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Abstract. – *We outline the history of Risk Measures from the original formulation given by Artzner Delbaen Eber and Heath until the more recent research on quasiconvex Risk Measures. We therefore present some novel results on quasiconvex Risk Measures in the conditional setting, focusing on two different approaches: the vector space compared to the module approach. In particular the second one will guarantee a complete duality theory which is a key ingredient in the representation of risk preferences.*

1. – Introduction: the axiomatic birth of risk measurement

What is Risk?

Due to the recent financial crisis which has sunk the trust and optimism of people towards financial markets, Risk has become one of the most used and misused words in everyday life. Nevertheless a rigorous mathematical definition of Risk is a delicate issue even for financial practitioners.
A seminal contribution to this topic was surprisingly given by Daniel Bernoulli in 1738, who perceived the role of the risk aversion in everyday life decision making. This was the starting point of the notion of *Expected Utility*.

'*The determination of the value of an item must not be based on its price, but rather on the utility it yields. The price of the item is dependent only on the thing itself and is equal for everyone; the utility, however, is dependent on the particular circumstances of the person making the estimate. Thus there is no doubt that a gain of one thousand ducats is more significant to a pauper than to a rich man though both gain the same amount.*

[...] *Now it is highly probable that any increase in wealth, no matter how insignificant, will always result in an increase in utility which is inversely proportionate to the quantity of goods already possessed.*

[...] *First, it appears that in many games, even those that are absolutely fair, both of the players may expect to suffer a loss; indeed this is Nature's admonition to avoid the dice altogether....This follows from the concavity of curve. [...] It is clear that the disutility which results from a loss will always exceed the expected gain in utility.*'

(D. Bernoulli, *Exposition of a new theory on the measurement of risk*, [3], English translation by L. Sommer).

The formulation of these ideas in a modern notation leads directly to the well known representation of Agents' beliefs in terms of an expected utility as studied by Von Neumann and Morgenstern [20]: let $L^\infty =: L^\infty(\Omega, \mathcal{F}, \mathbb{P})$ be the space of essentially bounded random variables over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ then for any $X, Y \in L^\infty$, X is preferred to Y if and only $E_{\mathbb{P}}[u(X)] \geq E_{\mathbb{P}}[u(Y)]$. Here $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is a monotone increasing concave utility function as suggested by Bernoulli, which describes the agent attitude towards Risk.

Later on, in a financial environment, many risk procedure were introduced. It was the case of the *Mean-Variance* criterion (Markovitz, 1952, [17]), the *Sharpe's ratio* (1964, [18]) and the *Value at Risk* ($V@R$), defined through the quantiles of a given distribution with a predefined level of probability. This last method is the most employed in credit institutes and has been pointed out as the reference parameter by the Basel Committee on Banking Supervision (Basel II 2006). More in detail, for a given random variable X and a parameter $\lambda \in (0, 1)$ (usually $\lambda = 0.01$) we have

$$V@R_\lambda(X) := -\sup\{m \in \mathbb{R} \mid P(X \leq m) \leq \lambda\}.$$

The capital requirement $V@R_\lambda(X)$ due to the risk exposure of position X is therefore the necessary amount of money to cover eventual losses greater than m which occur with probability $(1 - \lambda)$. Unfortunately we will be dangerously affected to all the other losses (those that are under level m), which are rare events of probability λ.

Modern Developments

At the end of the Nineties, Artzner, Delbaen, Eber and Heath [2] produced a rigorous axiomatic formalization of coherent risk measures, led by normative intent. The regulating agencies asked for computational methods to estimate the capital requirements, exceeding the unmistakable lacks showed by the extremely popular $V@R$. The key idea was to provide a set of axioms that any reasonable risk measure should have, instead of analyzing each single risk measure.

Risk Measures are real valued functionals defined on a space of random variables which encloses every possible financial position. It may seem naive to use a single number to describe the complexity of the distributions characterizing those random variables. On the other hand this appears as the only way to succeed in the assessment of the capital requirement needed to a bank to recover a high possible loss due to risky investments.

The definition of a *coherent risk measure* requires four main hypotheses to be satisfied:

DEFINITION 1. – *A coherent risk measure is a functional* $\rho : L^\infty \to \mathbb{R}$ *which satisfies*

(i) *monotonicity, i.e.* $X_1 \leq X_2$ $\mathbb{P}$*-a.s. implies* $\rho(X_1) \geq \rho(X_2)$ *for every* $X_1, X_2 \in L^\infty$,
(ii) *cash additivity, i.e.* $\rho(X + c) = \rho(X) - c$,
(iii) *positive homogeneity, i.e. for every* $\alpha > 0$, $\rho(\alpha X) = \alpha\rho(X)$,
(iv) *sublinearity*, i.e. $\rho(X + Y) \leq \rho(X) + \rho(Y)$.

Monotonicity represents the minimal requirement for a risk measure to model the preferences of a rational agent. Assumption (ii) allows an important characterization of Risk as

$$\rho(X) = \inf\{\alpha \in \mathbb{R} \mid X + \alpha \in \mathcal{A}\}. \tag{1}$$

The Risk of a financial position is thus the minimal amount of money that an institution will have to sum up to X in order to make it acceptable with respect to some *criterium* modelled by the Acceptance Set $\mathcal{A} = \{X \in L^\infty \mid \rho(X) \leq \rho(0)\}$.

Unfortunately both axioms (iii) and (iv) appear to be restrictive and unrealistic: the former does not sense the presence of liquidity risks, the latter does not describe the real intuition hidden behind the diversification process. For this reason in most of the literature (iii) and (iv) are substituted by

(v) *convexity*, i.e. $\rho(tX_1 + (1-t)X_2) \leq t\rho(X_1) + (1-t)\rho(X_2)$ for all $t \in [0, 1]$.

Axiom (v) has a natural interpretation: the risk of the diversified aggregated position $tX_1 + (1-t)X_2$ is surely smaller than the combination of the two single risks.

The class of convex Risk Measures was independently studied by Föllmer and Schied (2002, [9]) and Frittelli and Rosazza Gianin (2002, [14]).

Further developments were provided by El Karoui and Ravanelli relaxing the cash additivity axiom to cash subadditivity (2009, [7]) when the market presents a stochastic discount factor; finally Cerreia-Vioglio et al. (2010, [5]) showed how quasiconvexity better describes than convexity the principle of diversification, whenever cash additivity does not hold true. Following this trajectory we may conclude that the largest class of feasible Risk Measure is the following.

DEFINITION 2. – *A quasiconvex risk measure is a functional* $\rho : L^\infty \to \mathbb{R}$ *which satisfies*

(i) *monotonicity, i.e.* $X_1 \leq X_2$ *implies* $\rho(X_1) \geq \rho(X_2)$ *for every* $X_1, X_2 \in L^\infty$,
(vi) *quasiconvexity, i.e.* $\rho(tX_1 + (1-t)X_2) \leq \max\{\rho(X_1), \rho(X_2)\}$ *for all* $t \in [0, 1]$.

To better explain the quasiconvexity assumption we consider two random variable $X_1, X_2 \in L^\infty$ such that $\rho(X_1) \leq \rho(X_2)$. (vi) is the literal translation of the diversification principle: actually diversification over a proportion t of the position X_1 and the remaining proportion on the position X_2 (namely $tX_1 + (1-t)X_2$) should not increase the risk for more than $\rho(X_2)$.

On the other hand it is important to recall that under the cash additive assumption (ii) we deduce that quasiconvexity and convexity are equivalent. As a matter of fact quasiconvexity assumes a primary role in those markets for which the zero coupon bond is illiquid, and cash additivity needs to be dropped.

2. – Risk Measures in the Conditional Setting

Most of the applications that concern with decisions in the future are based on the notion of sigma algebra, which is the main probabilistic tool aimed to the description of the information available to the agent. As a consequence the Conditional Expectation, which is the simplest example of conditional map, appears as the 'red line' that distinguishes Probability from Analysis. The conditional expectation $E_{\mathbb{P}}[X|\mathcal{G}]$ filters a random variable X with the information provided by the sigma algebra $\mathcal{G}$, giving a sort of backward projection of X. In the dynamic description of Risk, we have the following situation: let $0 \leq t \leq T$

$$\begin{array}{lccccc} \text{Time Line} & 0 & \rightsquigarrow & t & \rightsquigarrow & T \\ \text{Filtration} & \sigma(\emptyset) & \rightsquigarrow & \mathcal{G} & \rightsquigarrow & \mathcal{F} \\ \text{Static Risk Measure} & \rho_0(X) & \leftleftarrows & \leftleftarrows & \leftleftarrows & X \\ \text{Conditional Risk Measure} & & & \rho_t(X) & \leftleftarrows & X \end{array} \tag{2}$$

where $\sigma(\emptyset)$ is the trivial σ-algebra. In the dynamic framework the risk assessment at the intermediate time t deals with a future decision and is thus described by a $\mathcal{G}$ measurable random variable $\rho_t(X)$. This means that any Risk Measure has to be a map that takes value in a set of random variables as will be properly formalized in the next paragraphs and in Section 4.

Notations

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is fixed throughout this paper and $\mathcal{G} \subseteq \mathcal{F}$ is any sigma algebra contained in $\mathcal{F}$. We denote with $L^0(\Omega, \mathcal{F}, \mathbb{P}) = L^0(\mathcal{F})$ (resp. $L^0(\mathcal{G})$) the space of $\mathcal{F}$ (resp. $\mathcal{G}$) measurable random variables that are $\mathbb{P}$ a.s. finite, whereas by $\bar{L}^0(\mathcal{F})$ the space of extended random variables which may take values in $\mathbb{R} \cup \{\infty\}$. In general since $(\Omega, \mathbb{P})$ are fixed we will always omit them in the notation. We define $L^0_+(\mathcal{F}) = \{X \in L^0(\mathcal{F}) \mid X \geq 0\}$ and

$L^0_{++}(\mathcal{F}) = \{X \in L^0(\mathcal{F}) \mid X > 0\}$. We remind that all equalities/inequalities among random variables are meant to hold $\mathbb{P}$-a.s.. As the expected value $E_{\mathbb{P}}[\,\cdot\,]$ is mostly computed w.r.t. the reference probability $\mathbb{P}$, we will often omit $\mathbb{P}$ in the notation.

Moreover the essential ($\mathbb{P}$ almost surely) *supremum* $ess\sup_\lambda(X_\lambda)$ of an arbitrary family of random variables $X_\lambda \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ will be simply denoted by $\sup_\lambda(X_\lambda)$, and similarly for the essential *infimum*. The symbol $\vee$ (resp. $\wedge$) denotes the essential ($\mathbb{P}$ almost surely) *maximum* (resp. the essential *minimum*) between two random variables, which are the usual lattice operations.

Capital Requirements in the conditional setting

As explained in the Introduction (see equation (1)) the cash additivity assumption guarantees the characterization of the Risk as a capital required to cover future losses or equivalently the minimal amount of money that we have to add so that the position becomes acceptable. Using the same notation as in (2) we consider $t \in [0, T]$ and a non empty convex set $C_T \in E \subset L^0(\mathcal{F})$ such that $C_T + L^0_+ \subseteq C_T$. The set C_T represents the future positions considered acceptable by the supervising agency. For all $m \in \mathbb{R}$ denote by $v_t(m, \omega)$ the price at time t of m euros at time T. The function $v_t(m, \cdot)$ will be in general $\mathcal{G}$ measurable as in the case of stochastic discount factor where $v_t(m, \omega) = D_t(\omega)m$. By adapting the definitions in the static framework of [2] and [5] we set:

$$\rho_{C_T,v_t}(X) := \inf_{Y \in L^0(\mathcal{G})} \{v_t(Y) \mid X + Y \in C_T\}. \tag{3}$$

Notice that the previous definition is well posed only if the sum $X + Y \in E$ for any $X \in E$ and any $Y \in L^0(\mathcal{G})$. In some sense $Y \in L^0(\mathcal{G})$ plays the role that $\alpha \in \mathbb{R}$ had in equation (1), but since it concerns a future decision then it has to be a $\mathcal{G}$ measurable random variable. For this reason we need to introduce the more complex structure of module over the ring $L^0(\mathcal{G})$ as will be soon explained. Once the opportune structure E is provided equation (3) defines a Risk Measure $\rho_{C_T,v_t} : E \to \overline{L}^0(\mathcal{G})$. When v_t is linear, then ρ_{C,v_t} is a convex cash additive dynamic risk measure, but the linearity of v_t may fail when zero coupon bonds with maturity T are illiquid. It seems anyway reasonable to assume that $v_t(\cdot, \omega)$ is increasing and upper semicontinuous and $v_t(0, \omega) = 0$, for $\mathbb{P}$ almost every $\omega \in \Omega$. In this case

$$\rho_{C_T,v_t}(X)(\omega) = v_t\Big(\inf_{Y \in L^0(\mathcal{G})} \{Y(\omega) \mid X + Y \in C_T\}, \omega\Big) = v_t(\rho_{C_T}(X)(\omega), \omega),$$

where $\rho_{C_T}(X)$ is the convex monetary dynamic risk measure induced by the set C_T namely

$$\rho_{C_T}(X) := \inf_{Y \in L^0(\mathcal{G})} \{Y \mid X + Y \in C_T\}.$$

Thus in general ρ_{C_T,v_t} is neither convex nor cash additive, but it is always quasiconvex.

3. – Current literature on L^0-modules

This section is inspired by the contribution given to the theory of L^0 -modules by Filipovic et al. [8] on one hand and on the other to the extended research provided by Guo from 1992 until today (see the references in [15]).

We will consider $L^0(\mathcal{G})$, with the usual operations among random variables, as a partially ordered ring and we will always assume in the sequel that τ_0 is a topology on $L^0(\mathcal{G})$ such that $(L^0(\mathcal{G}), \tau_0)$ is a topological ring. We do not require that τ_0 is a linear topology on $L^0(\mathcal{G})$ (so that $(L^0(\mathcal{G}), \tau_0)$ may not be a topological vector space) nor that τ_0 is locally convex.

DEFINITION 3 (Topological L^0-module). – *We say that (E, τ) is a topological L^0-module if E is a module over the ring $L^0(\mathcal{G})$ and τ is a topology on E such that the module operation*

(i) $(E, \tau) \times (E, \tau) \to (E, \tau)$, $(X_1, X_2) \mapsto X_1 + X_2$,

(ii) $(L^0(\mathcal{G}), \tau_0) \times (E, \tau) \to (E, \tau)$, $(\gamma, X_2) \mapsto \gamma X_2$

are continuous w.r.t. the corresponding product topology.

DEFINITION 4 (Duality for L^0-modules). – *For a topological L^0-module (E, τ), we denote*

$$E^* := \{\mu : (E, \tau) \to (L^0(\mathcal{G}), \tau_0) \mid \mu \text{ is a continuous module homomorphism}\}. \quad (4)$$

It is easy to check that $(E, E^*, \langle \cdot, \cdot \rangle)$ is a dual pair, where the pairing is given by $\langle X, \mu \rangle = \mu(X)$. Every $\mu \in E^*$ is $L^0(\mathcal{G})$-linear in the following sense: for all $\alpha, \beta \in L^0(\mathcal{G})$ and $X_1, X_2 \in E$

$$\mu(\alpha X_1 + \beta X_2) = \alpha\mu(X_1) + \beta\mu(X_2).$$

In particular, $\mu(X_1 \mathbf{1}_A + X_2 \mathbf{1}_{A^C}) = \mu(X_1)\mathbf{1}_A + \mu(X_2)\mathbf{1}_{A^C}$ for every $A \in \mathcal{G}$.

DEFINITION 5. – *A map $\|\cdot\| : E \to L^0_+(\mathcal{G})$ is a $L^0(\mathcal{G})$-seminorm on E if*

(i) $\|\gamma X\| = |\gamma| \|X\|$ *for all* $\gamma \in L^0(\mathcal{G})$ *and* $x \in E$,

(ii) $\|X_1 + X_2\| \leq \|X_1\| + \|X_2\|$ *for all* $X_1, X_2 \in E$.

The $L^0(\mathcal{G})$-seminorm $\|\cdot\|$ becomes a $L^0(\mathcal{G})$-norm if in addition

(iii) $\|X\| = 0$ *implies* $X = 0$.

We will consider families of $L^0(\mathcal{G})$-seminorms $\mathcal{Z}$ satisfying in addition the property:

$$\|X\| = 0,\ \forall \|\cdot\| \in \mathcal{Z} \ \text{ iff } \ X = 0, \tag{5}$$

As clearly pointed out in [15], one family $\mathcal{Z}$ of $L^0(\mathcal{G})$ -seminorms on E may induce on E more than one topology τ such that $\{X_\alpha\}$ converges to X in (E, τ) iff $\|X_\alpha - X\|$ converges to 0 in $(L^0(\mathcal{G}), \tau_0)$ for each $\|\cdot\| \in \mathcal{Z}$. Indeed, also the topology τ_0 on $L^0(\mathcal{G})$ play a role in the convergence.

DEFINITION 6 (L^0-module associated to $\mathcal{Z}$). – *We say that $(E, \mathcal{Z}, \tau)$ is a L^0-module associated to $\mathcal{Z}$ if:*

(i) *$\mathcal{Z}$ is a family of L^0-seminorms satisfying* (5),

(ii) *(E, τ) is a topological L^0-module,*

(iii) *A net $\{X_\alpha\}$ converges to X in (E, τ) iff $\|X_\alpha - X\|$ converges to* 0 *in (L^0, τ_0) for each $\|\cdot\| \in \mathcal{Z}$.*

Remark 2.2 in [15] shows that any random locally convex module over $\mathbb{R}$ with base $(\Omega, \mathcal{G}, \mathbb{P})$, according to Definition 2.1 [15], is a L^0-module $(E, \mathcal{Z}, \tau)$ associated to a family $\mathcal{Z}$ of L^0-seminorms, according to the previous definition.

EXAMPLE 1. – *Let $\mathcal{F}$ be a sigma algebra containing $\mathcal{G}$ and consider the generalized conditional expectation of $\mathcal{F}$ -measurable non negative random variables: $E[\,\cdot\,|\mathcal{G}] : L^0_+(\Omega, \mathcal{F}, \mathbb{P}) \to \bar{L}^0_+ := \bar{L}^0_+(\Omega, \mathcal{G}, \mathbb{P})$*

$$E[X|\mathcal{G}] =: \lim_{n\to+\infty} E[X \wedge n|\mathcal{G}].$$

Let $p \in [1, \infty]$ and consider the L^0-module defined as

$$L^p_{\mathcal{G}}(\mathcal{F}) =: \{X \in L^0(\Omega, \mathcal{F}, \mathbb{P}) \mid \|X|\mathcal{G}\|_p \in L^0(\Omega, \mathcal{G}, \mathbb{P})\}$$

where $\|\cdot|\mathcal{G}\|_p$ is the L^0-norm assigned by

$$\|X|\mathcal{G}\|_p =: \begin{cases} E[|X|^p|\mathcal{G}]^{\frac{1}{p}} & \text{if } p < +\infty \\ \inf\{Y \in \bar{L}^0(\mathcal{G}) \mid Y \geq |X|\} & \text{if } p = +\infty \end{cases} \tag{6}$$

Then $L^p_{\mathcal{G}}(\mathcal{F})$ becomes a L^0-normed module associated to the norm $\|\cdot|\mathcal{G}\|_p$ having the product structure:

$$L^p_{\mathcal{G}}(\mathcal{F}) = L^0(\mathcal{G})L^p(\mathcal{F}) = \{YX \mid Y \in L^0(\mathcal{G}),\ X \in L^p(\mathcal{F})\}.$$

For $p < \infty$, any L^0-linear continuous functional $\mu : L^p_{\mathcal{G}}(\mathcal{F}) \to L^0$ can be identi-

fied with a random variable $Z \in L^q_{\mathcal{G}}(\mathcal{F})$ *as* $\mu(\,\cdot\,) = E[Z \cdot |\mathcal{G}]$ *where* $\frac{1}{p} + \frac{1}{q} = 1$. *So we can identify* E^* *with* $L^q_{\mathcal{G}}(\mathcal{F})$.

Based on the results of Guo [15] and Filipovic et al. [8], we show that a family of seminorms on E may induce more than one topology on the L^0-module E and that these topologies guarantee a generalized version of the Hahn Banach Separation Theorem. The two different topologies on E depend on which topology is selected on L^0: either the uniform topology or the topology of convergence in probability.

These topologies on E will collapse to the same one whenever $\mathcal{G} = \sigma(\emptyset)$ is the trivial sigma algebra, but in general present different structural properties.

We set:

$$\|X\|_{\mathcal{S}} := \sup\{\|X\| \mid \|X\| \in \mathcal{S}\}$$

for any finite subfamily $\mathcal{S} \subset \mathcal{Z}$ of L^0 -seminorms. Recall from the assumption given in equation (5) that $\|X\|_{\mathcal{S}} = 0$ if and only if $X = 0$.

The uniform topology τ_c [8]

In this case, L^0 is equipped with the following uniform topology. For every $\varepsilon \in L^0_{++}$, the ball $B_\varepsilon := \{Y \in L^0 \mid |Y| \leq \varepsilon\}$ centered in $0 \in L^0$ gives the neighborhood basis of 0. A set $V \subset L^0$ is a neighborhood of $Y \in L^0$ if there exists $\varepsilon \in L^0_{++}$ such that $Y + B_\varepsilon \subset V$. A set V is open if it is a neighborhood of all $Y \in V$. A net converges in this topology, namely $Y_N \xrightarrow{|\cdot|} Y$ if for every $\varepsilon \in L^0_{++}$ there exists $\overline{N}$ such that $|Y - Y_N| < \varepsilon$ for every $N > \overline{N}$. In this case the space $(L^0, |\cdot|)$ looses the property of being a topological vector space. In this topology the positive cone L^0_+ is closed and the strictly positive cone L^0_{++} is open.

Under the assumptions that there exists an $X \in E$ such that $X\mathbf{1}_A \neq 0$ for every $A \in \mathcal{G}$ and that the topology τ on E is Hausdorff, Theorem 2.8 in [8] guarantees the existence of $X_0 \in E$ and $\mu \in E^*$ such that $\mu(X_0) > 0$.

A family $\mathcal{Z}$ of L^0-seminorms on E induces a topology on E in the following way. For any finite $\mathcal{S} \subset \mathcal{Z}$ and $\varepsilon \in L^0_{++}$ define

$$U_{\mathcal{S},\varepsilon} := \left\{X \in E \mid \|X\|_{\mathcal{S}} \leq \varepsilon\right\}$$

$$\mathcal{U} := \{U_{\mathcal{S},\varepsilon} \mid \mathcal{S} \subset \mathcal{Z} \text{ finite and } \varepsilon \in L^0_{++}\}.$$

$\mathcal{U}$ gives a convex neighborhood base of 0 and it induces a topology on E denoted by τ_c. We have the following properties:

1. $(E, \mathcal{Z}, \tau_c)$ is a $(L^0, |\cdot|)$-module associated to $\mathcal{Z}$, which is also a locally convex topological L^0-module (see Proposition 2.7 [15]);

2. on $(E, \mathcal{Z}, \tau_c)$ we can apply the generalization of Hahn Banach Separation theorem either for closed or open sets (see Theorems 2.6 and 2.8 [8]),
3. any topological $(L^0, |\cdot|)$ module (E, τ) is locally convex if and only if τ is induced by a family of L^0-seminorms, i.e. $\tau \equiv \tau_c$, (see Theorem 2.4 [8]).

A probabilistic topology $\tau_{\varepsilon,\lambda}$ [15]

The second topology on the L^0-module E is a topology of a more probabilistic nature and originated in the theory of probabilistic metric spaces.

Here L^0 is endowed with the topology $\tau_{\varepsilon,\lambda}$ of convergence in probability and so the positive cone L^0_+ is $\tau_{\varepsilon,\lambda}$-closed. According to [15], for every $\varepsilon, \lambda \in \mathbb{R}$ and a finite subfamily $\mathcal{S} \subset \mathcal{Z}$ of L^0 -seminorms we let

$$\mathcal{V}_{\mathcal{S},\varepsilon,\lambda} := \{X \in E \mid \mathbb{P}(\|X\|_{\mathcal{S}} < \varepsilon) > 1 - \lambda\}$$

$$\mathcal{V} := \{\mathcal{U}_{\mathcal{S},\varepsilon,\lambda} \mid \mathcal{S} \subset \mathcal{Z} \text{ finite}, \ \varepsilon > 0, \ 0 < \lambda < 1\}.$$

$\mathcal{V}$ gives a neighborhood base of 0 and it induces a linear topology on E, also denoted by $\tau_{\varepsilon,\lambda}$ (indeed if $E = L^0$ then this is exactly the topology of convergence in probability). This topology may not be locally convex, but has the following properties:

1. $(E, \mathcal{Z}, \tau_{\varepsilon,\lambda})$ becomes a $(L^0, \tau_{\varepsilon,\lambda})$-module associated to $\mathcal{Z}$ (see Proposition 2.6 [15]),
2. on $(E, \mathcal{Z}, \tau_{\varepsilon,\lambda})$ we can apply the generalization of Hahn Banach Separation theorem only for closed sets (see Theorems 3.6 and 3.9 [15]).

4. – Quasiconvex duality in the conditional setting

The relaxation of the convexity property for the risk maps has an immediate drawback in terms of the dual representation of the risk measures, since Fenchell-Moreau theorem fails for a non-convex functional. In the static case this problem has been addressed by Cerreia-Vioglio et al. in [5], providing a robust dual representation of the Risk Measures as a *supremum* over some probabilistic scenarios, based on the Penot-Volle dual representation (see [19]). The result obtained in [5] matches the one that holds for convex Risk Measures (see [14]) and can be stated as follows.

THEOREM 1 (Cerreia-Vioglio, Maccheroni, Marinacci, Montrucchio, [5]). – *A function $\rho : L^\infty \to \overline{\mathbb{R}}$ is quasiconvex monotone decreasing and $\sigma(L^\infty, L^1)$–lower* *semicontinuous if and only if*

$$\rho(X) = \sup_{Q \in \mathcal{P}} R(E_Q[-X], Q),$$

$$R(m, Q) = \inf\{\rho(\xi) \mid \xi \in L^{\infty} \text{ and } E_Q[-\xi] = m\}$$

where $\mathcal{P}$ is an opportune set of probability measures.

Notice that $R : \mathbb{R} \times \mathcal{P} \to \overline{\mathbb{R}}$ and $R(m, Q)$ can be interpreted as the reserve amount required today, under the scenario Q, to cover an expected loss m in the future.

In this section we consider the following type of Conditional Risk Measures and we compare the results obtained in two different frameworks. Here $(E, \geq)$ is a partially ordered convex lattice $E \subset L^0(\mathcal{F})$ and we assume that the following property holds:

$$X \in E \text{ and } A \in \mathcal{F} \Longrightarrow (X\mathbf{1}_A) \in E. \tag{7}$$

DEFINITION 7. – *A Conditional Quasiconvex Risk Measure is a map $\rho : E \to L^0(\mathcal{G})$ such that*

(MON) *monotone decreasing if for every $X, Y \in E$*

$$X \leq Y \quad \Rightarrow \quad \rho(X) \geq \rho(Y)\ ;$$

(QCO) *quasiconvex if for every $X, Y \in E$, $\Lambda \in L^0(\mathcal{G})$ and $0 \leq \Lambda \leq 1$*

$$\rho(\Lambda X + (1 - \Lambda)Y) \leq \rho(X) \vee \rho(Y);$$

(REG) *regular if for every $X, Y \in E$ and $A \in \mathcal{G}$*

$$\rho(X\mathbf{1}_A + Y\mathbf{1}_{A^C}) = \rho(X)\mathbf{1}_A + \rho(Y)\mathbf{1}_{A^C}.$$

The vector space approach [11]

We now adopt the standard approach in which E is any locally convex topological vector space of random variables satisfying the following assumptions

1. *The order continuous dual of $(E, \geq)$ denoted by E^*, is a lattice* ([1], *Th.* 8.28) *that satisfies $E^* \hookrightarrow L^1(\mathcal{F})$.*
2. *The space E endowed with the weak topology $\sigma(E, E^*)$ is a locally convex Riesz space.*

REMARK 1. – Many important classes of spaces satisfy these conditions, such as

– The L^p-spaces, $p \in [1, \infty]$: $E = L^p(\mathcal{F})$, $E^* \hookrightarrow L^1_{\mathcal{F}}$.
– The Orlicz spaces L^{Ψ} for any Young function Ψ: $E^* = L^{\Psi^*} \hookrightarrow L^1(\mathcal{F})$, where Ψ^* denotes the conjugate function of Ψ;
– The Morse subspace M^{Ψ} of the Orlicz space L^{Ψ}, for any continuous Young function Ψ.

In order to state the dual representation in the vector space case we set

$$\mathcal{P} =: \left\{ \frac{dQ}{d\mathbb{P}} \mid Q \ll \mathbb{P} \text{ and } Q \text{ probability} \right\}$$

and

$$R(Y, Q) := \inf_{\xi \in E} \{ \rho(\xi) \mid E_Q[-\xi|\mathcal{G}] =_Q Y \}. \tag{8}$$

THEOREM 2 (Frittelli - Maggis [11]). – *Suppose that E is order complete (with further topological assumptions). If* $\rho : E \to \overline{L}^0(\mathcal{G})$ *is (MON), (QCO), (REG) and* $\sigma(E, E^*)$*-lower semicontinuous then*

$$\rho(X) = \sup_{Q \in E^* \cap \mathcal{P}} R(E_Q[-X|\mathcal{G}], Q). \tag{9}$$

Sketch of the proof of Theorem 2

We here point out the essential arguments involved in the proof of Theorem 2 and we defer to the original article [11] for the details and the rigorous statements.

Unfortunately, we cannot prove directly that $\forall\, \varepsilon > 0$, $\exists\, Q_\varepsilon \in E^* \cap \mathcal{P}$ s.t.

$$\{\xi \in E \mid E_{Q_\varepsilon}[\xi|\mathcal{G}] \geq_{Q_\varepsilon} E_{Q_\varepsilon}[X|\mathcal{G}]\} \subseteq \{\xi \in E \mid \rho(\xi) > \rho(X) - \varepsilon\} \tag{10}$$

relying on Hahn-Banach Theorem, as it happened in the real case (see [19]). Indeed, the complement of the set in the right hand side of (10) is not any more a convex set – unless ρ is real valued – regardless of the continuity assumption made on ρ.

Also the idea applied in the conditional convex case [6] can not be used here, since the map $X \to E_{\mathbb{P}}[\rho(X)]$ there adopted preserves convexity but not quasiconvexity.

Then our method is to apply an approximation argument and the choice of approximating $\rho(\cdot)$ by

$$\rho^{\Gamma}(\cdot) = \sum_{A \in \Gamma} \rho_A(\cdot) \mathbf{1}_A,$$

is forced by the need to preserve quasiconvexity. Here Γ is any finite partition of

Ω and $\rho_A(X) = \sup_A \rho(X)$. Let

$$H(X) = \sup_{Q \in E^* \cap \mathcal{P}} \inf_{\xi \in E} \{\rho(\xi) \mid E_Q[\xi|\mathcal{G}] =_Q E_Q[X|\mathcal{G}]\}$$

$$H^{\Gamma}(X) = \sup_{Q \in E^* \cap \mathcal{P}} \inf_{\xi \in E} \{\rho^{\Gamma}(\xi) \mid E_Q[\xi|\mathcal{G}] =_Q E_Q[X|\mathcal{G}]\}.$$

I The first step is to prove that:

$$\pi^{\Gamma}(X) = H^{\Gamma}(X)$$

This is based on the representation of the *real valued* quasiconvex map ρ_A. Therefore, the assumptions (MON), (REG), (QCO) and (LSC) on ρ are here all needed.

II Then it is a simple matter to deduce $\rho(X) = \inf_{\Gamma} \rho^{\Gamma}(X) = \inf_{\Gamma} H^{\Gamma}(X)$, where the inf is taken with respect to all finite partitions.

III The last step, i.e. proving that $\inf_{\Gamma} H^{\Gamma}(X) = H(X)$, is more delicate. It can be shown easily that is possible to approximate $H(X)$ with $R(E_{Q_\varepsilon}[-X|\mathcal{G}], Q_\varepsilon)$ on a set A_ε of probability arbitrarily close to 1. However, we need the following *uniform* approximation: for any $\varepsilon > 0$ there exists $Q_\varepsilon \in E^* \cap \mathcal{P}$ such that for any finite partition Γ we have $H^{\Gamma}(X) - R(E_{Q_\varepsilon}[-X|\mathcal{G}], Q_\varepsilon) < \varepsilon$ on the same set A_ε. This key approximation result shows that the element Q_ε does not depend on the partition and allows us to conclude the proof.

The module approach [16]

We now replicate and empower the same results given in the previous paragraph using the L^0-module technology. The proof of our result is based on a version of the hyperplane separation theorem and not on some approximation or scalarization arguments, as it happened in the vector space setting. By carefully analyzing the proof in [16] one may appreciate many similarities with the original demonstration in the static setting by Volle [19]. One key difference with [19], in addition to the conditional setting, is the continuity assumption needed to obtain the representation. We choose to work, as in [4], with evenly quasiconvex functions, i.e. function having evenly convex lower level sets. This is an assumption weaker than quasiconvexity and lower semicontinuity. The notion of conditional evenly convexity has been studied in [13], matching the characterization originally given by Fenchel.

DEFINITION 8. – *Let $\mathcal{C}$ be a subset of E.*

(CSet) *$\mathcal{C}$ has the countable concatenation property if for every countable partition $\{A_n\}_n \subseteq \mathcal{G}$ and for every countable collection of elements $\{X_n\}_n \subset \mathcal{C}$ we have $\sum_n \mathbf{1}_{A_n} X_n \in \mathcal{C}$.*

We notice that an arbitrary set $\mathcal{C} \subset E$ may present some components which degenerate to the entire module. Basically it might occur that for some $A \in \mathcal{G}$, $\mathcal{C}\mathbf{1}_A = E\mathbf{1}_A$, i.e., for each $\xi \in E$ there exists $\eta \in \mathcal{C}$ such that $\eta\mathbf{1}_A = \xi\mathbf{1}_A$. In this case there are no chances to guarantee a separation on the set Ω as for the results given in [8]. Thus we need to determine the maximal $\mathcal{G}$-measurable set on which $\mathcal{C}$ reduces to E. The existence of the maximal element has been proved in [13] and the following definition is well posed.

DEFINITION 9. – *Fix a set $\mathcal{C} \subseteq E$ and $\mathcal{A}(\mathcal{C}) = \{A \in \mathcal{G} | \mathcal{C}\mathbf{1}_A = E\mathbf{1}_A\}$. We denote with $A_{\mathcal{C}}$ the $\mathcal{G}$-measurable maximal element of the class $\mathcal{A}(\mathcal{C})$ and with $D_{\mathcal{C}}$ the (P-a.s. unique) complement of $A_{\mathcal{C}}$. Hence $\mathcal{C}\mathbf{1}_{A_{\mathcal{C}}} = E\mathbf{1}_{A_{\mathcal{C}}}$.*

DEFINITION 10. – *Let $\mathcal{C}$ be a subset of E. We will say that*

(i) $X \in E$ is outside $\mathcal{C}$ *if $\mathbf{1}_A\{X\} \cap \mathbf{1}_A\mathcal{C} = \emptyset$ for every $A \in \mathcal{G}$ with $A \subseteq D_{\mathcal{C}}$ and $\mathbb{P}(A) > 0$.*

(iii) *$\mathcal{C}$ is conditional evenly convex if $\mathcal{C}$ satisfies (CSet) and for every X outside $\mathcal{C}$ there exists a $\mu \in E^*$ such that*

$$\mu(X) > \mu(\xi) \text{ on } D_{\mathcal{C}}, \ \forall\, \xi \in \mathcal{C}.$$

In [13] it is showed that any conditional evenly convex set is also L^0-convex and it can be characterized as intersection of half spaces.

DEFINITION 11. – *A map $\rho : E \to \bar{L}^0(\mathcal{G})$ is*

(EQC) *conditionally evenly quasiconvex if $U_Y = \{\xi \in E \mid \rho(\xi) \le Y\}$ are conditionally evenly convex for every $Y \in L^0(\mathcal{G})$.*

The following Theorem is the module counterpart of Theorem 2. One may appreciate the pretty weaker assumptions that allow the statement and the proof will be a natural application of functional analysis more that a mere matter of hard analysis as for the vector space approach.

THEOREM 3. – *If $\rho : E \to \bar{L}^0(\mathcal{G})$ is (REG), (EQC) then*

$$\rho(X) = \sup_{\mu \in E^*} \mathcal{R}(\mu(X), \mu), \tag{11}$$

where for $Y \in L^0(\mathcal{G})$ *and* μ,

$$\mathcal{R}(Y,\mu) := \inf_{\xi\in E}\{\rho(\xi) \mid \mu(\xi) \geq Y\}. \tag{12}$$

Sketch of the proof of Theorem 3

We here propose a non-rigorous simplification of the proof given in [13]. We defer to the original paper for all the technical details. Here we suppose for simplicity that $\rho(0) = 0$.

First there might exist a set $A \in \mathcal{G}$ on which the map ρ is constant, in the sense that $\rho(\xi)\mathbf{1}_A = \rho(\eta)\mathbf{1}_A$ for every $\xi,\eta \in E$. For this reason we introduce

$$\mathcal{A} := \{B \in \mathcal{G} \mid \rho(\xi)\mathbf{1}_B = \rho(\eta)\mathbf{1}_B \ \forall\, \xi,\eta \in L^p_{\mathcal{G}}(\mathcal{F})\}.$$

There exist two maximal sets $A \in \mathcal{G}$ and $A^{\vdash} \in \mathcal{G}$ for which $P(A \cap A^{\vdash}) = 0$, $P(A \cup A^{\vdash}) = 1$ and

$$\begin{aligned} \rho(\xi) = \rho(\eta) &\quad \text{on } A \text{ for every } \xi,\eta \in E,\\ \rho(\zeta_1) < \rho(\zeta_2) &\quad \text{on } A^{\vdash} \text{ for some } \zeta_1,\zeta_2 \in E. \end{aligned}$$

Fix $X \in E$ and $G = \{\rho(X) < +\infty\}$. For every $\varepsilon \in L^0_{++}(\mathcal{G})$ we set

$$Y_\varepsilon =: \rho(X)\mathbf{1}_A + (\rho(X) - \varepsilon)\mathbf{1}_{G\cap A^{\vdash}} + \varepsilon\mathbf{1}_{G^C\cap A^{\vdash}}$$

and for every $\varepsilon \in L^0(\mathcal{G})_{++}$ we set the evenly convex set

$$\mathcal{C}_\varepsilon =: \{\xi \in E \mid \rho(\xi) \leq Y_\varepsilon\} \neq \emptyset$$

This may be separated from X by $\mu_\varepsilon \in E^*$ i.e.

$$\mu_\varepsilon(X) > \mu_\varepsilon(\xi) \quad \text{ on } D_{\mathcal{C}_\varepsilon},\ \forall\, \xi \in \mathcal{C}_\varepsilon.$$

Since

$$\{\xi \in E \mid \mu_\varepsilon(X)\mathbf{1}_{A^{\vdash}} \leq \mu_\varepsilon(\xi)\mathbf{1}_{A^{\vdash}}\} \subseteq \{\xi \in E \mid \rho(\xi) > (\rho(X) - \varepsilon)\mathbf{1}_G + \varepsilon\mathbf{1}_{G^C} \text{ on } A^{\vdash}\},$$

one can easily deduce that on the set $A^{\vdash}$

$$\begin{aligned} \rho(X)\mathbf{1}_{A^{\vdash}} &\geq \inf_{\xi\in E}\{\rho(\xi)\mathbf{1}_{A^{\vdash}} \mid \mu_\varepsilon(\xi) \geq \mu_\varepsilon(X)\}\\ &\geq \inf_{\xi\in E}\{\rho(\xi)\mathbf{1}_{A^{\vdash}} \mid \rho(\xi) > (\rho(x) - \varepsilon)\mathbf{1}_G + \varepsilon\mathbf{1}_{G^C} \text{ on } A^{\vdash}\} &(13)\\ &\geq (\rho(X) - \varepsilon)\mathbf{1}_{G\cap A^{\vdash}} + \varepsilon\mathbf{1}_{G^C\cap A^{\vdash}} &(14) \end{aligned}$$

The representation (11) follows by taking ε arbitrary small on $G \cap A^{\vdash}$ and arbitrary big on $G^C \cap A^{\vdash}$ and observing that on A the representation trivially holds true.

Evidences of the power of the L^0-module approach: a Complete Duality result.

A *complete duality* for real valued quasiconvex functionals has been firstly established in [4]: the idea is to prove a *one to one* relationship between quasiconvex monotone functionals ρ and the function R in the dual representation. Obviously R will be unique only in an opportune class of maps satisfying certain properties. In Decision Theory the function R can be interpreted as the decision maker's index of uncertainty aversion: the uniquesness of R becomes crucial (see [4]) if we want to guarantee a robust dual representation of ρ characterized in terms of the unique R. In mathematical terms

DEFINITION 12. – *Let $\mathcal{P}$ be the set of probability measures on $(\Omega, \mathcal{F})$. There exists a complete duality between a class $\mathcal{R}$ of maps*

$$R : \mathbb{R} \times \mathcal{P} \to \overline{\mathbb{R}}$$

and a class $\mathcal{L}$ of functions

$$\rho : E \to \overline{\mathbb{R}}$$

if for every $\rho \in \mathcal{L}$ the only $R \in \mathcal{R}$ such that

$$\rho(X) = \sup_{Q \in \mathcal{P}} R(E_Q[-X], Q)$$

is given by

$$R(m, Q) = \inf_{\xi \in L_{\mathcal{F}}} \{\rho(\xi) \mid E_Q[-\xi] \geq m\};$$

and conversely for every $R \in \mathcal{R}$ there is a unique $\rho \in \mathcal{L}$ satisfying the above equations.

To the best of our knowledge the following Theorem is the first establishment of such a Complete Duality in the conditional framework for modules of the L^p-type, i.e. $E = L^p_{\mathcal{G}}(\mathcal{F})$. (see [12] for the complete proof)

THEOREM 4 (Frittelli-Maggis [12]). – *The map $\rho : L^p_{\mathcal{G}}(\mathcal{F}) \to L^0(\mathcal{G})$ satisfies (REG), (MON), (EVQ) if and only if*

$$\rho(X) = \sup_{Q \in \mathcal{P}^q} R\big(E_Q[-X|\mathcal{G}], Q\big) \tag{15}$$

where

$$R(Y, Q) = \inf_{\xi \in L^p_{\mathcal{G}}(\mathcal{F})} \big\{\rho(\xi) \mid E_Q[-\xi|\mathcal{G}] = Y\big\}$$

is unique in the class $\mathcal{M}(L^0(\mathcal{G}) \times \mathcal{P}^q)$.

DEFINITION 13. – *The class* $\mathcal{M}(L^0(\mathcal{G}) \times \mathcal{P}^q)$ *is composed by maps* $K : L^0(\mathcal{G}) \times \mathcal{P}^q \to \bar{L}^0(\mathcal{G})$ *s.t.*

(i) K *is increasing in the first component.*

(ii) $K(Y\mathbf{1}_A, Q)\mathbf{1}_A = K(Y, Q)\mathbf{1}_A$ *for every* $A \in \mathcal{G}$ *and* $\left(Y, \dfrac{dQ}{d\mathbb{P}}\right) \in \Sigma$.

(iii) $\inf_{Y \in L^0(\mathcal{G})} K(Y, Q) = \inf_{Y \in L^0(\mathcal{G})} K(Y, Q')$ *for every* $Q, Q' \in \mathcal{P}^q$.

(iv) K *is* $\diamond$*-evenly* $L^0(\mathcal{G})$*-quasiconcave: for every* $(Y^*, Q^*) \in L^0(\mathcal{G}) \times \mathcal{P}^q$, $A \in \mathcal{G}$ *and* $\alpha \in L^0(\mathcal{G})$ *such that* $K(Y^*, Q^*) < \alpha$ *on* A, *there exists* $(S^*, X^*) \in L^0_{++}(\mathcal{G}) \times L^p_{\mathcal{G}}(\mathcal{F})$ *with*

$$Y^*S^* + E\left[X^* \frac{dQ^*}{d\mathbb{P}} \mid \mathcal{G}\right] < YS^* + E\left[X^* \frac{dQ}{d\mathbb{P}} \mid \mathcal{G}\right] \text{ on } A$$

for every (Y, Q) *such that* $K(Y, Q) \geq \alpha$ *on* A.

(v) *the set* $\mathcal{K}(X) = \left\{ K(E[X \dfrac{dQ}{d\mathbb{P}} | \mathcal{G}], Q) \mid Q \in \mathcal{P}^q \right\}$ *is upward directed for every* $X \in L^p_{\mathcal{G}}(\mathcal{F})$.

(vi) $K(Y, Q_1)\mathbf{1}_A = K(Y, Q_2)\mathbf{1}_A$, *if* $\dfrac{dQ_1}{d\mathbb{P}}\mathbf{1}_A = \dfrac{dQ_2}{d\mathbb{P}}\mathbf{1}_A$, $Q_i \in \mathcal{P}^q$, *and* $A \in \mathcal{G}$.

REMARK 2. – In view of the establishment of a complete duality one could argue that the vector space approach should be set aside in favor of the module approach. This is not completely appropriate: as shown by Frittelli and Maggis [10] the class of quasiconcave maps given by Conditional Certainty Equivalents leads to a vector space approach using Orlicz spaces. It does not seem reasonable in such an example to extend the conditional maps to the L^0-module set up.

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