal{L}(\ L_c^p(B) \tag{3.6}$$

REMARK 3.4. – If the weight $c \equiv 1$, then the integral in (3.5) is infinite when $p = 1$ thus $D^{i\eta}$ doesn't processes in L^1 like it was said in Remark 3.2.

REMARK 3.5. – Let χ_T the characteristic function of $(0, T)$, then from (2.1) it is obvious that $u(t) = \dfrac{\chi_T(t)}{[c(t)]^{p'}} \in L_c^p(\boldsymbol{R}^+)$, and we can hope some necessary conditions on the weight from the fact that there exists a constant γ such that

$$|\tilde{u}|_{L_c^p(\boldsymbol{R}^+)} \leq \gamma |u|_{L_c^p(\boldsymbol{R}^+)} = \gamma \left(\int_0^T \frac{dt}{[c(t)]^{p'}} \right)^{1/p} \tag{3.7}$$

It is of interest to consider the particular case where $c(t) = t^\alpha$, with $\alpha p' < 1$. Obviousely we can check that $c \in \mathcal{H}(p)$ and also $c \in A(p)$, but forgetting this point for the time being, we want to check (3.7).

One has

$$\text{for } 0 < t \leq T,\ \tilde{u}(t) = g(\eta) \underset{\varepsilon \longrightarrow +0}{Lim} \left(\int_\varepsilon^t x^{-i\eta-1}(t-x)^{-\alpha p'} dx - \frac{\varepsilon^{-i\eta}}{i\eta} t^{-\alpha p'} \right),\ g(\eta) = \frac{1}{\Gamma(-i\eta)}.$$

tacking $x = \xi t$ in the integral and using eulerian integral properties, one has

$$\tilde{u}(t) = \frac{t^{-\alpha p' - i\eta}}{\Gamma(-i\eta)} \underset{\varepsilon \longrightarrow +0}{Lim} \left(\int_{\varepsilon/t}^1 \xi^{-i\eta-1}(1-\xi)^{-\alpha p'} + \frac{i}{\eta}\left(\frac{\varepsilon}{t}\right)^{-i\eta} \right) = \frac{\Gamma(1-\alpha p')}{\Gamma(1-\alpha p' - i\eta)} t^{-\alpha p' - i\eta}$$

and a simple computation gives

$$\int_0^T (|\tilde{u}(t)|c(t))^p dt \le \gamma(p',\eta,\alpha)\int_0^T (|u(t)|c(t))^p dt) = \int_0^T \frac{dt}{[c(t)]^{p'}}. \tag{3.8}$$

on the other hand

$$\textit{for } t>T,\ \tilde{u}(t)=g(\eta)\int_0^T (t-x)^{-i\eta-1}x^{-\alpha p'}dx = g(\eta)t^{-\alpha p'-i\eta}\int_0^{T/t}(1-\xi)^{-i\eta-1}\xi^{-\alpha p'}d\xi. \tag{3.9}$$

As the principal part of the last integral in (3.9) is $\gamma(p',\alpha,\eta)\left(\dfrac{T}{t}\right)^{1-\alpha p'}$, when $t \longrightarrow +\infty$, one deduces that there is a constant still denoted γ, such that

$$|\tilde{u}(t)| \simeq \gamma\,\frac{T^{1-\alpha p'}}{t},\ \textit{as } t>T. \tag{3.10}$$

Therefore

$$\int_T^{+\infty} (|\tilde{u}(t)|c(t))^p dt = \gamma_1\left[\frac{T^{1-\alpha p'}}{1-\alpha p'}\right]\int_T^{+\infty}\left[\frac{c(t)}{t}\right]^p dt$$

Like we assume that (3.7) holds, we must have a fortiori

$$\left[\frac{T^{1-\alpha p'}}{1-\alpha p'}\right]^{1/p}\left[\int_T^{+\infty}\left[\frac{c(t)}{t}\right]^p dt\right]^{1/p} \le \gamma_2\left(\int_0^T \frac{dt}{[c(t)\ ]^{p'}}\right)^{1/p}, \tag{3.11}$$

then, observing that $\int_0^T \dfrac{dt}{[c(t)]}^{p'} = \dfrac{T^{1-\alpha p'}}{1-\alpha p'}$, (3.11) implies (2.2).

Hence we obtain there an other direct proof of the necessity condition (2.2) for $c(t)=t^\alpha,\ \alpha p'<1$.

b) *The condition* (2.2) *is a sufficient condition.*

1) The key of the proof is

PROPOSITION 3.6. – *Let ϕ a locally integrable function with compact support $\subset (0,T)$ with values in B, then there is a constant $\gamma(\eta)$ such that*

$$\left|\tilde{\phi}(2T)\right|_B \le \gamma\frac{1}{T}\int_0^T |\phi(t)|_B dt \tag{3.12}$$

PROOF. – One uses two steps for the proof.

– First step.

Let a step function with compact support $[0,T]$ given by $f_k = \sum_{i=0}^{i=k-1} \beta_i \otimes \chi_{]a_i,a_{i+1}[}$ with $a_0 = 0,\ a_k = \ T$, and $supp.f_k \subset [0,T],\ \beta_i \in B$.

A simple computation for $D^{i\eta}(\phi_k) = \tilde{\phi}_k$, when $t > T$ gives

$$|\tilde{\phi}_k(t)|_B \le \gamma(\eta) \sum_{i=0}^{i=k-1} \frac{2}{|\eta|} \left| sin\left[\frac{\eta}{2} log\left(1+\frac{a_{i+1}-a_i}{(t-a_{i+1})}\right)\right]\right| |\beta_i|_B \le \gamma(\eta) \sum_{i=0}^{i=k-1} \frac{a_{i+1}-a_i}{t-a_{i+1}} |\beta_i|_B, \forall t > T.$$

(we have used $|sinu| \le |u|$, and $log(1+v) \le v,\ 0<v<1$).

Now if we choose $t = 2T$, then $T - a_{i+1} \ge 0,\ 0 \le i \le k-1$, and one has

$$\left|\tilde{\phi}_k(2T)\right|_B \le \gamma(\eta) \frac{1}{T} \sum_{i=0}^{i=k-1} (a_{i+1} - a_i)|\beta_i|_B = \gamma(\eta) \frac{1}{T} \int_0^T |\phi_k(\tau)|_B d\tau. \tag{3.13}$$

and the proposition is proved for f_k. (with $\gamma(\eta) = \left(\frac{\eta sh\eta}{\pi}\right)^{1/2}$).

– Second step.

For a given $\phi \in L^1(O,T;B)$ with compact support contained in $[0,T]$, we can always find step functions with compact support $\phi_k \longrightarrow \phi$, *a.e. and in* $L^1(0,T;B)$ norm as $k \longrightarrow +\infty$.

If we start with the corresponding inequality (3.13), we can replace in the last integral of the right member ϕ_k *by* ϕ when one pass to the limit as $k \longrightarrow +\infty$;

Now, observing that the kernell of $\tilde{\phi}_k(2T)$ *is bounded* because $2T - x \ge T$, $0 \le x \le T$, the Lebesgue dominated convergence theorem gives (3.12).

2) The proof

Let, $f \in L^1_{loc}(R^+)$ *and fix* $t > 0$. One introduces, for $n \in \boldsymbol{N}$, the truncating sequence θ_n:

$$\theta_n(\tau) = 1\ ,\ 0 \le \tau \le t - 1/n,$$
$$\theta_n(\tau) = n(t - \frac{1}{2n} - \tau),\ t - \frac{1}{n} \le \tau \le t - \frac{1}{2n}$$
$$\theta_n(\tau) = 0\ ,\ \tau \ge t - \frac{1}{2n}$$

then $f_n = \theta_n f$ has a compact support $\subset (0, t - \frac{1}{2n})$. We can apply (3.12) to f_n and pass to the limit by Lebesgue theorem as $n \longrightarrow +\infty$, to obtain

$$\left|\tilde{f}(2t)\right|_B \le \gamma(\eta)\mathcal{H}(|f|_B)(t),\ \textit{for all } t > 0. \tag{3.14}$$

If we assume $f \in L^p_c(B)$ and $c \in \mathcal{H}(p)$ then $\mathcal{H}(|f|_B) \in L^p_c(B)$, accordingly, one has

$$\left|\mathcal{H}(|f|_B\right|_{L^p_c(R^+)} \le k|f|_{L^p_c(B)} \tag{3.15}$$

Multiplying the two sides of (3.14) by $c(2t)$ and observing that $\mathcal{H}(|f|_B)(t) \leq 2\mathcal{H}(|f|_B(2t)$, we can take the integral on R^+ of the power p of each side in (3.14.), to obtain the wanted result.

REMARK 3.7. – Thus we have proved that $Y_{-i\eta}$ is a convolutor for $L^p_c(B)$ if and only if $c \in \mathcal{H}(p)$. That is the so called condition $(\mathcal{C})$ used in [4].

An other main result obtained in the same way is

THEOREM 3.8. – *Assume* $c \in \mathcal{H}(p)$ *and* $f \in L^1(\boldsymbol{R}^+)$, $u \in L^p_c(B)$. *Setting* $\tilde{v} = Y_{-i\eta} * v$, *one has* $f * \tilde{u} \in L^p_c(B)$ *and there is a constant* γ *such that*

$$|f * \tilde{u}|_{L^p_c(B)} \leq \gamma |f|_1 |u|_{L^p_c(B)} \,. \tag{3.16}$$

PROOF. – We introduce, for $t > 0$ fixed, the truncating sequence θ_n and letting $u_n = \theta_n u$, we can write $g_n = f * \tilde{u}_n = \tilde{f} * u_n$, so that

$$|g_n(2t)|_B = \left| \int_0^{2t} \tilde{f}(2t-\sigma) u_n(\sigma) d\sigma \right| \leq \int_0^{t-1/2n} \left| \tilde{f}(2t-\sigma) \right| |u_n(\sigma)|_B d\sigma,$$

from the definition of u_n.

Now as we have see before that $\left|\tilde{f}(s)\right| \leq \frac{\gamma}{s} \int_0^s |f(\zeta) d\zeta|$, one deduces from the last inequality

$$|g_n(2t)|_B \leq \gamma \int_0^{t-1/2n} [\frac{u_n(\sigma)}{2t-\sigma} \left(\int_0^{2t-\sigma} |f(\zeta)| d\zeta \right)] d\sigma \leq \gamma |f|_1 \frac{1}{t+1/2n} \int_0^{t-1/2n} |u_n(\sigma)|_B d\sigma$$

Like at the end of the proof of theorem (3.3), we can pass to the limit as $n \longrightarrow +\infty$ to get

$$\forall t > 0, \ |g(2t)|_B \leq \gamma |f|_1 \mathcal{H}(|u|_B)(t). \tag{3.17}$$

and the proof is ended as before to obtain the wanted result.

Now, let A_0, A_1, X_0, X_1 given like in subsection 2.2, satisfying (2.4), (2.5), (2.7), then from theorem 3.3, using real (4) or complex (5) interpolation methods, one has

(4) see [21], [28], [33].
(5) see [10], [18].

PROPOSITION 3.9. – *Let*

$$\theta \in]0,1[,\ \frac{1}{p}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1},\ 1\le p_i\le\infty,\ (i=0,1),\ c_\theta=c_0^{1-\theta}c_1^{\theta}$$

then $\{Y_{-i\eta}*\}_{\eta\in R}$ *belongs to* $\mathcal{L}\big(L^p_{c_\theta}\big[(A_0,A_1)_{\theta,p}\big]\big)$ *and to* $\mathcal{L}\big(L^p_{c_\theta}\big[(A_0,A_1)_{\theta}\big]\big)$.

PROOF. – Theorem 3.3 means $\{Y_{-i\eta}*\}_{\eta\in R}\in\mathcal{L}(X_i),\ i=0,1$ and by real or complex interpolation method, one has $\{Y_{-i\eta}*\}_{\eta\in R}\in\mathcal{L}\big[\big(X_0,X_1\big)_{\theta,p}\big]$ or $\in\mathcal{L}\big[(X_0,X_1)_\theta\big]$ and we know [10], [18], [21], [28], [33],

$$(X_0,X_1)_{\theta,p}=L^p_{c_\theta}\Big[(A_0,A_1)_{\theta,p}\Big],\ (X_0,X_1)_\theta=L^p_{c_\theta}\big[(A_0,A_1)_\theta\big].$$

REMARK 3.10. – For $\eta\in R$, consider the space $W^{(m)}_\eta=\{u;\ Y_{-i\eta}*u\in W^{(m)}\}$. From theorem 3.3 , like $\big[Y_{-i\eta*}\big]^{-1}=Y_{i\eta}*$, one has $W^{(m)}\subset W^{(m)}_\eta$ hence we can say that $W^{(m)}\simeq W^{(m)}_\eta$ with equivalence of norms.

4. – Interpolation with complex method for the spaces $W^{(m)}$

4.1 – *Intermediate derivatives properties.*

Now we can give the main theorem of intermediate derivatives for $W^{(m)}$ improving results of [4] and [5].

THEOREM 4.1. – *Assume* $c_i\in\mathcal{H}(p_i)$ *then the mapping* $u\longrightarrow D^ju$, $0\le j\le m-1$, *is continuous from* $W^{(m)}$ *into* $L^{p_j}_{c_j}[(A_0,A_1)_{j/m}]$, *with*

$$c_j=c_0^{1-j/m}c^{j/m},\quad \frac{1}{p_j}=\frac{1-j/m}{p_0}+\frac{j/m}{p_1}. \tag{4.1}$$

morever one has

$$N_j(D^ju)\le\gamma N_0^{1-j/m}(u)N_1^{j/m}(D^mu) \tag{4.2}$$

REMARK 4.2. – Theorem 4.1 stated here with j an integer $0\le j\le m-1$, is also naturally valid for any derivative of order $r\le m$.

PROOF OF THEOREM 4.1. – We are going to proceed in two steps.

1) *First step.*

Letting $\Lambda=D^m$, one has

$$\begin{array}{ccc} W^{(m)} & \xrightarrow{I} & X_0\ \subset\ X_0+X_1\\ W^{(m)} & \Lambda\longrightarrow & X_1\ \subset\ X_0+X_1\end{array}$$

(where $\subset$ means always continuous embedding)

For $z \in \boldsymbol{C}$, let

$$\Lambda^z = Y_{-mz}*,$$

and we claim

LEMMA 4.3. – *When* $z=\rho+i\zeta$, $0\leq\rho\leq 1$, $\zeta\in\boldsymbol{R}$, *the operator* $\Lambda^z\in\mathcal{L}(W^{(m)}, X_0+X_1)$.

PROOF OF THE LEMMA 4.3. – If $u \in W^{(m)}$, obviouslely $\Lambda^z u(t)$ takes its values (a.e.) in Y and from theorem 3.3, the lemma is true for $\rho = 0$ *and* $\rho = 1$.

From lemma 4.1 it is sufficient to prove the result, when $0<\rho<1$, for $u \in \mathcal{D}(\overline{\boldsymbol{R}}^+; X)$. Accordingly with (3.2), we can write

(4.3) $$\Lambda^z = Y_{-m+(1-\rho)m-i\eta}* = Y_s * Y_{-m} * Y_{-i\eta}*,\ s = (1-\rho)m,\ \eta = m\zeta$$

and one has

(4.4) $$\Lambda^z u = Y_s * D^m\tilde{u},\ \textit{with}\ D^m\tilde{u} \in X_1$$

so that

(4.5) $$\Lambda^z u(t) = \frac{1}{\Gamma(s)}\int_0^t (t-\sigma)^{s-1} D^m\tilde{u}(\sigma)d\sigma$$

where the integral in (4.8) take a sense because $s-1 > -1$.

Now we can find two smooth functions $\phi_i,\ i = 0, 1$ defined on $\overline{\boldsymbol{R}}^+$, satisfying

(4.6) $$\phi_0(t) + \phi_1(t) = 1\ \textit{for}\ t \geq 0,$$

and

(4.7) $$D^m\left[t^{s-1}\phi_0\right] \in L^1(\boldsymbol{R}^+),\ t^{s-1}\phi_1 \in L^1(\boldsymbol{R}^+)$$

In order to do that we can take $\phi_0 = 1 - e^{-t^k},\ \phi_1 = 1 - \phi_0, k$ sufficiently large.

Then we obtain a decomposition of $\Lambda^z u$ *in* $X_0 + X_1$ by

(4.8) $$\Lambda^z u(t) = \psi_0(t) + \psi_1(t),\ \psi_i \in X_i,\ i = 0, 1$$

with

$$\psi_0 = \frac{1}{\Gamma(s)}\left[D^m\left(t^{s-1}\phi_0\right)\right] * \tilde{u},\quad \psi_1 = \frac{1}{\Gamma(s)} t^{s-1}\phi_1 * \tilde{(D^m u)}.$$

The kernels of ψ_0 and ψ_1 being in $L^1(\boldsymbol{R}^+)$ one can apply theorem 3.8 to obtain

(4.9) $$N_0(\psi_0) \leq \gamma_0(m,\rho)N_0(u),\quad N_1(\psi_1) \leq \gamma_1(m,\rho)N_1(D^m u)$$

and the proof of lemma 4.3 follows by density.

2) *Second step.*

We are going to use interpolation by complex method. We recall that if $\tilde{A}_0, \tilde{A}_1, \tilde{\mathcal{A}}$, , $\tilde{X},\ \tilde{Y}$, are like $A_0,\ A_0, \mathcal{A}$ an intermediate space between $\tilde{A}_0,\ \tilde{A}_1$ is any normed space $\tilde{A}$ such that $\tilde{X} \subset \tilde{A} \subset \tilde{Y}$ with continuous embedding.

Such a space $\tilde{A}$ is a space of interpolation, if every linear mapping from $\tilde{Y}$ into itself which is continuous from $\tilde{A}_0$ into itself and from $\tilde{A}_1$ into itself is automatically continuous from $\tilde{A}$ into itself.The complex method (see [10], [18]) is a method which allows to construct interpolation spaces from two normed spaces, Banach spaces or Hilbert spaces, Lorentz spaces and so on...

Let $\tilde{A}_0, \tilde{A}_1$ two Banach spaces and consider the space $\mathcal{K}(\tilde{A}_0, \tilde{A})$ of analytic functions

$$f \ : \ z = \zeta + i\eta \longrightarrow f(z)$$

with values in $\tilde{Y}$, which is defined on the open strip $B = \{0 < \zeta < 1, \ \eta \in \boldsymbol{R}\}$, continuous on the closed strip $\overline{B} = \{0 \leq \zeta \leq 1, \ \eta \in \boldsymbol{R}\}$ and such that

$$(4.10)\quad f(i\eta) \in \tilde{A}_0, \ \textit{and } \eta \longrightarrow f(i\eta) \textit{ is continuous and bounded from } \boldsymbol{R}_\eta \longrightarrow \tilde{A}_0,$$

$$(4.11)\quad f(1+i\eta) \in \tilde{A}_1, \ \eta \longrightarrow f(1+i\eta) \textit{ is continuous, bounded from } \boldsymbol{R}_\eta \longrightarrow \tilde{A}_1.$$

equipped with the norm

$$(4.12)\qquad |f|_{\mathcal{K}(\tilde{A}_0,\tilde{A}_1)} = max\Big[\sup_\eta |f(i\eta)|_{\tilde{A}_0}, \ \sup_\eta |f(1+i\eta)|_{\tilde{A}_1}\Big]$$

and from a theorem of Carleman [23] (known also as “Theorem of three lines”) one see that $\mathcal{K}(\tilde{A}_0, \tilde{A}_1)$ is a Banach space.

For $0 < \theta < 1$, one defines

$$(4.13)\qquad \big(\tilde{A}_0, \ \tilde{A}_0\big)_\theta = \{a \in \tilde{Y}; \ a = f(\theta)\}$$

which is a Banach space equipped with the norm

$$(4.14)\qquad \|a\|_\theta = \inf_{f(\theta)=a} |f|_{\mathcal{K}(\tilde{A}_0,\tilde{A}_1)}.$$

REMARK 4.4. – Boundeness in (4.13), (4.14) is not essential for the theory (see [20]). In fact we can consider also the case

$$(4.15)\qquad |f(i\eta)|_{\tilde{A}_0} \leq \gamma e^{k|\eta|}, \quad |f(1+i\eta)|_{\tilde{A}_1} \leq \gamma e^{k|\eta|},$$

one obtains a space $\mathcal{K}_{esp}(\tilde{A}_0, \tilde{A}_1)$ normed by (4.15) where $e^{-k|\eta|}f$ is subsituted to f. However one has

$$(4.16)\qquad \mathcal{K}_{exp}(\tilde{A}_0, A_1) = \mathcal{K}(\tilde{A}_0, \tilde{A}_1), \ \text{with equivalence of norms.}$$

Like $Y_{-i\eta}*$ is a group of operators, from the properties of semi-groups of class C^0 one has

$$(4.17)\qquad \forall \eta \in R, \big\|Y_{-i\eta}*\big\|_{\mathcal{L}(X_i)} \leq \gamma e^{\alpha|\eta|}, \ \gamma, \alpha = \textit{constants} > 0$$

(in the case $c_i \equiv 1$, we can make γ, α more precise (see [3]).

Thus we are in the framework of remark 4.4, if we choose $f(z) = \Lambda^z$. In order to take in account (4.19) we define, with $z = \rho + i\zeta$, a function

$$(z,t) \longrightarrow f(z,t) = \chi(z)\Lambda^z u(t), \;\; u \in W^{(m)} \tag{4.18}$$

where

$$\chi(z) = exp[m(z-\rho)^2] \tag{4.19}$$

then

$$\chi(\rho) = 1, \;\; |\chi(i\eta)|e^{\alpha|\eta|} < +\infty, \;\; |\chi(1+i\eta|e^{\alpha|\eta|} < +\infty, \;\; \eta = m\zeta, \tag{4.20}$$

Now, let

$$f(z) = t \longrightarrow f(z,t)$$

therefore $f : z \longrightarrow f(z)$ is analytic bounded from $B = \{0<\rho<1, \;\eta\in R\}$ *to* $X_0 + X_1$, and continuous bounded from $\overline{B}$ *to* $X_0 + X_1$, then by complex interpolation

$$f(\rho) \in \;(X_0, X_1))_\rho, \;\; \rho \in]0,1[. \tag{4.21}$$

But, we know

$$(X_0, X_1)_\theta = L^{p_\theta}_{c_\theta}\left[(A_0, A_1)_\theta\right], \tag{4.22}$$

where

$$c_\theta = c_0^{1-\theta} c_0^\theta, \;\; \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \tag{4.23}$$

with

$$\|f(\theta)\|_\theta \leq [N_0(u)]^{1-\theta}[N_1(D^m u)]^\theta. \tag{4.24}$$

hence one can take $\theta = \rho = j/m, \;\; 1 \leq j \leq m-1$, and theorem 4.2 is proved.

REMARK 4.5. – The theorem 4.1, valid for $c_j \in \mathcal{H}(p)$, improve too, of the point of view of the spaces, the result of [5] (Th. 5.7), which gives the same result (with only $c_j \in A(p)$) but we have only

$$D^j u \in L^{p_i}_{c_j}[T_j^{(m)}(\infty, A_0; \infty, A_1)] \tag{4.25}$$

where $T_j^{(m)}$ is the space of trace of order j of the space $W^{(m)}(\infty, 1, A_0; \infty, 1, \; A_1)$.

From [21] $T_j^{(m)} \simeq S(\infty, j/m, A_0; \; \infty, 1-j/m, A_1)$, so that from [28] $T_j^{(m)} \simeq (A_0, A_1)_{\theta,\infty}$. Now again from [21] one has

$$(A_0, A_1)_{\theta,1} \subset (A_0, A_1)_\theta \subset (A_0, A_1)_{\cdot\theta,\infty}.$$

The result follows.

4.2 – *Fractional derivatives.*

First, consider the case $m=1$. In this case, lemma 4.3 being true, theorem 4.1 gives

THEOREM 4.6. – *Assume* $m=1$, *and let* $0 \le \mu \le 1$, *then if* $u \in W^{(1)}$ *one has*

$$D^{\mu}u = Y_{-\mu} * u \in L^{p_\mu}_{c_\mu}\left[(A_0, A_1)_\mu\right] \tag{4.26}$$

with

$$N_\mu(D^\mu u) \le [N_0(u)]^{1-\mu} N_0(Du)]^{\mu}. \tag{4.27}$$

REMARK 4.7. – Consider the case $A_0 = A_1 = \boldsymbol{R}$, and $c_0 = c_1 = 1$, then the corresponding space $W^{(1)}$ is the Sobolev space $W^{1,1}(\boldsymbol{R}^+) = W^1(\boldsymbol{R}^+)$ and theorem 4.5 gives

$$D^\mu u = Y_{-\mu} * u \in L^1(\boldsymbol{R}^+),\ |D^\mu u|_1 \le \gamma |u|_1^{1-\mu} |Du|_1^{\mu} \le \gamma \|u\|_{W^1},\ 0<\mu<1. \tag{4.28}$$

The result can be found also using Fourier's transform (see section 5).

Now we consider for $0<\mu<1$ the space

$$W^{(\mu)} = \{u;\ u \in L^{p_0}_{c_0}(A_0),\ Y_{-\mu} * u \in L^{p_1}_{c_1}(A_1)\}$$

Which is a Banach space provided with the natural norm.

We claim

THEOREM 4.8. – *Assume* $c_i \in \mathcal{H}(p_i),\ i = 0,1$ *then for all* $u \in W^{(\mu)}$, *one has*

$$D^\nu(u) \in L^{p_\nu}_{c_\nu}[(A_0, A_1)_\nu],\ 0 \le \nu \le \mu, \tag{4.29}$$

with

$$c_\nu = c_0^{1-\nu} c_1^{\nu},\ \frac{1}{p_\nu} = \frac{1-\nu}{p_0} + \frac{\nu}{p_1}$$

and

$$N_\nu(D^\nu u) \le \gamma\, [N_0(u)]^{1-\nu} [N_1(D^\mu u)]^{\nu}. \tag{4.30}$$

PROOF. – Like for theorem 4.1 we consider $\Lambda^z = Y_{-\mu z}*$ and to prove lemma 4.3 we have only to check that (4.11) holds. This is obvious for $t^{s-1}\phi_1$ and like the choice of ϕ_0 implies $t^{s-1}\phi_0 \in W^1(\boldsymbol{R}^+)$, the remark 4.6 gives the result.

5. – Return to unweighted spaces

When $c_0 = c_1 = 1$ théorem 4.1 and similar results for fractionnal derivatives are obviously true without conditions on the spaces excepted the fact that $Y_{-i\eta}$ does not works for $p = 1$ and one extends simply the result to R^N.

In fact if $t = (t_1, \dots, t_N),\ \eta = (\eta_1, \dots, \eta_N)$, one defines

$$Y_{-i\eta} = Y_{-i\eta_1} \otimes \dots \otimes Y_{-i\eta_N},$$

and the associated function $f(z)$. Therefore if

$$\begin{aligned} &W^{(m)} = [u;\ u \in L^{p_0}(R^N, A_0)),\ D^\beta(u) \in L^{p_1}(R^N, A_1), \\ &\beta = (\beta_1, \dots, \beta_N),\ |\beta| = |\beta_1| + \dots + |\beta_N| = m, \end{aligned} \tag{5.1}$$

Which is a Banach space provided with the norm

$$\|u\|_m = N_0(u) + \sum_{|\beta|=m} N_1(D^\beta u), \tag{5.2}$$

one has for $j = (j_1, \dots, j_N),\ 1 < |j| < m,\ \theta = |j|/m$,

$$D^j(u) \in L^{p_\theta}\big[R^N, (A_0, A_1)_\theta\big] \ = \ X_\theta,\quad \frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \tag{5.3}$$

with

$$\|D^j(u)\|_{X_\theta} \le \gamma_j\ [N_0(u)]^{1-\theta}[\sum_{|\beta|=m} N_1(D^\beta u)]^\theta. \tag{5.4}$$

Like Fourier transform is a good tool when $A_i,\ i = 0, 1$, are Hilbert spaces (6) and gives the same results that the direct method of theorem 4.1, we can hope to use this transformation also in the case where A_i are Banach space. However the case is more complicated and we are leading to restrict the class of the spaces considered to do that.Now we return to simplify to the case of $N = 1$. Letting as above $X_i = L^{p_i}(A_i),\ i = 0, 1$, and if

$$\mathcal{F} :\ f \longrightarrow \hat{f} = \mathcal{F}(f) \tag{5.5}$$

is the Fourier transform of the vectorial distribution (7) f defined by

$$\hat{f}(\xi) = \int_R e^{i\xi x} f(x) dx$$

(6) see [17], [20].
(7) see [31].

(when f is a function), we let

$$\widehat{X}_i = \mathcal{F}(X_i) =, \quad i = 0, 1 \tag{5.6}$$

which is a Banach space for the topology carried out by $\mathcal{F}$.

Let Φ be any functor ([8]) of interpolation

LEMMA 5.1. – *$\mathcal{F}$ is an isomorphism from $\Phi(X_0, X_1)$ onto $\hat{\Phi}(X_0, X_1)$ and one has*

$$\hat{\Phi}(X_0, X_1) = \Phi(\widehat{X}_0, \widehat{X}_1). \tag{5.7}$$

PROOF. – See [3].

But we know ([9]) that if $f \in L^p(B)$, B being a Banach space, one has generally $\hat{f} \notin L^1_{loc}(B)$, so that, we don't have an Hausdorff-Young theorem like for functions with scalar values, that is: *if* $f \in L^p$, $1 \leq p \leq 2$, then $\hat{f} \in L^{p'}$, $2 \leq p' \leq \infty$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$. Then it is natural to introduce the

DEFINITION 5.2. – *We shall said that B is of type p, $1 \leq p \leq 2$, if $f \in L^p(B)$ implies $\hat{f} \in L^{p'}(B)$, $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.*

One has (see [27]): every Banach space is of type 1, every Hilbert space is of type 2 and if B is reflexive and of type p, then the dual B' is also of type p. [27].

Now if we consider the spaces A_i, $i = 0, 1$, like in subsection 2.2, we have in particular

PROPOSITION 5.3. – *Assume A_i of type p_i, $(i = 0, 1)$ and suppose that $L^{p_0}(A_0)$ or $L^{p_1}(A_1)$ to be reflexive, then for $0 \leq \theta \leq 1$, $(A_0, A_1)_\theta$ is of type p_θ, given by* $\dfrac{1}{p_\theta} = \dfrac{1-\theta}{p_0} + \dfrac{\theta}{p_1}$.

PROOF. – From assumptions we have

$$\mathcal{F}: \quad X_i = L^{p_i}(A_i) \longrightarrow \tilde{X}_i = L^{p'_i}(A_i), \quad i = 0, 1,$$

then from lemma 5.1

$$\mathcal{F}: \quad (X_0,\ X_1)_\theta \longrightarrow \Big(L^{p'_0}(A_0),\ L^{p'_1}(A_1)\Big)_\theta \tag{5.8}$$

([8]) that is a method to construct interpolation space ([3], [13]).
([9]) for example see [27].

But, from ([10], [18]), the assumption of reflexivity implies

$$(X_0, X_1)_\theta = L^{p_\theta}\left[(A_0, A_1)_\theta\right] \tag{5.9}$$

and

$$\left(L^{p'_0}(A_0), L^{p'_1}(A_1)\right)_\theta \subset L^{p'_\theta}\left[(A_0, A_1)_\theta\right], \ \frac{1}{p_\theta} + \frac{1}{p'_\theta} = 1.$$

The result follows.

REMARK 5.4. – From lemma 5.1 and some properties in [21] the result of Proposition 5.3 is also valid by real method for $(A_0, A_1)_{\theta,p}$ which is of type $p = p_\theta$, but here without the assumption of reflexivity.

Now if $W_1^{(m)} = \{u;\ u \in X_0,\ D^m u \in X_1\}$ provided with the natural topologie we can state

THEOREM 5.5. – *Assume A_i of type p_i $(i = 0, 1)$, and reflexive, then for every j, $0 \le j \le m$, the mapping $u \longrightarrow D^j u$ is continuous from $W_1^{(m)}$ into $X_\theta = L^{p_\theta}\left[(A_0, A_1)_\theta\right]$, $\theta = j/m$ and one has*

$$\left\|D^j u\right\|_{X_\theta} \le \gamma_j [N_0(u)]^{1-\theta} [N_1(D^m u)]^\theta.$$

PROOF. – Consider with $\widehat{X}_i = L^{p'_i}(A_i),\ i = 0, 1$:

$$\widehat{W}_1^{(m)} = \{v;\ v \in \widehat{X}_0,\ |\xi|^m v \in \widehat{X}_1\}; \tag{5.10}$$

provided with the norm

$$\|v\|_{\widehat{W}_1^{(m)}} = \|v\|_{\widehat{X}_0} + \||\xi| v\|_{\widehat{X}_1}, \tag{5.11}$$

$\widehat{W}_1^{(m)}$ is a Banach space isomorphic to $\mathcal{F}\left(W_1^{(m)}\right)$.

Like from assumptions, Lemma 5.1 and Proposition 5.3, one has (5.8), (5.9), and also that $(A_0, A_1)_\theta$ is of type p_θ, we have only to prove the continuity of

$$\hat{u} \longrightarrow |\xi|^j \hat{u} = T_j \hat{u},\ \widehat{W}_1^{(m)} \longrightarrow \left(\widehat{X}_0, \widehat{X}_1\right)_{j/m},\ 0 < j < m. \tag{5.12}$$

We can take notice that one has the scheme $(\mathcal{S})$

$$\begin{array}{ccccc} \widehat{W}_1^{(m)} & \xrightarrow{T_0} & \widehat{X}_0 & \subset & \widehat{X}_0 + \widehat{X}_1 \\ \widehat{W}_1^{(m)} & \xrightarrow{T_m} & \widehat{X}_1 & \subset & \widehat{X}_0 + \widehat{X}_1 \end{array}$$

then introducing the function of the complex variable $z = \rho + i\eta$,

$$z \longrightarrow U(z) = T_{mz} \hat{u}$$

we have to check that

U *is analytic bounded from* $B \longrightarrow \widehat{X}_0 + \widehat{X}_1$, *and continuous from* $\overline{B} \longrightarrow \widehat{X}_0 + \widehat{X}_1$,

where B and $\overline{B}$ are the strips defined in section 4.

Analyticity and continuity are obvious and for boundeness we look on the decomposition

$$T_{mz}\hat{u}(\xi) = \frac{|\xi|^{mz}}{1+|\xi|^m}[\psi_0 + \Psi_1] = \widehat{\Psi}_0 + \widehat{\Psi}_1,\ \Psi_i \in \widehat{X}_i,\ i = 0, 1. \tag{5.13}$$

Like the factor $\mu(\xi) = \dfrac{|\xi|^{m\rho}}{1+|\xi|^m}$ is bounded as $0 \le \rho \le 1$, and because $\Psi_i \in \widehat{X}_i = L^{p'_i}(A_i)$, one has $\widehat{\Psi}_i \in L^{p'_i}(A_i),\ i = 0, 1$.

On the other hand, from the scheme $(\mathcal{S})$:

$$\underset{\eta \in R}{sup}\ \|U(i\eta)\|_{\widehat{X}_0} \le c_0 \|\hat{u}\|_{\widehat{X}_0},\quad \underset{\eta \in R}{sup}\ \|U(1+i\eta)\| \le c_1 \||\xi|^m \hat{u}\|_{\widehat{X}_1} \tag{5.14}$$

therefore one deduces by complex interpolation

$$U(j/m) \in \left(\widehat{X}_0, \widehat{X}_1\right)_{j/m},\ 0 \le j \le m, \tag{5.15}$$

with

$$\|U(j/m)\|_{\left(\widehat{X}_0,\widehat{X}_1\right)_{j/m}} \le c[\|\hat{u}\|_{\widehat{X}_0}]^{1-j/m}[\||\xi|^m \hat{u}\|]^{j/m}, \tag{5.16}$$

and the wanted result follows with the help of proposition 5.3.

REMARK 5.6. – We can extend obviously the result to R^N to obtain (5.3), (5.4) for the space $W^{(m)}$ defined by (5.1), (5.2).

REMARK 5.7. – If we do not assume the conditions of p type, we have to work with $\widehat{X}_i = \mathcal{F}[L^{p_i}(A_i)]$ and one arrives to formula (5.13) where we need to have

$$\widehat{\Psi}_i \in \mathcal{F}[L^{p_i}(A_i)],\ i = 0, 1. \tag{5.17}$$

Like the factor $\mu(\xi)$ satisfies the Michlin condition

$$\underset{\xi \in R^N}{sup}\ ||\xi| D\mu(\xi)| < +\infty \tag{5.18}$$

it is natural to assume that A_i is such that the theorem of Michlin on multipliers was true in $\mathcal{F}[L^{p_i}(A_i)]$, that is the condition $\mathcal{M}$ of [3] (see introduction).

Therefore the result of theorem 3.5 and the Remark 5.6 can be extended to that general case. Like one don' thave a characterisation of spaces satisfying the condition $\mathcal{M}$, spaces of p type gives a realistic example for applications.

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