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A Note on Some Nonlinear Fourth Order Differential Equations

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Abstract. – *For a family of fourth order semilinear ordinary differential equations we discuss some fundamental issues, such as global continuation of solutions and their qualitative behavior. The note is the summary of a communication given at the XIX Congress of U.M.I. (Bologna - September* 12-17, 2011*).*

1. – Introduction

The present note is the summary of a communication held at the XIX Congress of U.M.I. (Bologna - September 12-17, 2011). It contains results proved in [5] complemented with some remarks and comments.

Consider the following fourth order nonlinear equation

$$w''''(s) + kw''(s) + f(w(s)) = 0 \qquad (s \in \boldsymbol{R}) \tag{1}$$

where $k \in \boldsymbol{R}$ and f is a locally Lipschitz function. When k is negative, (1) is known as the extended Fisher-Kolmogorov equation, whereas when k is positive it is usually referred to as Swift-Hohenberg equation.

Equation (1) serves as model in several contexts such as the pattern formation in many physical, chemical and biological systems or the spreading of deformation in a strut confined by an elastic foundation, see [7, 22]. In particular, if $f(t) = (t+1)^+ - 1$ and $k = c^2$, where c is the wave speed, (1) describes traveling waves in a suspension bridge according to the model suggested by McKenna-Walter [20]. Here, w denotes the vertical deflection of the bridge and f represents the forces on the bridge composed by gravity and by the action of the suspension cables. The positive part in f reflects the fact that cables obey the Hooke's law, with a restoring force proportional to the displacement if they are stretched and no restoring force if they are compressed. Later, Chen-McKenna [10] suggest to replace the piecewise linear function with the nonlinearity $f(t) = e^t - 1$ sharing many qualitative properties with the original one but being analytic. See also [13, 14] for further comments on this model.

Our interest in (1) is also due to the fact that it arises when dealing with a suitable family of biharmonic coercive equations. For instance, when $k = -4$ and f is the smoothed nonlinearity in the suspension bridge model, equation (1) comes from a suitable transformation of the biharmonic pde

$$\Delta^2 u + e^u = \frac{1}{|x|^4} \qquad \text{in } \boldsymbol{R}^4 \setminus \{0\}, \tag{2}$$

namely a fourth order coercive nonautonomous version of the celebrated Gelfand problem [15, 16]. For other values of n and k further biharmonic equations arise, see Section 3. While not much is now about (2), its noncoercive versions have recently attracted much interest both in $\boldsymbol{R}^4$ (see [4, 9, 19]) and in $\boldsymbol{R}^n$ for $n \geq 5$ (see [1, 2, 3, 4, 6, 11, 12]). The study of (1), though lying in an apparently different context, may contribute to reach a better knowledge to (2).

Two fundamental issues related to (1) are global continuation of local solutions and their qualitative behavior. In Section 2 we collect some results concerning these topics. The problem of existence of homoclinics is also briefly discussed there. In Section 3 we show how the results given in Section 2 can be applied to the family of biharmonic coercive equations to which (2) belongs. In the last Section we describe the main tools applied in the proofs referring to [5] for the details.

2. – Global continuation of solutions and their qualitative behavior

First we discuss global continuation of solutions to (1). From [5] we know

THEOREM 1. – *Let $k \in \boldsymbol{R}$ and assume that f satisfies*

$$f \in \mathrm{Lip}_{\mathrm{loc}}(\boldsymbol{R}), \quad f(t)\,t > 0 \quad \textit{for every } t \in \boldsymbol{R} \setminus \{0\}. \tag{3}$$

(*i*) *If a local solution w to* (1) *blows up at some finite $R \in \boldsymbol{R}$, then*

$$\liminf_{s \to R} w(s) = -\infty \qquad \textit{and} \qquad \limsup_{s \to R} w(s) = +\infty\,. \tag{4}$$

(*ii*) *If f also satisfies*

$$\limsup_{t \to +\infty} \frac{f(t)}{t} < +\infty \qquad \textit{or} \qquad \limsup_{t \to -\infty} \frac{f(t)}{t} < +\infty, \tag{5}$$

then any local solution to (1) *exists for all $s \in \boldsymbol{R}$.*

In other words, from Theorem 1-(i) we infer that, under assumption (3), the only way that finite time blow up can occur is with wide oscillations of the solution. This clearly suggests a possible strategy to prove the second part of the statement: to get global continuation one only needs to rule out solutions blowing up in finite time with this kind of behavior. The proof of Theorem 1 is quite long and technical, see [5, Section 4] for the details. It is based on elementary calculus arguments but applied in a refined way. One of the crucial tools exploited is a suitable energy function, see our Section 4.1.

A few comments on (5) are in order. If both the conditions in (5) are satisfied then global existence follows from classical theory of ODEs. But (5) merely requires that f is "one-sided at most linear". Of course, (5) does not cover functions f (satisfying (3)) with uncontrolled behaviors at both $\pm\infty$. Nevertheless, if we exclude these cases, assumption (5) appears general enough to include many interesting models satisfying (3). In particular, (5) is satisfied if f is either concave or convex.

Let us now turn to assumption (3). One may easily find examples of functions showing that if it is violated global continuation may fail. For instance, if $f(t) = 24(t - t^5)$ then $w(s) = \tan s$ solves (1) for $k = -20$ and blows up in finite time. Here, (5) holds at $+\infty$ while (3) is not satisfied. See [5] for further examples.

A more difficult problem is to find examples showing that when (3) holds and (5) is violated, (4) really occurs. The following result is due to [13]

THEOREM 2. – *Fix any integer* $n \geq 5$ *and consider equation* (1) *with* $k = -\dfrac{n^2 - 4n + 8}{2} < 0$ *and*

$$f(t) = \left(\frac{n(n-4)}{2}\right)^2 t + |t|^{8/(n-4)} t. \tag{6}$$

There exists a solution to (1) *which is defined in a neighborhood of* $s = -\infty$ *and such that* (4) *holds for some finite* $R \in \mathbf{R}$.

The nonlinearity given in (6) satisfies (3) but not (5). As far as we are aware, this is the first explicit example where the finite time blow up with wide and thinning oscillations (4) occurs. In [13] the authors also showed that this phenomenon is typical of (at least) fourth order problems such as (1) since it does not occur in related lower order equations. The proof of Theorem 2 is achieved by studying some applications of Theorem 1 to suitable semilinear biharmonic problems at critical growth. This justifies the special form (6), see [13, Section 3.2].

Once global continuation of solutions to (1) has been discussed the natural subsequent step consists in studying the behavior of global solutions as $s \to \pm\infty$. Here we recall some statements about this topic. All the proofs can be found in [5]. If $k \geq 0$, then solutions to (1) have oscillations.

THEOREM 3. – *Let* $k \geq 0$ *and* f *satisfy* (3) *with* $\limsup_{t\to-\infty} f(t) < 0$ *and* $\liminf_{t\to+\infty} f(t) > 0$. *If* w *is a global solution to* (1), *then*

$$\liminf_{s\to+\infty} w(s) \leq 0 \leq \limsup_{s\to+\infty} w(s)\ ,$$

so that if $\lim_{s\to+\infty} w(s)$ *exists then*

$$\lim_{s\to+\infty} w(s) = 0 \ .$$

Furthermore, if $w \not\equiv 0$ *then* $w(s)$ *changes sign infinitely many times as* $s \to +\infty$. *Similar statements hold for* $s \to -\infty$.

Figure 1 shows that a completely different behavior may occur when $k<0$.

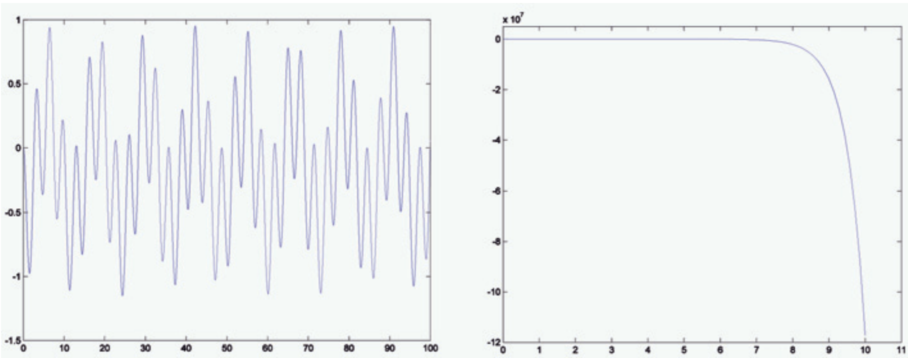

Fig. 1. – The plot of the solution to (1) with $f(t) = e^t - 1$ and $k = 4$ or $k = -4$ (right) corresponding to the initial values $w(0) = w'(0) = w'''(0) = 0$ and $w''(0) = -2$.

What is suggested by the figure can be proved rigorously. Indeed, we have

THEOREM 4. – *Let* $k<0$ *and assume that* f *satisfies* (3) *and*

$$\inf_{t\in \mathbf{R}} f(t) = -M > -\infty, \tag{7}$$

then there exists a global solution w *of* (1) *which is eventually negative, decreasing, and concave as* $s \to +\infty$; *in particular,*

$$\lim_{s\to+\infty} w(s) = -\infty \ .$$

If, instead of (7), f *satisfies*

$$\sup_{t\in \mathbf{R}} f(t) = M < +\infty. \tag{8}$$

Then there exists a global solution w *of* (1) *which is eventually positive, increasing, and convex as* $s \to +\infty$; *in particular,*

$$\lim_{s\to+\infty} w(s) = +\infty \ .$$

Similarly, there exist solutions having the above mentioned behaviors as $s \to -\infty$.

We point out that (7) (respectively (8)) implies the second (respectively the first) condition in (5) so that Theorem 1 states that all the solutions are global. We also emphasize that assumption (7) is essential in the previous statement. The next statement shows that if it is violated, then the solution is bounded from above.

THEOREM 5. – *Let $k<0$ and f satisfy* (3), (8) *and*

$$\lim_{t\to-\infty}\frac{f(t)}{t}=+\infty\,. \tag{9}$$

Then any solution w of (1) *is global and*

$$\sup_{s\in\boldsymbol{R}} w(s)=+\infty \quad \textit{and} \quad \inf_{s\in\boldsymbol{R}} w(s)>-\infty\ .$$

On the contrary, suppose that (8) *is replaced by* (7) *and* (9) *holds as $t\to+\infty$. Then any solution w of* (1) *is global and*

$$\sup_{s\in\boldsymbol{R}} w(s)<+\infty \quad \text{and} \quad \inf_{s\in\boldsymbol{R}} w(s)=-\infty\ .$$

Important global solutions are the so-called *homoclinics*. These are nontrivial solutions w such that

$$\lim_{s\to\pm\infty} w(s)=0.$$

In literature, one may find different definitions which also involve the derivatives of the solutions. For instance, by successive integrations of the equation it can be proved that if $f:\boldsymbol{R}\to\boldsymbol{R}$ is a continuous function such that $f(0)=0$ and w is a global solution to (1) such that $\lim_{s\to+\infty} w(s)=0$, then

$$\lim_{s\to+\infty} w'(s)=\lim_{s\to+\infty} w''(s)=\lim_{s\to+\infty} w'''(s)=\lim_{s\to+\infty} w''''(s)=0\ . \tag{10}$$

The same result also holds with $-\infty$ in place of $+\infty$. See e.g. [5, Proposition 11] for a proof.

Existence of homoclinics to (1) is a tricky problem (see [21, Problem 6.2]) and, as far we are aware, only partial results are known. We collect some of them in Theorem 6 below. Here and in the sequel we denote

$$F(t):=\int_0^t f(\tau)\,d\tau.$$

In particular, by (3) we see that $F(t)>0$ for all $t\neq 0$.

THEOREM 6. – *Let f satisfy* (3). *The following statements hold*

(*i*) *if $k\le 0$ equation* (1) *has no nontrivial bounded solutions. In particular, it has no homoclinic solution.*

(*ii*) *if* $0<k\leq 2$ *and* f *satisfies*

$$\frac{f(t)}{t}\geq 1 \qquad \forall t\neq 0\,, \tag{11}$$

then equation (1) *has no homoclinic solution.*

(*iii*) *If* f *is differentiable at* $t=0$ *with* $f'(0)=1$ *and* $\lim_{t\to-\infty}\frac{F(t)}{t^2}=0$, *then there exists a homoclinic solution to* (1) *for almost every* $k\in(0,2)$.

The proof of statement (*i*) can be obtained by repeating the proof of [22, Theorem 10.1.1] where the explicit nonlinearity $f(t)=e^t-1$ is considered. On the other hand, a very simple proof of the second part of statement (*i*) can be obtained by exploiting a suitable energy function, see our Section 4.1. The same energy function has been used to prove statement (*ii*) in [5, Section 11], see again Section 4.1. Statement (*iii*) has been proved in [24] by using a mountain-pass procedure. We note that an alternative approach in getting existence of homoclinics consists in studying a suitable constrained minimization problem. This has been developed in the work [23]. We refer the interested reader to [23, Sections 2 and 7]. Finally we mention that, when $f(t)=e^t-1$, a multiplicity result for homoclinics is obtained in [8] by means of a computer-assisted proof.

We conclude our short survey on homoclinic solutions with a couple of complementary results from [5]. See also [17, 18] for further results.

THEOREM 7. – *Let* $k>0$. *Assume that* f *satisfies* (3) *and it is differentiable at* $t=0$ *with* $f'(0)=1$. *The following statements hold*

(*i*) *if* w *is a homoclinic solution to* (1), *and if* $\{s_m\}_{m\geq 1}$ *denotes the increasing sequence of zeroes of* w *as* $s\to+\infty$, *then*

$$\liminf_{m\to+\infty}\,(s_{m+1}-s_m)\geq\frac{\pi\sqrt{k+\sqrt{k^2+12}}}{\sqrt{6}}\,. \tag{12}$$

A similar statement holds as $s\to-\infty$.

(*ii*) *If* $k<2$, *then any homoclinic solution* w *to* (1) *satisfies* $w\in H^2(\boldsymbol{R})$.

From Theorem 3 we know that any homoclinic solution changes sign infinitely many times as $s\to\pm\infty$ (cfr. Theorem 6-(*i*)). Theorem 7-(*i*) states that the distance between two consecutive zeroes of an homoclinic solution stays bounded away from zero as $s\to\pm\infty$. Finally, we note that the limit value $k=2$ comes from the dynamical system analysis, see Section 4.2.

3. – Related biharmonic Gelfand-type problems

Here we give an interpretation of the results stated in Section 2 in terms of suitable biharmonic equations. Let $k \in \boldsymbol{R}$ and consider the equation

$$(13)\quad \Delta^2 u - 2(n-4)\frac{x \cdot \nabla \Delta u}{|x|^2} + (n^2 - 6n + 12 + k)\frac{\Delta u}{|x|^2} - (n-2)\left[(n-2)^2 + k\right]\frac{x \cdot \nabla u}{|x|^4} + e^u = \frac{1}{|x|^4}$$

where $x \in \boldsymbol{R}^n \setminus \{0\}$ $(n \geq 2)$. For any $k \in \boldsymbol{R}$, (13) admits an explicit global radial solution which is given by $\overline{u}(x) = -4 \log |x|$. To see this, one may write (13) in its radial form, that is

$$u''''(r) + 6\,\frac{u'''(r)}{r} + (7+k)\,\frac{u''(r)}{r^2} + (1+k)\,\frac{u'(r)}{r^3} + e^{u(r)} = \frac{1}{r^4}\ ,$$

where $r = |x| \in (0, +\infty)$. Then, with the change of variables

$$s = \log r \qquad w(s) := u(e^s) + 4s \qquad s \in \boldsymbol{R}\ ,$$

one finds that $w = w(s)$ solves (1) with $f(t) = e^t - 1$ and the singular solution $\overline{u}(x) = -4 \log |x|$ to (13) corresponds to the trivial solution $w \equiv 0$.

For problem (13), Theorem 1 reads

COROLLARY 1. – *Let $k \in \boldsymbol{R}$ and B_R be the ball in $\boldsymbol{R}^n$ $(n \geq 2)$ with radius $0 < R < +\infty$ and center the origin. Then, any radial solution to* (13) *in $B_R \setminus \{0\}$ admits a radial extension to $\boldsymbol{R}^n \setminus \{0\}$. In particular, equation* (13) *in $B_R \setminus \{0\}$ subject to the boundary condition*

$$\lim_{|x| \to R} u(x) = \infty\ ,$$

admits no radial solution.

On the other hand, Theorem 6 reads

COROLLARY 2. – *Let $k \leq 0$ and let u be a radial solution to* (13). *If*

$$\lim_{|x| \to 0} (u(x) + 4 \log |x|) = 0 = \lim_{|x| \to +\infty} (u(x) + 4 \log |x|)\,,$$

then $u(x) \equiv -4 \log |x|$.

Let us consider some meaningful values of n and k in (13).

If $n = 4$, (13) becomes

$$\Delta^2 u + (4+k)\left(\frac{\Delta u}{|x|^2} - 2\,\frac{x \cdot \nabla u}{|x|^4}\right) + e^u = \frac{1}{|x|^4}\,, \qquad x \in \boldsymbol{R}^4 \setminus \{0\}.$$

Hence, if furthermore $k = -4$ we get equation (2).

If $n = 2$, (13) corresponds to the equation

$$\Delta^2 u + 4\,\frac{x \cdot \nabla \Delta u}{|x|^2} + (4+k)\,\frac{\Delta u}{|x|^2} + e^u = \frac{1}{|x|^4}\,, \qquad x \in \boldsymbol{R}^2 \setminus \{0\}.$$

Thus, by taking $k = -4$, the equation reduces to

$$\Delta^2 u + 4\,\frac{x \cdot \nabla \Delta u}{|x|^2} + e^u = \frac{1}{|x|^4}\,, \qquad x \in \boldsymbol{R}^2 \setminus \{0\}.$$

Finally, for any $n \geq 2$, taking $k = -(n^2 - 6n + 12) \in (-\infty, -3]$ we have

$$\Delta^2 u - 2(n-4)\left[\frac{x \cdot \nabla \Delta u}{|x|^2} + (n-2)\frac{x \cdot \nabla u}{|x|^4}\right] + e^u = \frac{1}{|x|^4}\,, \qquad x \in \boldsymbol{R}^n \setminus \{0\},$$

while taking $k = -(n-2)^2 \in (-\infty, 0]$ leads to

$$\Delta^2 u - 2(n-4)\left[\frac{x \cdot \nabla \Delta u}{|x|^2} + \frac{\Delta u}{|x|^2}\right] + e^u = \frac{1}{|x|^4}\,, \qquad x \in \boldsymbol{R}^n \setminus \{0\}.$$

4. – About the proofs

As already remarked, the proofs of the statements given in Section 2 are mainly based on elementary calculus arguments but applied in a refined way. We refer to [5] for the details. Nevertheless it is worth mentioning two fundamental tools which turn out to be crucial in most of the proofs. Namely, the exploitation of suitable energy functions and the analysis of the dynamical system associated to (1). The next two subsections are devoted to a description of these tools.

4.1 – *Energy functions*

To equation (1) we may associate the following three different energy functions each one characterized by suitable properties:

$$E(s) := \frac{1}{2}\,w''(s)^2 - \frac{k}{2}\,w'(s)^2 - F(w(s)),$$

$$\mathcal{E}(s) := \frac{1}{2}\, w''(s)^2 - \frac{k}{2}\, w'(s)^2 - w'(s)w'''(s) - F(w(s)) = E(s) - w'(s)w'''(s)$$

and

$$H(s) := w'(s)w''(s) - w(s)w'''(s) - kw(s)w'(s)$$

for any $s \in \boldsymbol{R}$.

$E(s)$ turns out to be constant on critical points of solutions. Indeed, let $w = w(s)$ be a solution to (1) and let s_1 and s_2 be such that $w'(s_1) = w'(s_2) = 0$. By differentiating we obtain

$$E'(s) = w'''(s)w''(s) - kw''(s)w'(s) - f(w(s))w'(s).$$

Hence, if $s_2 > s_1$, an integration by parts yields

$$\begin{aligned} E(s_2) - E(s_1) &= \int_{s_1}^{s_2} E'(s)\, ds \\ &= \int_{s_1}^{s_2} \big(w'''(s)w''(s) - kw''(s)w'(s) - f(w(s))w'(s)\big)\, ds \\ &= -\int_{s_1}^{s_2} \big(w''''(s) + kw''(s) + f(w(s))\big)w'(s)\, ds = 0, \end{aligned}$$

where we used $w'(s_1) = w'(s_2) = 0$ and (1).

The second energy function is constant on solutions. Indeed, if w solves (1), there holds

$$\mathcal{E}'(s) = w'''(s)w''(s) + w''''(s)w'(s) - (w'(s)w'''(s))' = 0 \quad \Longrightarrow \quad \mathcal{E}(s) = C\,,$$

for some $C \in \boldsymbol{R}$. Therefore, if one is interested in homoclinics, by (10), $\mathcal{E}(s) = 0$ for all $s \in \boldsymbol{R}$. This property has been exploited to prove Theorem 7-(i).

Next we turn to $H(s)$. Let w be a solution to (1), a short computation gives

$$H'(s) = w''(s)^2 - kw'(s)^2 + w(s)f(w(s)).$$

If $k \leq 0$ and f satisfies (3), then it is readily deduced that H is nondecreasing. This gives a simpler proof of the second part of Theorem 6-(i). Indeed, let w be a homoclinic solution to (1), (10) yields $\lim_{s\to\pm\infty} H(s) = 0$. Since H is nondecreasing, $H = H' \equiv 0$ and we have a simple proof of the fact that no homoclinic solution exists.

The energy H may also be exploited when $k > 0$. See e.g. the proof of Theorem 6-(ii) as given in [5].

4.2 – *Dynamical system analysis*

In some situations, it may be useful to transform the fourth order ode (1) into a *first order system* of four equations. Let $w = w(s)$ be a solution to (1) and put

$$Y(s) = (y_1(s), y_2(s), y_3(s), y_4(s)) = (w(s), w'(s), w''(s), w'''(s))$$

so that (1) may be rewritten as a system

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = y_4 \\ y_4' = -ky_3 - f(y_1). \end{cases} \tag{14}$$

If we define $\Phi : \boldsymbol{R}^4 \to \boldsymbol{R}^4$ by

$$\Phi(y_1, y_2, y_3, y_4) = (y_2, y_3, y_4, -ky_3 - f(y_1))$$

then any solution $Y(s) = (y_1(s), y_2(s), y_3(s), y_4(s))$ of (14) may be rewritten as

$$Y'(s) = \Phi(Y(s)).$$

In view of (3), $f(s)$ admits a unique zero at $s = 0$. Therefore, the dynamical system (14) admits a unique stationary point which is $O = (0, 0, 0, 0)$. This point corresponds to the solution $w \equiv 0$ to (1). We now study the stability of O.

If we assume that f is differentiable at $t = 0$ with $f'(0) = 1$, then the linearized problem at O for (14) reads

$$Y'(s) = AY(s)\ , \qquad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -k & 0 \end{pmatrix}.$$

The eigenvalues λ of A satisfy the equation $\lambda^4 + k\lambda^2 + 1 = 0$ and therefore also

$$\lambda^2 = \frac{-k \pm \sqrt{k^2 - 4}}{2}.$$

Hence, if $k < -2$ the eigenvalues of A are all real and are given by

$$\lambda \in \left\{ \pm\sqrt{\frac{|k| + \sqrt{k^2 - 4}}{2}}, \pm\sqrt{\frac{|k| - \sqrt{k^2 - 4}}{2}} \right\}.$$

If $-2 < k < 2$, the eigenvalues of A are

$$\lambda \in \left\{ \pm\frac{\sqrt{2 - k}}{2} \pm i\,\frac{\sqrt{2 + k}}{2}\ , \pm\frac{\sqrt{2 - k}}{2} \mp i\,\frac{\sqrt{2 + k}}{2} \right\}.$$

If $k > 2$, the eigenvalues of A are given by

$$\lambda \in \left\{ \pm i\sqrt{\frac{k+\sqrt{k^2-4}}{2}}, \pm i\sqrt{\frac{k-\sqrt{k^2-4}}{2}} \right\}.$$

If $k = -2$ the eigenvalues of A are $\lambda \in \{\pm 1\}$, they both have multiplicity 2, and the corresponding eigenvectors are $v_+ = (1,1,1,1)$ and $v_- = (1,-1,1,-1)$.

If $k = 2$ the eigenvalues of A are $\lambda \in \{\pm i\}$ (with multiplicity 2) and the corresponding eigenvectors are $v = (1, i, -1, -i)$ and $\overline{v} = (1, -i, -1, i)$.

Summarizing, we have

PROPOSITION 1. – *Assume* (3) *and that* f *is differentiable at* $t = 0$ *with* $f'(0) = 1$. *For any* $k \in \boldsymbol{R}$, (14) *has a unique stationary point* $O = (0,0,0,0)$ *which satisfies*

(*i*) *if* $k < -2$, *O has a 2-dimensional stable manifold and a 2-dimensional unstable manifold, both not oscillating near O;*
(*ii*) *if* $k = -2$, *O has a 2-dimensional stable manifold (tangent to* v_- *near O) and a 2-dimensional unstable manifold (tangent to* v_+ *near O);*
(*iii*) *if* $-2 < k < 2$, *O has a 2-dimensional stable manifold and a 2-dimensional unstable manifold, both having locally the form of a spiral near O;*
(*iv*) *if* $k = 2$, *the linearized problem at O has 2 (opposite) double purely imaginary eigenvalues;*
(*v*) *if* $k > 2$, *the linearized problem at O has 4 purely imaginary eigenvalues.*

If $k \geq 0$, by Theorem 3 we know that any global solution to (1) changes sign infinitely many times as $s \to \pm\infty$. The dynamical system analysis performed above suggests that oscillations may also occur when $-2 < k < 0$ since A has complex eigenvalues. On the other hand when $k \leq -2$ we expect solutions having eventually one sign. More precisely, from [5] one has

THEOREM 8. – *Assume that* f *satisfies* (3).

(i) *If* $k \leq -2$ *and* f *also satisfies one of the following*

$$f(t) \geq t \quad \text{near } t = 0 \quad \text{or} \quad f(t) \leq t \quad \text{near } t = 0 \tag{15}$$

then any global solution w *to* (1) *such that* $\lim_{s\to+\infty} w(s) = 0$ *is of one sign as* $s \to +\infty$. *Moreover, a similar statement holds with* $+\infty$ *replaced by* $-\infty$.

(ii) *If* $-2 < k < 0$, f *is differentiable at* $t = 0$ *with* $f'(0) = 1$ *and also satisfies*

$$\liminf_{|t|\to+\infty} \frac{f(t)}{t} > k^2 \,, \tag{16}$$

then any global nontrivial solution w *to* (1) *changes sign infinitely many times both as* $s \to \pm\infty$.

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