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Antonio Tortora

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The Power Mapping as Endomorphism of a Group

ANTONIO TORTORA

Abstract. – *Let $n \neq 0, 1$ be an integer. A group G is said to be n-abelian if the mapping $f_n : x \mapsto x^n$ is an endomorphism of G. Then $(xy)^n = x^n y^n$ for all $x, y \in G$, from which it follows $[x^n, y] = [x, y]^n = [x, y^n]$. In this paper we investigate groups G such that f_n is a monomorphism or an epimorphism of G. We also deal with the connections between n-abelian groups and groups satisfying the identity $[x^n, y] = [x, y]^n$ or $[x^n, y] = [x, y^n]$. Finally, we provide an arithmetic description of the set of all integers n such that f_n is an automorphism of a given group G.*

1. – Introduction

Let $n \neq 0, 1$ be an integer. A group G is said to be *n-abelian* if the mapping $f_n : x \mapsto x^n$ is an endomorphism of G. Then $(xy)^n = x^n y^n$ for all $x, y \in G$, from which it follows $[x^n, y] = [x, y]^n = [x, y^n]$. Clearly, every group of finite exponent dividing n or $n - 1$ is n-abelian. It is also easy to see that a group G is n-abelian if and only if it is $(1 - n)$-abelian. The structure of n-abelian groups has been described in [2] and [1] (see [7] for an account). If $n \neq 0, 1$ and G is an n-abelian group, then the quotient group $G/Z(G)$ has finite exponent dividing $n(n - 1)$. This implies that every torsion-free n-abelian group is abelian. Of course 2-abelian groups are precisely all abelian groups, whereas 3-abelian groups are all 2-Engel groups (i.e. groups satisfying the identity $[x, y, y] = 1$) with commutator subgroup of exponent 3 (Levi [13], Kappe and Morse [10]).

It is well-known that torsion abelian or nilpotent groups are direct products of their primary components. Our first result shows that a similar factorization is also valid for torsion n-abelian groups. Here and throughout the paper, except the last section, we assume that $n \neq 0, 1$ is an integer, π_n and π_{n-1} are the sets of all primes dividing n and $n - 1$, respectively, and $\mathbb{P}$ is the set of all primes. Moreover, we set $\pi'_{n-1} = \mathbb{P} \setminus \pi_{n-1}$ and $\pi'_{n(n-1)} = \mathbb{P} \setminus (\pi_n \cup \pi_{n-1})$.

THEOREM 1.1 (Baer [2]). – *The elements of finite order in any n-abelian group form a subgroup which is the direct product of a π_n-group, a π_{n-1}-group and an abelian $\pi'_{n(n-1)}$-group.*

For finite groups a stronger result holds: a finite group is n-abelian if and only if it is a homomorphic image of a subgroup of the direct product of a finite abelian group, a finite group of exponent dividing n and a finite group of exponent dividing $n-1$ (see [1]). This is a consequence (but not immediate!) of the following characterization of n-abelian groups.

THEOREM 1.2 (Alperin [1]). – *A group is n-abelian if and only if it is a homomorphic image of a subgroup of the direct product of an abelian group, a group of exponent dividing n and a group of exponent dividing $n-1$.*

In this paper we are mainly concerned with some questions related to n-abelian groups. More precisely, in Section 2, we investigate groups G such that f_n is either a monomorphism or an epimorphism of G while, in Section 3, we deal with the connections between n-abelian groups and groups satisfying the identity $[x^n, y] = [x, y]^n$ or $[x^n, y] = [x, y^n]$. Finally, in Section 4, we provide an arithmetic description of the set of all integers n such that f_n is an automorphism of a given group G.

2. – Some classes of n-abelian groups

In this section, following [8], we denote by $\mathfrak{B}_n$ and $\mathfrak{C}_n$ the classes of all groups G for which $f_n : x \mapsto x^n$ is a monomorphism and an epimorphism of G, respectively. Since $\mathfrak{B}_{-1} = \mathfrak{C}_{-1}$ is the class of all abelian groups, we may assume $|n| > 1$. Then, $G \in \mathfrak{B}_n$ if and only if G is an n-abelian group without elements of order dividing n. Similarly, $G \in \mathfrak{C}_n$ if and only if G is n-abelian and for any $g \in G$ there exists an element $x \in G$ such that $g = x^n$. It follows that, if $G \in \mathfrak{B}_n \cup \mathfrak{C}_n$, then $G^{n-1} \leq Z(G)$ and G is $(n-1)$-abelian [8, Proposition 2.2]. Hence, G' has finite exponent dividing $n-1$. We also set $\mathfrak{A}_n = \mathfrak{B}_n \cap \mathfrak{C}_n$. This class has been studied in [16], [18] and [15]. Of course, groups of exponent dividing $n-1$ are in $\mathfrak{A}_n$.

For all integers $n \neq 0$, every divisible abelian group is in $\mathfrak{C}_n$. In particular, the additive group $\mathbb{Q}$ of rational numbers is in $\mathfrak{A}_n$, as well as every Prüfer group $\mathbb{Z}(p^\infty)$, with $\gcd(p, n) = 1$. The class $\mathfrak{B}_n$ is subgroup closed, but the class $\mathfrak{C}_n$ is not: in fact the group $\mathbb{Z}$ of all integers is not in $\mathfrak{C}_n$. The class $\mathfrak{C}_n$ is quotient closed, but the class $\mathfrak{B}_n$ is not: for example $\mathbb{Q}/\mathbb{Z}$ is not in $\mathfrak{B}_n$. Each of these classes is closed under forming direct products of its members. However, they are both not closed under extensions. For example, let G be the wreath product of a cyclic group of order p by $\mathbb{Z}(p^\infty)$. Then G is an extension of groups in $\mathfrak{A}_n$, for all integers n with $\gcd(p, n) = 1$. But $Z(G)$ is trivial and so G is not n-abelian when $n \neq 0, 1$.

Applying Theorem 1.1, we can characterize torsion groups in $\mathfrak{B}_n \cup \mathfrak{C}_n$, as follows.

THEOREM 2.1 (Delizia, Tortora [8]). – *Let G be a torsion group. Then:*

(*i*) *$G \in \mathfrak{B}_n$ if and only if $G = A \times B$ where A is an n-abelian π_{n-1}-group and B is an abelian $\pi'_{n(n-1)}$-group;*
(*ii*) *$G \in \mathfrak{C}_n$ if and only if $G = A \times B$ where A is an n-abelian π_{n-1}-group and $B = B^n$ is an abelian π'_{n-1}-group.*

If $\mathfrak{T}$ denotes the class of all torsion groups, we deduce by Theorem 2.1 that

$$(*) \qquad \mathfrak{A}_n \cap \mathfrak{T} = \mathfrak{B}_n \cap \mathfrak{T} \subseteq \mathfrak{C}_n \cap \mathfrak{T}$$

for all integers n, where the inclusion can be proper: for each prime p, the group $\mathbb{Z}(p^\infty)$ is in $\mathfrak{C}_p \setminus \mathfrak{B}_p$. Nevertheless, for groups of finite exponent we have:

PROPOSITION 2.2 (Delizia, Tortora [8]). – *Let $G \in \mathfrak{C}_n$ be a group of finite exponent. Then $G \in \mathfrak{A}_n$.*

Now we use Theorem 2.1 to obtain characterizations of groups in $\mathfrak{B}_n$ and in $\mathfrak{C}_n$.

THEOREM 2.3 (Delizia, Tortora [8]). – *Let G be a group. Then $G \in \mathfrak{B}_n$ if and only if G is isomorphic to a subgroup of the direct product of an n-abelian π_{n-1}-group by an abelian group without elements of order dividing n.*

PROOF. – Assume $G \in \mathfrak{B}_n$. Let V be a maximal torsion-free subgroup of $Z(G)$ and let W/V be the subgroup consisting of all π_n-elements of $Z(G)/V$. Notice that G/W is torsion because so are $G/Z(G)$ and $Z(G)/V$. Furthermore, if $x \in W \cap G'$, then there exists a π_n-number m such that $x^m \in V$. As $\gcd(m, n-1) = 1$ and $x^{n-1} = 1$, we get $x = 1$. So, W has trivial intersection with G'.

Let $x \in G$. First, suppose $(xW)^n = W$. Then $x^n V \in W/V$ and $x^{ns}V = V$ for some π_n-number s. But $x^{n-1} \in Z(G)$, so that $x \in Z(G)$. Thus $xV \in Z(G)/V$ is a π_n-element. Therefore $x \in W$ and G/W has no elements of order dividing n, that is $G/W \in \mathfrak{B}_n$. Suppose now $(xG')^n = G'$. Then $x^n \in G'$, so $x^{n(n-1)} = 1$. This implies $x^{n-1} = 1$. Hence $x \in G'$ and $G/G' \in \mathfrak{B}_n$.

Finally, since G is isomorphic to a subgroup of $G/W \times G/G'$, the claim follows by (*i*) of Theorem 2.1.

The converse is clear. □

THEOREM 2.4 (Delizia, Tortora [8]). – *Let G be an n-abelian group and denote by T its torsion group. Then $G \in \mathfrak{C}_n$ if and only if $T = A \times B$, where A is a π_{n-1}-group, $B = B^n$ is an abelian π'_{n-1}-group, and G/T is a p-divisible abelian group for any prime p dividing n.*

PROOF. – Assume $G \in \mathfrak{C}_n$ and let p be a prime dividing n. Then $n = ap$ for some integer a and, for all $g \in G$, there exists $x \in G$ such that $g = x^n = (x^a)^p \in G^p$. This means that G/T is in $\mathfrak{C}_p$. The rest follows by (*ii*) of Theorem 2.1.

Conversely, given $g \in G$, we have $gT = x^nT$ for some $x \in G$: in fact, G/T is n-divisible. Thus $g^{-1}x^n \in T$ and, by $(*)$, there exists $y \in T$ such that $g^{-1}x^n = y^n$. Therefore $g = x^ny^{-n} = (xy^{-1})^n$, that is $G = G^n$. □

Notice that in Theorem 2.4 one cannot replace the hypothesis that G is n-abelian by the weaker hypothesis that A is n-abelian. For example, consider the wreath product G of the cyclic group of order 2 by $\mathbb{Q}$. The torsion part T of G is the base group, that is an infinite group of exponent 2. Moreover, $G/T = \mathbb{Q}$. Finally $Z(G) = 1$ and so G is not n-abelian for all $n \neq 0, 1$. In particular, $G \notin \mathfrak{C}_n$.

3. – n-Levi and n-Bell groups

In [9], a group G is said to be *n-Levi* if $[x^n, y] = [x, y]^n$ for all $x, y \in G$, and *n-Bell* if $[x^n, y] = [x, y^n]$ for all $x, y \in G$. Following [5], we also say that G is a *Levi group* (resp. *Bell group*) if it is n-Levi (resp. n-Bell) for some integer $n \neq 0, 1$. The class of n-Levi groups clearly includes all 2-Engel groups, for any integer n, and all groups with n-abelian normal closures. Consequently, n-abelian groups are examples of n-Levi groups. The converse is not true in general: there exist n-Levi groups which are not m-abelian for any $m \neq 0, 1$. For example, if G is a (non-abelian) torsion-free 2-Engel group, then G is n-Levi for any n. On the other hand, if G were m-abelian for some $m \neq 0, 1$, the identity $(xy)^m = x^my^m[y, x]^{m(m-1)/2}$ would imply $[x, y] = 1$ for all $x, y \in G$, which is impossible.

It is also obvious that any n-Levi group is n-Bell. Then, for a group G, each of the following conditions is a consequence of the previous one:

(*i*) *the normal closure x^G is n-abelian for all $x \in G$;*
(*ii*) *G is an n-Levi group;*
(*iii*) *G is an n-Bell group.*

If $n = 2$ it can be easily seen that these three conditions are equivalent, since each of them is equivalent to the 2-Engel condition. A similar result holds when $n = 3$.

THEOREM 3.1 (Kappe, Morse [10]). – *For a group G the following conditions are equivalent:*

(*i*) *x^G is 3-abelian for all $x \in G$;*
(*ii*) *G is 3-Levi;*
(*iii*) *G is 3-Bell;*
(*iv*) *G is 3-Engel (i.e. $[x, y, y, y] = 1$) and $[x, y, y]^3 = 1$ for all $x, y \in G$.*

In [11], Kappe and Morse proved that the equivalence is true even when $n = p$ is a prime and G is a metabelian p-group. However, in general, such conditions are not equivalent: Brandl and Kappe [4] constructed a metabelian 2-group in which the law $[x^4, y] = [x, y^4]$ does not imply $[x^4, y] = [x, y]^4$. An easier counterexample is then $SL(2, 5)$. It has a 6-Bell subgroup which is not 6-Levi and it is 30-Bell but not 30-Levi [9, Kappe]. We also point out that Delizia, Moravec and Nicotera showed in [6] that there exist n-Bell groups which are not m-Levi for any $m \neq 0, 1$. Therefore the class of Levi groups is properly contained in the class of Bell groups. Nevertheless, these two classes coincide if we consider only locally graded groups [6].

Recall that a group is locally graded if every non-trivial finitely generated subgroup has a non-trivial finite image. The class of locally graded groups is rather wide, since it includes locally (soluble-by-finite) groups, as well as residually finite groups. In the realm of locally graded groups, every torsion n-Bell groups is locally finite.

THEOREM 3.2 (Delizia, Moghaddam, Rhemtulla [5]). – *If G is a locally graded n-Bell group, then the elements of finite order in G form a locally finite subgroup. In particular, G is an extension of a locally finite group by a torsion-free nilpotent group of class at most* 2 (see also [3]).

In any group G the set $R_2(G) = \{x \in G : [x, y, y] = 1 \text{ for any } y \in G\}$ of all right 2-Engel elements of G is always a characteristic subgroup [12]. It plays the same role for n-Bell groups as the centre does for n-abelian groups.

THEOREM 3.3 (Kappe [9] and Brandl, Kappe [4]). – *Let G be an n-Bell group. Then $G/R_2(G)$ has finite exponent dividing $n(n-1)$.*

The question then arises whether $R_2(G)$ can be replaced by the second centre in Theorem 3.3. In fact, given an n-Bell group, the exponent of $G/Z_2(G)$ is always finite [5, Delizia, Moghaddam, Rhemtulla]. It divides $3n^2(n-1)^2/2$, and $3n(n-1)$ when the group is n-Levi [17, Tortora]. But, in this latter case, the bound is also the best possible [5]. Notice also that, if G is a torsion-free nilpotent group of class two, then G is an n-Levi group such that $G/Z(G)$ is torsion-free [17].

The next result shows that locally finite n-Bell groups can be represented similarly to torsion n-abelian groups (compare with Theorem 1.1).

THEOREM 3.4 (Brandl, Kappe [4] and Tortora [17]). – *Let G be a locally finite n-Bell group. Then $G = A \times B \times C$ where A is a π_n-group, B is a π_{n-1}-group and C is a 2-Engel $\pi'_{n(n-1)}$-group. Furthermore, $A^n \leq R_2(G)$ and $B^{n-1} \leq R_2(G)$*

As a consequence of Theorem 3.4 we obtain structural results about n-Bell groups for special values of n.

COROLLARY 3.5. – *Let G be an n-Bell group.*

(i) *If $n = 3$, then G is nilpotent of class at most* 4 (Kappe, Morse [10]).
(ii) *If $n = 4$, then G is locally nilpotent* (Brandl, Kappe [4]).

COROLLARY 3.6 (Tortora [17]). – *Let G be a locally graded n-Bell group.*

(i) *If $|n|$ and $|n-1|$ are both prime powers, then G is locally nilpotent.*
(ii) *If either $|n|$ or $|n-1|$ is equal to $2^a p^b$ where p is a prime and a, b are non-negative integers, then G is locally soluble.*

4. – An arithmetic approach

We begin this section with some information on the sets $\mathbb{E}(G)$, $\mathbb{L}(G)$ and $\mathbb{B}(G)$ of all integers n for which a given group G is n-abelian, n-Levi or n-Bell respectively, i.e.:

$$\mathbb{E}(G) = \{n \in \mathbb{Z} \mid (xy)^n = x^n y^n \text{ for all } x, y \in G\},$$

$$\mathbb{L}(G) = \{n \in \mathbb{Z} \mid [x^n, y] = [x, y]^n \text{ for all } x, y \in G\},$$

$$\mathbb{B}(G) = \{n \in \mathbb{Z} \mid [x^n, y] = [x, y^n] \text{ for all } x, y \in G\}.$$

These subsets of $\mathbb{Z}$ are multiplicative semigroups containing 0 and 1. It is also easy to see that if n belongs to one of them, then so does $1 - n$.

The set $\mathbb{E}(G)$ is called the *exponent semigroup* of G. An arithmetic characterization of $\mathbb{E}(G)$ for an arbitrary group G was obtained by Levi ([14], see also [9]). Following [9], we have that $\mathbb{E}(G)$ is either $\{0, 1\}$ or a *Levi system*, that is a subset W of $\mathbb{Z}$ satisfying the following conditions (here $[n]_w$ denote the residue class of n modulo w):

(i) $n, m \in W$ implies $nm \in W$;
(ii) $n \in W$ implies $1 - n \in W$;
(iii) $0 \in W$;
(iv) there exists $w \in W, w > 0$, such that for all $n \in W$ we have $n^2 \equiv n$ (mod w) and $[n]_w \subseteq W$;
(v) $[n]_w, [n+1]_w \subseteq W$ implies $[n]_w = [0]_w$.

In other words, $\mathbb{E}(G)$ is either $\{0, 1\}$ or $\mathbb{Z}$ or a set of residue classes modulo some integers depending on G. More precisely, let $q_1, q_2, \dots, q_t$ be integers with $q_i > 1$ and $\gcd(q_i, q_j) = 1$ for $i \neq j$. Let $B(q_1, q_2, \dots, q_t)$ be the subset of $\mathbb{Z}$ which is the union of 2^t residue classes modulo q_i satisfying each a system of congruences $m \equiv \delta_i$ (mod q_i), where $i = 1, \dots, t$ and $\delta_i \in \{0, 1\}$. Then $\mathbb{E}(G) = \{0, 1\}$, or $\mathbb{Z}$, or $B(q_1, \dots, q_t)$ with $q_i > 2$ (see [9]).

Concerning $\mathbb{L}(G)$ and $\mathbb{B}(G)$, they were introduced by Kappe in [9] and are called respectively the *Levi semigroup* and the *Bell semigroup* corresponding to G. Surprisingly, their characterization is the same as the one that we have for $\mathbb{E}(G)$.

THEOREM 4.1 (Kappe [9]). – *Let W be a subset of $\mathbb{Z}$. Then the following statements are equivalent:*

(*i*) $W = \mathbb{L}(H)$ *for some group* H*;*
(*ii*) $W = \mathbb{B}(K)$ *for some group* K*;*
(*iii*) $W = \{0, 1\}$ *or a Levi system;*
(*iv*) $W = \{0, 1\}, \mathbb{Z}$ *or* $B(q_1, \dots, q_t)$ *with* $q_i > 2$.

Following [8], with $f_n : x \in G \mapsto x^n \in G$, now we introduce the set

$$\mathbb{A}(G) = \{n \in \mathbb{Z} : f_n \in \mathrm{Aut}(G)\}.$$

This is a subsemigroup of $\mathbb{E}(G)$ containing 1. Obviously $0 \in \mathbb{A}(G)$ if and only if $G = \{1\}$; in that case $\mathbb{A}(G) = \mathbb{Z}$. We may therefore assume $G \neq \{1\}$ in the sequel.

LEMMA 4.2 (Delizia, Tortora [8]). – *Let G be a group and suppose that $\mathbb{A}(G)$ satisfies one of the following conditions:*

(*i*) $2 \in \mathbb{A}(G)$*;*
(*ii*) $3 \in \mathbb{A}(G)$*;*
(*iii*) $n \in \mathbb{A}(G)$ *and* $-n \in \mathbb{A}(G)$ *for some* $n \neq 0$*;*
(*iv*) $n \in \mathbb{A}(G)$ *and* $m \in \mathbb{A}(G)$ *with* $\gcd(n-1, m-1) \leq 2$.

Then G is abelian.

Notice that $\mathbb{E}(G) = \mathbb{Z}$ if and only if G is abelian. On the other hand, it is easy to show that $\mathbb{A}(G) = \mathbb{Z} \setminus \{0\}$ if and only if G is isomorphic to a direct sum of copies of $\mathbb{Q}$. However, G is abelian if and only if $-1 \in \mathbb{A}(G)$. So, in that case, $-n \in \mathbb{A}(G)$ for all $n \in \mathbb{A}(G)$. Furthermore, if G is abelian, $n \in \mathbb{A}(G)$ if and only if $p \in \mathbb{A}(G)$ for all primes p dividing n. The semigroup $\mathbb{A}(G)$ is thus generated by -1 and all primes in $\mathbb{A}(G)$. Therefore, if $\pi(G)$ denotes the set of all primes involved in the decomposition of orders of elements of G, and $\delta(G)$ denotes the set of all primes p such that G is p-divisible, one can easily realize that:

THEOREM 4.3 (Delizia, Tortora [8]). – *Let G be an abelian group. Then $\mathbb{A}(G)$ is the multiplicative subsemigroup of $\mathbb{Z}$ generated by $(\delta(G) \setminus \pi(G)) \cup \{-1\}$. In particular:*

(*i*) *if G is torsion, then $\mathbb{A}(G)$ is generated by* $(\mathbb{P} \setminus \pi(G)) \cup \{-1\}$*;*
(*ii*) *if G is torsion-free, then $\mathbb{A}(G)$ is generated by* $\delta(G) \cup \{-1\}$.

By Theorem 4.3, $\mathbb{A}(\mathbb{Z}) = \{-1, 1\} = \mathbb{A}(\mathbb{Q}/\mathbb{Z})$ and, if G is a torsion abelian group, then $\mathbb{A}(G) = \mathbb{Z} \setminus \cup_{p\in\pi(G)} p\mathbb{Z}$. We also point out that, given a set π of primes, there always exists a torsion-free abelian group G such that $\mathbb{A}(G)$ is the multiplicative subsemigroup of $\mathbb{Z}$ generated by $\pi \cup \{-1\}$. For example, the additive group of all rational numbers with a π-number as denominator has the above properties.

Finally, let G be a non-abelian group and suppose that G is n-abelian for some $n \neq 0, 1$, so $\{0, 1\} \subset \mathbb{E}(G) \subset \mathbb{Z}$. By [14] (see also [9]), the set $\mathbb{E}_0(G)$ of all integers n such that G is n-abelian and $G^n \leq Z(G)$ is an ideal of $\mathbb{Z}$. Let $\mathbb{E}_0(G) = w\mathbb{Z}$. Thus $w > 2$, G is w-abelian, $G^w \leq Z(G)$ and w is the least positive integer with such properties. Moreover, if $w = q_1q_2 \dots q_t$ is a factorization of w (with $t \geq 1$, $q_i > 2$ for all $i = 1, 2, \dots, t$ and $\gcd(q_i, q_j) = 1$ for $i \neq j$), then $\mathbb{E}(G) = B(q_1, q_2, \dots, q_t)$. Since $n, n-1 \in \mathbb{E}(G)$ for all $n \in \mathbb{A}(G)$, we have $n - 1 \equiv 0 \pmod{w}$. Hence:

THEOREM 4.4 (Delizia, Tortora [8]). – *For each group G, $\mathbb{A}(G) \subseteq [1]_w$.*

In general, the equality does not hold in the theorem above. For example, consider a non-abelian group H of exponent 3, and the direct product G of H by the cyclic group of order 4. Clearly, $\mathbb{E}_0(G) = 3\mathbb{Z}$. But, $4 \notin \mathbb{A}(G)$ since G has an element of order 4. Thus $\mathbb{A}(G) \neq [1]_3$.

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Dipartimento di Matematica, Università di Salerno
Via Giovanni Paolo II, 132 - 84084 - Fisciano (SA), Italy
E-mail: antortora@unisa.it

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