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Multigrid Methods for (Multilevel) Structured Matrices Associated with a Symbol and Related Applications (*)

MARCO DONATELLI - STEFANO SERRA CAPIZZANO

Abstract. – *When dealing with large linear systems with a prescribed structure, two key ingredients are important for designing fast solvers: the first is the computational analysis of the structure which is usually inherited from an underlying infinite dimensional problem, the second is the spectral analysis which is often deeply related to a compact symbol, again depending on the infinite dimensional problem of which the linear system is a given approximation. When considering the computational viewpoint, the first ingredient is useful for designing fast matrix-vector multiplication algorithms, while the second ingredient is essential for designing fast iterative solvers (multigrid, preconditioned Krylov etc.), whose convergence speed is optimal in the Axelsson, Neytcheva sense, i.e., the number of iterations for reaching a preassigned accuracy can be bounded by a pure constant independent of the matrix-size.*

In this review paper we consider in some details the specific case of multigrid-type techniques for Toeplitz related structures, by emphasizing the role of the structure and of the compact spectral symbol. A sketch of several extensions to other (hidden) structures as those appearing in the numerical approximation of partial differential equations and integral equations is given and critically discussed.

1. – Introduction

We investigate multigrid methods for multilevel linear systems whose coefficient matrices are generated by a real and nonnegative multivariate polynomial f and belong to multilevel matrix algebras like circulant, tau, Hartley, or are of Toeplitz type.

In the case of linear systems, where the coefficient matrix belongs to one of the above mentioned matrix algebras, we prove that the convergence rate is independent of the system dimension even in presence of asymptotical ill-conditioning (this happens iff f takes the zero value). More precisely, if the d-level coefficient matrix has partial dimension n_r at level r, with $r = 1, \ldots, d$, then the size of the system is $N(\boldsymbol{n}) = \prod_{r=1}^{d} n_r$, $\boldsymbol{n} = (n_1, \ldots, n_d)$, and $O(N(\boldsymbol{n}))$ operations are required by the considered V-cycle multigrid in order to compute the solution

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within a fixed accuracy. Since the total arithmetic cost is asymptotically equivalent to the one of a matrix-vector product, the proposed method is optimal in the sense stated in [3]. As a specific important application, we consider the image deblurring problem in the case of a known space invariant blur and proper boundary conditions. In particular, we show that the spectral information given by the symbol can be used for defining a regularizing multigrid which improves the restoration, when an iterative regularization method is used as smoother.

1.1 – *Multigrid methods for structured matrices*

Multigrid methods for tau and Toeplitz matrices were firstly introduced in [23] and then applied to two-level Toeplitz matrices in [24]. These early works led to several generalizations, other matrix algebras in [40, 12], different projecting strategies for Toeplitz matrices in [30, 9, 13], and to further developments in the theoretical analysis on the convergence rate [9, 13, 36, 2]. In [2] it was proved that in the one-dimensional case the Algebraic Multi-Grid (AMG) firstly introduced in [23] for tau and Toeplitz matrices and in [40] for circulant matrices generated by nonnegative polynomial functions, under slightly stronger conditions, is optimal when using the V-cycle and only one iteration of relaxed Richardson as post-smoother. The optimality is in the sense of Axelsson and Neytcheva [3], i.e., the problem of solving a linear system with coefficient matrix $A_{\boldsymbol{n}}$ is asymptotically of the same cost as the direct problem of multiplying $A_{\boldsymbol{n}}$ by a vector. These slightly stronger conditions are translated into the choice of a projector which results more powerful than the previous proposals, when the coefficient matrices possess a generating function with zeros of order greater than two.

The multidimensional setting was studied in [1], by considering structures belonging to multilevel circulant, tau or Hartley algebras and generated by nonnegative multivariate polynomial functions. More specifically, under slightly stronger conditions and for the class of matrices mentioned above, the AMG proposed in [24, 36, 40] is optimal, when using the V-cycle and at least one (pre or post) smoothing iteration of relaxed Richardson. A similar analysis can be done for other stationary methods.

Here we review such results by showing that the total cost of the considered AMG is given by $O(N(\boldsymbol{n}))$ arithmetic operations, since:

1. all the matrices appearing in the AMG have a number of non-zero diagonals independent of $\boldsymbol{n}$ and they can be computed within a number of operations proportional to $\log(N(\boldsymbol{n}))$,
2. each iteration requires the same computational cost of the matrix-vector product, i.e., $O(N(\boldsymbol{n}))$ arithmetic operations,
3. the number of iterations required for the convergence is bounded by a constant which does not depend on $\boldsymbol{n}$.

We observe that the last point means that the convergence rate is independent of $N(\boldsymbol{n})$ which is the main ingredient for the optimality of the resulting method. Furthermore, the matrices at each multigrid level belong to the same algebra and then the recursive V-cycle procedure is well defined.

In the case of the considered matrix algebras the cost by direct methods using fast transforms is $O(N(\boldsymbol{n}) \log N(\boldsymbol{n}))$ operations, while an optimal technique would require just $O(N(\boldsymbol{n}))$ operations. This kind of linear systems associated with matrix algebras is widely encountered when preconditioning more complicated problems (dense multilevel Toeplitz systems, discretization of multidimensional differential and/or integral equations, etc. [10, 35]) or directly arises in some image restoration problems with shift-invariant kernel and suitable boundary conditions (see [28, 38]).

The tau algebra is also the key to extend the proposed AMG to multilevel Toeplitz matrices. In this direction there are several proposals [30, 36, 9, 13, 2]. The proposal in [30] does not follow a Galerkin strategy and hence there are not many useful tools for a theoretical analysis, while the proposal in [2] extends the one in [36] preserving more information at each recursion level when the generating function has a zero of arbitrary finite order. In [36] the optimality is proved in the Two-Grid case, while in [9, 13] the level independency in the one and twodimensional case is also proved with generating function having zeros of order at most two. The latter implies the optimality using the W-cycle, but, as shown in [2], it is not enough for the optimality of the V-cycle. We emphasize that for multilevel Toeplitz matrices with nonnegative generating functions having a zero of order at most two, all the generalizations described in [30, 36, 9, 13, 2] of the original idea contained in [23, 24] define exactly the same multigrid procedure. Furthermore for generating functions having zeros of order greater than two there are no results on the optimality of the cited proposals. We recall that in [36] the level independency is implicitly proved but not explicitly stated for a zero of arbitrary order. However the experimentation in [1, 2] confirms numerically an optimal behavior of our proposal also for zeros of order greater than two already in the V-cycle case. In this case the fast direct techniques require a computational cost of $O([N(\boldsymbol{n})]^{\frac{3d-1}{d}})$ [32] and need further stabilization tricks, while the most popular preconditioning strategies can be far from being optimal [41].

We remark that generally it is not true that if we have optimality for a given iterative solver in the one-dimensional case, then the same property transfers to the multidimensional case. A notable example is the preconditioning of multilevel Toeplitz systems using multilevel algebras like the circulant algebra: indeed many optimal preconditioners can be found in the one dimensional case while in the multidimensional case this has been shown to be theoretically impossible, see [41]. On the other hand, by using multilevel band Toeplitz preconditioners (see e.g. [35]), it is possible to reduce the computation with dense Toeplitz systems to the case of Toeplitz linear systems whose coefficient matrices are generated by

nonnegative polynomials. Therefore it is of special interest to be able to solve in an optimal way these preconditioned systems. This numerical solution can be performed with the proposed AMG.

We expect that the theoretical tools introduced in this paper for the multilevel matrix algebra case can be employed for proving the AMG optimality in the multilevel Toeplitz context as well. However, the simplification of considering the case of matrix algebras, instead of the Toeplitz case, is the same simplification that is done in the classical Local Fourier Analysis (LFA) for the geometric multigrid (see [17] for a comparison between our analysis and the LFA). As an example, some discretizations of PDEs with periodic boundary conditions lead to circulant matrices while Dirichlet boundary conditions lead to Toeplitz matrices.

1.2 – *Multigrid methods for regularization purposes*

We consider the de-blurring problem of noisy and blurred images in the case of known space invariant point spread functions with four choices of boundary conditions. We combine our algebraic multigrid defined ad hoc for structured matrices related to space invariant operators (Toeplitz, circulants, trigonometric matrix algebras, etc.) and the classical geometric multigrid studied in the partial differential equations context. The resulting technique is parameterized in order to have more degrees of freedom: a simple choice of the parameters allows us to devise a quite powerful regularizing method. It defines an iterative regularizing method where the smoother itself has to be an iterative regularizing method (e.g., conjugate gradient, Landweber, conjugate gradient for normal equations, etc.). More precisely, with respect to the smoother, the regularization properties are improved and the total complexity is lower. Furthermore, in several cases, when it is directly applied to the system $A\boldsymbol{f} = \boldsymbol{g}$, the quality of the restored image is comparable with that of all the best known techniques for the normal equations $A^TA\boldsymbol{f} = A^T\boldsymbol{g}$, but the related convergence is substantially faster. Finally, the associated curves of the relative errors versus the iteration numbers are "flatter" with respect to the smoother (the estimation of the stopping iteration is less crucial). Therefore, we can choose multigrid procedures which are much more efficient than classical techniques without losing accuracy in the restored image (as often occurs when using preconditioning). The numerical experiments show the effectiveness of our proposals.

1.3 – *Plan of the paper*

The paper is organized as follows. § 2 is devoted to present multilevel circulant, tau and Hartley algebras and the multilevel Toeplitz matrices, by

emphasizing their main common properties and especially the role of the symbol for describing spectral features. In § 3 the V-cycle AMG procedure is presented while in § 4 three constraints for the AMG optimality are discussed. In § 5 we prove the convergence and optimality property of our AMG and we briefly present generalizations and extensions, where the notion of spectral symbol is of paramount importance. Section § 6 is devoted to discuss the image restoration problem, by the viewpoints of the multigrid and of the spectral symbol. Finally, § 7 contains concluding remarks.

2. – Multilevel algebra and Toeplitz matrices

We will consider a multigrid method to solve linear systems whose matrices belong to multilevel circulant, tau and Hartley algebras. We will provide a uniform approach that in fact can be extended to other matrix algebras (for DCTIII see [12]).

Let $d, n \in \mathbb{N} \setminus \{0\}$, $\mathcal{F}_d = \{f : \mathbb{R}^d \to \mathbb{R}\}$ and let $\mathrm{Diag}(\boldsymbol{z})$ be the diagonal matrix with principal diagonal equal to $\boldsymbol{z} \in \mathbb{R}^n$. With any unitary matrix Q_n (i.e. $Q_n^{-1} = Q_n^H$), we can associate the Hermitian algebra $\mathcal{G}(Q_n) = \{Q_n \cdot \mathrm{Diag}(\boldsymbol{z}) \cdot Q_n^H \mid \boldsymbol{z} \in \mathbb{R}^n\}$ and hence the map $\mathcal{A}_n$ defined by

$$\begin{array}{rccc} \mathcal{A}_n : & \mathcal{F}_1 & \longrightarrow & \mathcal{G}(Q_n) \\ & f & \longrightarrow & Q_n \cdot \mathrm{Diag}\, f(\boldsymbol{w}^{[n]}) \cdot Q_n^H \end{array}$$

is an algebra homomorphism where $\boldsymbol{w}^{[n]}$ is a fixed vector of $\mathbb{R}^n$ and $f(\boldsymbol{w}^{[n]})$ denotes the vector with components $f(w_i^{[n]})$. As a consequence $\boldsymbol{u}_i^{[n]} = Q_n \boldsymbol{e}_i$ is a unitary eigenvector of $\mathcal{A}_n(f)$ related to the eigenvalue $f(w_i^{[n]})$. The circulant matrix $\mathcal{C}_n(f)$, the tau matrix $\tau_n(f)$ and the Hartley matrix $\mathcal{H}_n(f)$ with $f \in \mathcal{F}_1$ can be written as $\mathcal{A}_n(f)$, $\mathcal{A} \in \{\mathcal{C}, \tau, \mathcal{H}\}$, by means of the objects Q_n and $\boldsymbol{w}^{[n]}$ defined in Table 1.

TABLE 1. – Basics of trigonometric algebras: index range, sampling points, eigenvectors.

	$\mathcal{A}$	$\mathcal{I}_n$	$\boldsymbol{w}^{[n]}$	Q_n
Circulants	$\mathcal{C}$	$0, \ldots, n-1$	$w_i^{[n]} = \dfrac{2\pi i}{n}$	$F_n = \dfrac{1}{\sqrt{n}} \left[\mathrm{e}^{\mathrm{i} j w_i^{[n]}}\right]_{i,j \in \mathcal{I}_n}$
Hartley	$\mathcal{H}$	$0, \ldots, n-1$	$w_i^{[n]} = \dfrac{2\pi i}{n}$	$H_n = \mathrm{Re}(F_n) + \mathrm{Im}(F_n)$
Tau	τ	$1, \ldots, n$	$w_i^{[n]} = \dfrac{\pi i}{n+1}$	$S_n = \sqrt{\dfrac{2}{n+1}} \left[\sin\left(j w_i^{[n]}\right)\right]_{i,j \in \mathcal{I}_n}$

In [2] we proposed an AMG (improving the one studied in [23]) to solve the linear system $\mathcal{A}_n(f)\boldsymbol{x} = \boldsymbol{b}$ with total arithmetic cost linear in n under the assumption that f is a trigonometric polynomial that vanishes in zero and is positive in the open interval $(0, 2\pi)$ (we required f even in the tau case): then it was shown how to extend the result to the case where the unique root is not at zero and in the case where f has more than one root. In [1] we extended this analysis to the multilevel case improving the previous proposals and the theoretical results in [24, 36, 40]. Here we review the main results directly in the multilevel case.

A d-level matrix $A_{\boldsymbol{n}}$ of partial dimension $\boldsymbol{n} = (n_1, n_2, \ldots, n_d) \in (\mathbb{N} \setminus \{0\})^d$ can be described (see [46]) as an $n_1 \times n_1$ block matrix whose elements are $n_2 \times n_2$ block matrices and so on with d nesting levels; its true dimension is $N(\boldsymbol{n}) = \prod_{r=1}^{d} n_r$. We will refer to the elements of such matrix by using a pair $(\boldsymbol{i},\boldsymbol{j})$ of d-indices: $A_{\boldsymbol{n}} = [a_{\boldsymbol{i},\boldsymbol{j}}]$, and the selected element $a_{\boldsymbol{i},\boldsymbol{j}}$ is the one in position (i_r, j_r) at the r-th level, for every $r = 1, \ldots, d$.

Circulant, tau and Hartley d-level matrix algebras of partial dimension $\boldsymbol{n}$ can be defined as the matrix algebra $\mathcal{G}(Q_{\boldsymbol{n}})$ associated with the transform $Q_{\boldsymbol{n}} = Q_{n_1} \otimes \cdots \otimes Q_{n_d}$. All Q_{n_i} have to be selected in the same row of Table 1: it is possible to deal with mixed structures and the corresponding multigrid analysis is straightforward (see e.g. [40]), but we will not emphasize this point hereafter. Of course we can associate multilevel matrices $\mathcal{C}_{\boldsymbol{n}}(f)$, $\tau_{\boldsymbol{n}}(f)$ and $\mathcal{H}_{\boldsymbol{n}}(f)$ with each multivariate function $f \in \mathcal{F}_d$, thus we extend the map $\mathcal{A}_n$ to $\mathcal{A}_{\boldsymbol{n}}$ as follows

$$\begin{array}{rccc} \mathcal{A}_{\boldsymbol{n}}: & \mathcal{F}_d & \longrightarrow & \mathcal{G}(Q_{\boldsymbol{n}}) \\ & f & \longrightarrow & Q_{\boldsymbol{n}} \cdot \mathrm{Diag}\, f(\boldsymbol{w}^{[\boldsymbol{n}]}) \cdot Q_{\boldsymbol{n}}^H, \end{array}$$

with the sampling point multilevel vector $\boldsymbol{w}^{[\boldsymbol{n}]} \in \mathbb{R}^{N(\boldsymbol{n})}$ defined as

$$w_{\boldsymbol{i}}^{[\boldsymbol{n}]} = (w_{i_1}^{[n_1]}, \ldots, w_{i_d}^{[n_d]}), \quad \boldsymbol{i} \in \mathcal{I}_{\boldsymbol{n}} := \mathcal{I}_{n_1} \times \ldots \times \mathcal{I}_{n_d}.$$

It follows that $\boldsymbol{u}_{\boldsymbol{i}}^{[\boldsymbol{n}]} = \boldsymbol{u}_{i_1}^{[n_1]} \otimes \cdots \otimes \boldsymbol{u}_{i_d}^{[n_d]}$ is an eigenvector related to the sampling at $w_{\boldsymbol{i}}^{[\boldsymbol{n}]}$ and $\mathcal{A}_{\boldsymbol{n}}$ is an algebra homomorphism as well, so that $\mathcal{A}_{\boldsymbol{n}}(f)\mathcal{A}_{\boldsymbol{n}}(g) = \mathcal{A}_{\boldsymbol{n}}(fg)$.

In this article we are interested in linear systems $\mathcal{A}_{\boldsymbol{n}}(f)\boldsymbol{x} = \boldsymbol{b}$ with $\mathcal{A} \in \{\mathcal{C}, \tau, \mathcal{H}, T\}$, f being a nonnegative multivariate trigonometric polynomial. Once again, we require f to be even (with respect to each variable) in the tau case. Here $T_{\boldsymbol{n}}(f)$ is the Toeplitz d-level matrix of partial dimension $\boldsymbol{n}$ defined as

$$T_{\boldsymbol{n}}(f) = \sum_{|\boldsymbol{j}| \leqslant \boldsymbol{n}-\boldsymbol{e}} a_{\boldsymbol{j}} J_{\boldsymbol{n}}^{[\boldsymbol{j}]} = \sum_{|j_1| < n_1} \cdots \sum_{|j_d| < n_d} a_{(j_1,\ldots,j_d)} J_{n_1}^{[j_1]} \otimes \cdots \otimes J_{n_d}^{[j_d]}$$

($\boldsymbol{e} = (1, \ldots, 1) \in \mathbb{N}^d$) by means of the Fourier coefficients of f

$$a_{\boldsymbol{k}} = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} f(\boldsymbol{x}) \mathrm{e}^{-\boldsymbol{i}\langle \boldsymbol{k}|\boldsymbol{x}\rangle} \mathrm{d}\boldsymbol{x}, \qquad \boldsymbol{i}^2 = -1, \qquad \boldsymbol{k} \in \mathbb{Z}^d. \tag{1}$$

Here $J_n^{[j]} \in \mathbb{R}^{n\times n}$ is the matrix whose (s,t)-th entry equals 1 if $s-t=j$ and is 0 elsewhere.

We assume $f \in \mathbb{R}_{\boldsymbol{z}}$ with $\boldsymbol{z} \in (\mathbb{N}\setminus\{0\})^d$, where $\mathbb{R}_{\boldsymbol{\eta}}$, $\boldsymbol{\eta}\in\mathbb{N}^d$, is the set of d-variate real-valued trigonometric polynomials with degree up to $\boldsymbol{\eta}$:

$$\mathbb{R}_{\boldsymbol{\eta}}[\boldsymbol{x}] = \left\{ \sum_{|\boldsymbol{k}|\leq\boldsymbol{\eta}} a_{\boldsymbol{k}} \mathrm{e}^{\boldsymbol{i}\langle \boldsymbol{k}|\boldsymbol{x}\rangle} \ \text{ s.t. } \ a_{-\boldsymbol{k}} = \overline{a_{\boldsymbol{k}}} \in \mathbb{C} \right\}, \qquad \langle \boldsymbol{k}|\boldsymbol{x}\rangle = \sum_{i=1}^{d} \boldsymbol{k}_i \boldsymbol{x}_i.$$

In this case it is known (see [46, 5, 6]) that all the matrices $\mathcal{A}_{\boldsymbol{n}}(f)$ are Hermitian, banded (in the way induced from the considered structure) and semipositive definite if $f\geqslant 0$. Moreover $\mathcal{A}_{\boldsymbol{n}}(f)$ is ill-conditioned whenever f takes the zero value; it is singular if $\mathcal{A}\in\{\mathcal{C},\tau,\mathcal{H}\}$ and f vanishes in a grid point $w_{\boldsymbol{i}}^{[\boldsymbol{n}]}$.

If $f\geqslant 0$ vanishes in the grid point $w_{\boldsymbol{i}}^{[\boldsymbol{n}]}$ then it is usually replaced by a positive function that leads to a rank-1 correction of $\mathcal{A}_{\boldsymbol{n}}(f)$ [45]. Such correction should be taken into account also in the definition of the projector. However, for brevity, we will not discuss further this case (refer to [2]).

3. – Algebraic MultiGrid and Ruge–Stüben theory

Let $A\in\mathbb{C}^{N\times N}$ be a Hermitian positive definite matrix, $\boldsymbol{b}\in\mathbb{C}^N$, m integer with $0<m<N$. Fix integers $N_0=N>N_1>N_2>\ldots>N_m>0$, take $R_i\in\mathbb{C}^{N_{i+1}\times N_i}$ full-rank matrices and consider two classes $\mathcal{S}_i$, $\widetilde{\mathcal{S}}_i$ of iterative methods for N_i-dimensional linear systems, $i=0,\ldots,m-1$. The related AMG in the V-cycle version produces the sequence $\{\boldsymbol{x}^{(k)}\}_{k\in\mathbb{N}}\subset\mathbb{C}^N$ according to the rule $\boldsymbol{x}^{(k+1)}=\mathcal{AMG}(0,\boldsymbol{x}^{(k)},\boldsymbol{b})$, with $\mathcal{AMG}$ recursively defined as follows (where $A_0=A$, $\boldsymbol{b}_0=\boldsymbol{b}$):

(2)

$$\mathbf{x}_i^{(\mathrm{out})} := \mathcal{AMG}(\,i,\mathbf{x}_i^{(\mathrm{in})},\mathbf{b}_i\,)$$

If $(i=m)$ Then Solve $(A_m\mathbf{x}_m^{(\mathrm{out})}=\mathbf{b}_m)$

Else

1. $\mathbf{x}_i^{(\mathrm{pre})} := \mathcal{S}_i^{\nu_i}\left(\mathbf{x}_i^{(\mathrm{in})}\right)$
2. $\mathbf{r}_i := \mathbf{b}_i - A_i\mathbf{x}_i^{(\mathrm{pre})}$
3. $\mathbf{b}_{i+1} := R_i\mathbf{r}_i$
4. $A_{i+1} := R_iA_i(R_i)^H$
5. $\mathbf{x}_{i+1}^{(\mathrm{out})} := \mathcal{AMG}(i+1,\mathbf{0}_{N_{i+1}},\mathbf{b}_{i+1})$
6. $\mathbf{x}_i^{(\mathrm{int})} := \mathbf{x}_i^{(\mathrm{pre})} + R_i^H\mathbf{x}_{i+1}^{(\mathrm{out})}$
7. $\mathbf{x}_i^{(\mathrm{out})} := \widetilde{\mathcal{S}}_i^{\vartheta_i}\left(\mathbf{x}_i^{(\mathrm{int})}\right)$

Step **1** performs some (ν_i) iterations of a "pre-smoother"; step **2** calculates the residue of presmoother approximation; steps **3**, **4**, **5** and **6** define the *recursive coarse grid correction* by restriction (**3**) of the residue, coarse grid correction (**4**, **5**) and interpolation (**6**), while step **7** performs some (ϑ_i) iterations of a "post-smoother".

The restrictors R_i have to be full-rank, thus all A_i are nonsingular, Hermitian and positive definite. Most of the times smoothers are one-point methods:

$$\begin{cases} \mathcal{S}_i(\boldsymbol{x}) &= S_i\boldsymbol{x} + (I_{N_i} - S_i)A_i^{-1}\boldsymbol{b}_i \\ \widetilde{\mathcal{S}}_i(\boldsymbol{x}) &= \widetilde{S}_i\boldsymbol{x} + (I_{N_i} - \widetilde{S}_i)A_i^{-1}\boldsymbol{b}_i \end{cases}, \quad \boldsymbol{x} \in \mathbb{C}^{N_i}, \quad i = 0, \ldots, m-1. \tag{3}$$

Steps **2**-**6** allow us to define on each level i the *exact coarse grid correction operator* :

$$CGC_i = I_{N_i} - R_i^H A_{i+1}^{-1} R_i A_i, \qquad i = 0, \ldots, m-1. \tag{4}$$

Under these assumptions, it is possible to prove that the AMG is a one-point method and its linear part AMG_0 is recursively defined as

$$\begin{cases} AMG_m &= O_{N_m \times N_m} \\ AMG_i &= \widetilde{S}_i^{\vartheta_i} \cdot \left[I_{N_i} - R_i^H \left(I_{N_{i+1}} - AMG_{i+1} \right) A_{i+1}^{-1} R_i A_i \right] \cdot S_i^{\nu_i}, \\ & \qquad\qquad\qquad\qquad i = m-1, \ldots, 0. \end{cases} \tag{5}$$

This shows that, unless we are in the two-grid case, by swapping the order of smoothers (or else by applying both before or after the recursive coarse grid correction) we affect the spectra of AMG_0.

In the following, whenever X is a Hermitian positive definite matrix we define $\| \cdot \|_X = \| X^{1/2} \cdot \|_2$, where we denote by $\| \cdot \|_2$ the usual Euclidean norm on $\mathbb{C}^n$ and also the induced matrix norm on $\mathbb{C}^{n \times n}$. If X and Y are Hermitian matrices then $X \leqslant Y$ means that $Y - X$ is positive.

To prove the Multigrid convergence we resorted to an approach proposed by Ruge and Stüben that is based on the following theorem.

THEOREM 1 ([34]). – *Let m, N be integers satisfying $0 < m < N$ and suppose that $A \in \mathbb{C}^{N \times N}$ is a positive definite Hermitian matrix and $\boldsymbol{b} \in \mathbb{C}^N$; given a sequence of $m+1$ positive integers $N = N_0 > N_1 > \ldots > N_m$, let $R_i \in \mathbb{C}^{N_{i+1} \times N_i}$ be full-rank matrices for each $i = 0, \ldots, m-1$. Define $A_0 = A$, $\boldsymbol{b}_0 = \boldsymbol{b}$ and choose two classes of iterative methods $\mathcal{S}_i$, $\widetilde{\mathcal{S}}_i$ as in* (3). *If there exist two real positive numbers δ_{pre}, δ_{post} satisfying*

$$\| S_i^{\nu_i} \boldsymbol{x} \|_{A_i}^2 \leqslant \| \boldsymbol{x} \|_{A_i}^2 - \delta_{\text{pre}} \| CGC_i \, S_i^{\nu_i} \boldsymbol{x} \|_{A_i}^2 \qquad \forall \boldsymbol{x} \in \mathbb{C}^{N_i} \tag{6a}$$

and

$$\|\widetilde{S}_i^{\vartheta_i}\boldsymbol{x}\|_{A_i}^2 \leqslant \|\boldsymbol{x}\|_{A_i}^2 - \delta_{\text{post}}\,\|CGC_i\boldsymbol{x}\|_{A_i}^2 \qquad \forall\,\boldsymbol{x}\in\mathbb{C}^{N_i} \tag{6b}$$

both for every $i=0,\dots,m-1$, *then it holds* $\delta_{\text{post}}\leqslant 1$ *and*

$$\|AMG_0\|_A \leqslant \sqrt{\frac{1-\delta_{\text{post}}}{1+\delta_{\text{pre}}}} < 1. \tag{7}$$

REMARK 2. – From Theorem 1 the sequence $\{\boldsymbol{x}^{(k)}\}_{k\in\mathbb{N}}$ converges to the solution of $A\boldsymbol{x}=\boldsymbol{b}$ and if at least one between δ_{pre} and δ_{post} is independent of N and m, it converges with a constant error reduction not depending on N and m.

We split (6a) and (6b), namely

$$\text{(6.a-bis)}\qquad \begin{cases} \|S_i^{\nu_i}\boldsymbol{x}\|_{A_i}^2 & \leqslant \|\boldsymbol{x}\|_{A_i}^2 - \alpha\,\|S_i^{\nu_i}\boldsymbol{x}\|_{A_i^2}^2 \\ \|CGC_i\boldsymbol{x}\|_{A_i}^2 & \leqslant \gamma\,\|\boldsymbol{x}\|_{A_i^2}^2 \\ \delta_{\text{pre}} = \alpha/\gamma & \end{cases}$$

for (6a) and

$$\text{(6.b-bis)}\qquad \begin{cases} \|\widetilde{S}_i^{\vartheta_i}\boldsymbol{x}\|_{A_i}^2 & \leqslant \|\boldsymbol{x}\|_{A_i}^2 - \beta\,\|\boldsymbol{x}\|_{A_i^2}^2 \\ \|CGC_i\boldsymbol{x}\|_{A_i}^2 & \leqslant \gamma\,\|\boldsymbol{x}\|_{A_i^2}^2 \\ \delta_{\text{post}} = \beta/\gamma & \end{cases}$$

and for (6b), see [2, 1]

The coefficients α, β and γ can be different when i changes, since the step from (6.a-bis) to (6a) and from (6.b-bis) to (6b) is purely algebraic and does not affect the proof of Theorem 1. This means that we can use the inequalities

$$\|S_i^{\nu_i}\boldsymbol{x}\|_{A_i}^2 \leqslant \|\boldsymbol{x}\|_{A_i}^2 - \alpha_i\,\|S_i^{\nu_i}\boldsymbol{x}\|_{A_i^2}^2 \qquad (\alpha_i\geqslant 0),\quad \forall\,\boldsymbol{x}\in\mathbb{C}^{N_i}, \tag{8a}$$

$$\|\widetilde{S}_i^{\vartheta_i}\boldsymbol{x}\|_{A_i}^2 \leqslant \|\boldsymbol{x}\|_{A_i}^2 - \beta_i\,\|\boldsymbol{x}\|_{A_i^2}^2 \qquad (\beta_i\geqslant 0),\quad \forall\,\boldsymbol{x}\in\mathbb{C}^{N_i}, \tag{8b}$$

$$\|CGC_i\boldsymbol{x}\|_{A_i}^2 \leqslant \gamma_i\,\|\boldsymbol{x}\|_{A_i^2}^2 \qquad (\gamma_i>0),\quad \forall\,\boldsymbol{x}\in\mathbb{C}^{N_i}. \tag{8c}$$

which are not weaker than (6), provided that it holds

$$\delta_{\text{pre}} = \min_{0\leqslant i<m}\frac{\alpha_i}{\gamma_i}, \qquad \delta_{\text{post}} = \min_{0\leqslant i<m}\frac{\beta_i}{\gamma_i} \tag{9}$$

for every $i=0,\dots,m-1$.

We refer to (8a) as the *presmoothing property*, (8b) as the *postsmoothing property* and (8c) as the *approximation property* (see [34]). The approximation property depends exclusively on the choice of the projectors R_i but not on

smoothers, whereas the smoothing properties are not related to R_i. The separate study of these properties allows us to cope with the more difficult part of the procedure, the verification of condition (8c), which involves the projectors but is independent of the smoothers.

However, in order to fulfil conditions (6a) and (6b) with δ_{pre}, δ_{post} independent of $\boldsymbol{n}$ and m (which in turn imply the AMG optimal convergence by Theorem 1), we will show (see § 5.2) that positive sequences $\{\alpha_i\}$, $\{\beta_i\}$ and $\{\gamma_i\}$ can be found such that the two ratios α_i/γ_i, β_i/γ_i converge to two positive constants if i goes to infinity. It follows that *the optimality is characterized by satisfaction of at least one of the two next* inf – min *conditions:*

$$\inf_t \min_{0\leqslant i<m_{\max}(t)} \frac{\alpha_i}{\gamma_i} > 0, \qquad \inf_t \min_{0\leqslant i<m_{\max}(t)} \frac{\beta_i}{\gamma_i} > 0. \tag{10}$$

4. – The AMG for matrix algebras

To reach convergence and optimality, and, more meaningfully, to write a good algorithm, we have to answer three requests of different nature: algebraic, computational, and convergence-optimality.

The *algebraic* requirement (§ 4.1) is the following: every matrix A_i generated from step **4** of AMG algorithm (2) has to be in the same algebra $\mathcal{G}$ of A_0 and hence

$$A_i = \mathcal{A}_{\boldsymbol{n}_i}(f_i) \in \mathcal{G}(Q_{\boldsymbol{n}_i}) \tag{11}$$

has to hold, f_i being a suitable function (in the following $f = f_0$, $\boldsymbol{z} = \boldsymbol{z}_0$ and $\boldsymbol{n} = \boldsymbol{n}_0$ by choice) and $\boldsymbol{n}_i$ a suitable multiindex. This means that the matrices A_i generated from step **4** of algorithm (2) have all to be circulant, or all tau or all Hartley, each one of the right partial order. It is obvious that the algebraic requirement does not imply by itself convergence and optimality, but it is necessary to define a recursive technique and also to obtain a good method: since the coarse grid matrix has to approximate the fine grid matrix, if they are of the same matrix algebra type (e.g. circulant) then the approximation would likely be better.

The *computational* requirement (§ 4.2) is related to optimality: the computational cost on each iteration has to be as low as possible, i.e.,

$$O(N_0) = O(N(\boldsymbol{n}_0))$$

since we deal with banded matrices. This is reached if the following three conditions are guaranteed:

1. $\{R_i\}_{i=0}^{m-1}$ and $\{A_i\}_{i=1}^{m}$ can be (pre)computed with cost at most $O(N_0)$;
2. the products $A_i\boldsymbol{x}$, $R_i\boldsymbol{r}$ and ${R_i}^H\boldsymbol{y}$ (steps **2**, **3** and **6**) and smoothers (steps **1** and **7**) have linear cost with respect to the dimension $N(\boldsymbol{n}_i)$;
3. the cost of solving $A_m\boldsymbol{x}_m^{(\text{out})} = \boldsymbol{b}_m$ is at most $O(N_0)$.

The *convergence-optimality* requirement (§ 5) is the following: the error reduction on each iteration has to be smaller than one (convergence) and also uniformly bounded (optimality), with respect to the dimension of the problem, by a constant smaller than one and independent of N_0 and m. It follows that this constant will depend only on the generating function:

$$\rho(AMG_0) \leqslant \text{const}(f_0) < 1,$$

$\rho(M)$ being the spectral radius of M. It is possible to prove convergence and optimality for AMG algorithm (2) if all inequalities (8a,b,c) and at least one of (10) hold. Convergence and optimality will be proved in § 5.

TABLE 2. – Scalar case: dimensions, cutting operators and relations ($M_n = \text{Diag}_{r=1}^{n}(-1)^r$).

	Circulant & Hartley algebra	τ algebra
N_0	2^t	$2^t - 1$
N_i	$\frac{N_{i-1}}{2} = 2^{t-i}$	$\frac{N_{i-1}-1}{2} = 2^{t-i} - 1$
K_{N_i}	$\begin{bmatrix} 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & \ddots & \ddots & & \\ & & & & 1 & 0 \end{bmatrix}_{N_{i+1} \times N_i}$	$\begin{bmatrix} 0 & 1 & 0 & & & \\ & 0 & 1 & 0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & 0 & 1 & 0 \end{bmatrix}_{N_{i+1} \times N_i}$
$K_{N_i} Q_i$	$[Q_{N_{i+1}} \vert Q_{N_{i+1}}]$	$[Q_{N_{i+1}} \vert 0_{N_{i+1}} \vert M_{N_{i+1}} Q_{N_{i+1}}]$
R_i	$K_{N_i}\ \mathcal{A}_{N_i}(p_i)$	

4.1 – *Algebraic requirement*

Here we describe how to satisfy the algebraic requirement (11). We simply give the multilevel version of the arguments defined in the one-level case in [2] according to Table 2. In [2] we fix $N_i = 2^{t-i}$ for circulants and Hartley and $N_i = 2^{t-i} - 1$ for tau (t is an integer number) and we choose as projector (restrictor) R_i the product between a cutting matrix K_{N_i} (defined in Table 2) and a matrix $\mathcal{A}_{N_i}(p_i)$ in the algebra $\mathcal{G}(Q_{N_i})$, each $p_i \in \mathcal{F}_1$ being a trigonometric polynomial. By means of cutting relations we obtained $A_i = \mathcal{A}_{N_i}(f_i)$, being $\{f_i\}_{i=0}^{m} \subset \mathcal{F}_1$ defined by $f_{i+1} = \Psi_1(p_i^2 f_i)$, with $\Psi_1 : \mathcal{F}_1 \longrightarrow \mathcal{F}_1$ defined as follows [23]:

$$\Psi_1[g(x)] = \frac{1}{2}\left[g\left(\frac{x}{2}\right) + g\left(\frac{x}{2} + \pi\right)\right].$$

Now we deal with the d-level case, starting with $A_0 = \mathcal{A}_{\boldsymbol{n}_0}(f_0)$ whose partial order is $\boldsymbol{n}_0 = 2^t \boldsymbol{e} \in \mathbb{N}^d$ for circulants and Hartley and $\boldsymbol{n}_0 = (2^t - 1)\boldsymbol{e}$ for tau, where $\boldsymbol{e} = (1, \ldots, 1) \in \mathbb{N}^d$ and t is still a positive integer. We essentially halve each partial

order on each level, by defining $\boldsymbol{n}_i = 2^{t-i}\boldsymbol{e}$ for circulants and Hartley and $\boldsymbol{n}_i = (2^{t-i}-1)\boldsymbol{e}$ for tau. As projector R_i we choose again a product, between a d-level cutting matrix $K_{\boldsymbol{n}_i} = K_{(n_i)_1} \otimes \cdots \otimes K_{(n_i)_d}$ and a matrix $\mathcal{A}_{\boldsymbol{n}_i}(p_i)$ in the d-level algebra $\mathcal{G}(Q_{\boldsymbol{n}_i})$, $p_i \in \mathcal{F}_d$ (see Table 3).

TABLE 3. – Multilevel case ($d > 1$): dimensions and cutting operators ($\boldsymbol{e} = (1, \ldots, 1)$).

	Circulant & Hartley algebra	τ algebra
$\boldsymbol{n}_0$	$2^t\boldsymbol{e}$	$(2^t-1)\boldsymbol{e}$
$\boldsymbol{n}_i$	$\frac{\boldsymbol{n}_{i-1}}{2} = 2^{t-i}\boldsymbol{e}$	$\frac{\boldsymbol{n}_{i-1}-\boldsymbol{e}}{2} = (2^{t-i}-1)\boldsymbol{e}$
$m_{\max}$	t	$t-1$
$K_{\boldsymbol{n}_i}$	$K_{(n_i)_1} \otimes \cdots \otimes K_{(n_i)_d}$	
R_i	$K_{\boldsymbol{n}_i}\ \mathcal{A}_{\boldsymbol{n}_i}(p_i)$	

These choices preserve a d-level structure in each A_i, because of the inductive step $A_i = \mathcal{A}_{\boldsymbol{n}_i}(f_i) \ \Rightarrow\ A_{i+1} = \mathcal{A}_{\boldsymbol{n}_{i+1}}(f_{i+1})$:

$$\begin{aligned} A_{i+1} &= R_i\, A_i\, R_i^H \\ &= K_{\boldsymbol{n}_i}\ \mathcal{A}_{\boldsymbol{n}_i}(p_i^2 f_i)\, K_{\boldsymbol{n}_i}^T \\ &= K_{\boldsymbol{n}_i} Q_{\boldsymbol{n}_i}\ \mathrm{Diag}\Big((p_i^2 f_i)(\boldsymbol{w}^{[\boldsymbol{n}_i]})\Big)\ Q_{\boldsymbol{n}_i}^H K_{\boldsymbol{n}_i}^T \\ &= Q_{\boldsymbol{n}_{i+1}}\ \mathrm{Diag}\Big(\big[\Psi_d(p_i^2 f_i)\big](\boldsymbol{w}^{[\boldsymbol{n}_{i+1}]})\Big)\ Q_{\boldsymbol{n}_{i+1}}^H \end{aligned}$$

(see [24, 40] for details on last equality, where we assume all functions to be even in the tau case). This leads to the following

PROPOSITION 3. – *With notations of Table* 3, *let* t, $m \in \mathbb{N}$ *be such that* $0 < m < t$ *and let* f_0, $p_i \in \mathcal{F}_d$ *be* 2π*-periodic functions (even in tau case) for* $i = 0, \ldots, m-1$. *Define also* $A_{i+1} = R_i A_i (R_i)^H$ *for* $i = 0, \ldots, m-1$. *Then it holds* $A_i = \mathcal{A}_{\boldsymbol{n}_i}(f_i)$, $i = 0, \ldots, m$, $\{f_i\}_{i=0}^m$ *being defined as*

$$f_{i+1} = \Psi_d(p_i^2 f_i) \tag{12}$$

and $\Psi_d : \mathcal{F}_d \longrightarrow \mathcal{F}_d$ *defined as*

$$\Psi_d\big[g(\boldsymbol{x})\big] = \frac{1}{2^d} \sum_{\boldsymbol{s} \in \{0;1\}^d} g\Big(\frac{\boldsymbol{x}}{2} + \pi\boldsymbol{s}\Big). \tag{13}$$

Moreover the projector R_i *is full-rank if* $\Psi_d[p_i^2(\boldsymbol{x})] > 0$ *holds for every* $\boldsymbol{x}$.

4.2 – *Computational requirement*

As we stated in § 1, we are interested in linear systems generated by a polynomial $f_0 \in \mathbb{R}_{\boldsymbol{z_0}}$: this means that the first matrix of the sequence $\{A_i\}_{i=0}^m$ is structured (i.e. $A_0 = \mathcal{A}_{\boldsymbol{n}_0}(f_0) \in \mathcal{G}(Q_{\boldsymbol{n}_0})$) and sparse ($f_0 \in \mathbb{R}_{\boldsymbol{z}_0}$), while all A_i are still structured ($A_i = \mathcal{A}_{\boldsymbol{n}_i}(f)$ for § 4.1), even if they can be dense. We assert that under the simple assumption that all the projector's generators p_i are polynomials, then all the matrices A_i have a number of non-zero diagonals lower than a constant independent of $\boldsymbol{n}$ and m. As a consequence, it is possible to guarantee that each iteration of the AMG (2) has a cost proportional to $N(\boldsymbol{n})$.

To show this result we have to analyze in detail how Ψ_d acts on polynomials: applying Ψ_d to a generic polynomial $\sum_{\boldsymbol{c}_1 \leqslant \boldsymbol{k} \leqslant \boldsymbol{c}_2} a_{\boldsymbol{k}} \mathrm{e}^{\boldsymbol{i}\langle \boldsymbol{k}|\boldsymbol{x}\rangle}$ we obtain

$$\begin{aligned} \Psi_d\left(\sum_{\boldsymbol{c}_1 \leqslant \boldsymbol{k} \leqslant \boldsymbol{c}_2} a_{\boldsymbol{k}} \mathrm{e}^{\boldsymbol{i}\langle \boldsymbol{k}|\boldsymbol{x}\rangle}\right) &= \sum_{\boldsymbol{c}_1 \leqslant \boldsymbol{k} \leqslant \boldsymbol{c}_2} \left(\frac{1}{2^d} \sum_{\boldsymbol{s} \in \{0;1\}^d} \mathrm{e}^{\pi \boldsymbol{i}\langle \boldsymbol{k}|\boldsymbol{s}\rangle}\right) a_{\boldsymbol{k}} \mathrm{e}^{\boldsymbol{i}\langle \frac{\boldsymbol{k}}{2}|\boldsymbol{x}\rangle} \\ &= \sum_{\left\lceil \frac{\boldsymbol{c}_1}{2} \right\rceil \leqslant \boldsymbol{k} \leqslant \left\lfloor \frac{\boldsymbol{c}_2}{2} \right\rfloor} a_{2\boldsymbol{k}} \mathrm{e}^{\boldsymbol{i}\langle \boldsymbol{k}|\boldsymbol{x}\rangle} \end{aligned} \tag{14}$$

with componentwise floor and ceiling. The second equality follows from an orthogonality result:

$$\sum_{\boldsymbol{s} \in \{0;1\}^d} \mathrm{e}^{\pi \boldsymbol{i}\langle \boldsymbol{k}|\boldsymbol{s}\rangle} = \begin{cases} 2^d & \text{if } k_r \equiv 0 \bmod 2 \quad \forall r \in \{1, \ldots, d\}, \\ 0 & \text{if } \exists\, \overline{r} \in \{1, \ldots, d\} \text{ s.t. } k_{\overline{r}} \equiv 1 \bmod 2. \end{cases}$$

In particular we get

$$\Psi_d(\mathbb{R}_{\boldsymbol{\eta}}[\boldsymbol{x}]) \subseteq \mathbb{R}_{\left\lfloor \frac{\boldsymbol{\eta}}{2} \right\rfloor}[\boldsymbol{x}]. \tag{15}$$

PROPOSITION 4 ([1]). – *Under the same assumptions of Proposition* 3, *let* p_i *be polynomials such that* $p_i \in \mathbb{R}_{\boldsymbol{q}_i}$ *and assume* $f_0 \in \mathbb{R}_{\boldsymbol{z}_0}$. *Then the following properties hold:*

1. *each* f_i *is a polynomial;*
2. $f_i \in \mathbb{R}_{\boldsymbol{z}_i}$, *being* $\boldsymbol{z}_{i+1} \leqslant \boldsymbol{q}_i + \left\lfloor \dfrac{\boldsymbol{z}_i}{2} \right\rfloor$;
3. $\boldsymbol{z}_i \leqslant \max\{\boldsymbol{z}_0, 2\boldsymbol{q}_j \,:\, 0 \leqslant j < i\}$;
4. *if* $\boldsymbol{q}_i = \boldsymbol{q}$ *for each i, then* $\boldsymbol{z}_i \leqslant 2\boldsymbol{q}$ *for i large enough (it depends on* $\boldsymbol{z}_0 - 2\boldsymbol{q}$*).*

Here multiindex inequalities and max*imum hold componentwise as usual.*

From Proposition 4, it follows that if $p_i \in \mathbb{R}_{\boldsymbol{q}}$ for all i (we will show in § 5.2 that this happen in our case) then the number of nonzero diagonals of the

coefficient matrix at each multigrid recursion level is lower than $\prod_{r=1}^{d} 2\boldsymbol{q}_r + 1$. Therefore, it is easy to prove that, with a suitable choice of the smoother, one iteration of the algorithm $\mathcal{AMG}$ (2) is linear in N_0. This is done in the following lemma.

LEMMA 5 ([1]). – *Under the same assumptions of Proposition* 3 *and if*

1. $p_i \in \mathbb{R}_{\boldsymbol{q}}$, *for* $i = 0, \ldots, m-1$,
2. *pre and post-smoother are Richardson with* $\nu_i + \vartheta_i \leqslant h \cdot (2^d - 1)^i$, *where* $h \geqslant 1$ *is a constant,*

then each iteration of the $\mathcal{AMG}$ *in* (2) *has a computational cost linear in* N_0.

The above lemma does not consider the cost $C_{\{A_i\}_{i=1}^m}$ of calculating the matrices $\{A_i\}_{i=1}^m$, i.e., of calculating the functions $\{f_i\}_{i=1}^m$. This can be done before the first iteration with logarithmic cost in N_0. Indeed, from Proposition 3 and from equation (14), it follows that we can get the coefficients of each f_{i+1} by computing the product $p_i^2 f_i$, where $p_i \in \mathbb{R}_{\boldsymbol{q}}$ and the relative $f_i \in \mathbb{R}_{\max\{\boldsymbol{z}_0, 2\boldsymbol{q}\}}$ holds for each i. Since $\boldsymbol{q}$ and $\boldsymbol{z}_0$ do not depend on $\boldsymbol{n}_0$ but only on f_0, and we have to repeat this calculation $m-1$ times, it follows that there exists a constant $c(f_0)$ such that $C_{\{A_i\}_{i=1}^m} \leqslant c(f_0) \cdot m$ and m is less than $\log(N(\boldsymbol{n}_0))$.

Concluding, under the assumptions of § 4.1 (Table 3) and of Lemma 5 ($p_i \in \mathbb{R}_{\boldsymbol{q}}$ and $\nu_i + \vartheta_i \leqslant h\,(2^d - 1)^i$), we know that each iteration of $\mathcal{AMG}$ has linear cost, but it still remains to show the convergence and to check that the error reduction is constant with respect to the dimension $\boldsymbol{n}_0$. For this purpose we use the Theorem 1 and we will show the validity of its hypotheses using linear algebra and functional tools (this is done in § 5).

5. – Convergence and optimality

In this section we show how to ensure (8a,b,c) that imply the convergence of our AMG (§ 5.1) and how to satisfy (10) that implies the optimality (§ 5.2).

5.1 – *Convergence*

In the following proposition we consider smoothers at a fixed recursion level and therefore, in order to simplify the notation, we do not use the grid index i.

PROPOSITION 6 ([1]). – *Let $A = \mathcal{A}_{\boldsymbol{n}}(f)$ being $f \in \mathcal{F}_d$ nonnegative and not identically zero and let ω be a real number. If we define $S = I_{N(\boldsymbol{n})} - \omega A$, then*

$$\|S^{\nu}\boldsymbol{x}\|_A^2 \leqslant \|\boldsymbol{x}\|_A^2 - \alpha\,\|S^{\nu}\boldsymbol{x}\|_{A^2}^2, \qquad (\alpha \geqslant 0), \qquad \forall \boldsymbol{x} \in \mathbb{R}^{N(\boldsymbol{n})} \tag{16}$$

holds with $\nu \in \mathbb{N}$ if one of the following two is satisfied:

1. $0 \leqslant \omega \leqslant 1/\|f\|_\infty$ *and* $\alpha \leqslant 2\,\omega\,\nu$;
2. $1/\|f\|_\infty < \omega \leqslant 2/\|f\|_\infty$ *and*

$$\alpha \leqslant \min\left\{2\,\omega\,\nu,\ \frac{1}{\|f\|_\infty}\left[\frac{1}{\left(1-\omega\,\|f\|_\infty\right)^{2\nu}} - 1\right]\right\}.$$

Moreover if we define $\widetilde{S} = I_{N(\boldsymbol{n})} - \omega A$, then

$$\|\widetilde{S}^{\vartheta}\boldsymbol{x}\|_A^2 \leqslant \|\boldsymbol{x}\|_A^2 - \beta\,\|\boldsymbol{x}\|_{A^2}^2, \qquad (\beta \geqslant 0), \qquad \forall \boldsymbol{x} \in \mathbb{R}^{N(\boldsymbol{n})} \tag{17}$$

holds with $\vartheta \in \mathbb{N}$ if $0 \leqslant \omega \leqslant 2/\|f\|_\infty$ and

$$\beta \leqslant \frac{1 - \left(1-\omega\,\|f\|_\infty\right)^{2\vartheta}}{\|f\|_\infty}$$

are satisfied.

To prove the convergence it remains to prove the *approximation property* (8c).

DEFINITION 7. – *Let $x \in \mathbb{R}^d$, the set of all* corners *is given by*

$$\Omega(x) = \left\{y_j \mid (y_j)_i \in \{x_i,\ \pi + x_i\},\ i = 1,\dots,d,\ j = 1,\dots,2^d\right\}$$

which has cardinality 2^d. The set of mirror points *(see* [24]*) is denoted as*

$$\mathcal{M}(x) = \Omega(x) \setminus \{x\},$$

e.g. for $d = 1$ it is $\mathcal{M}(x) = \{\pi + x\}$.

Furthermore, we define

$$g[x] = (g(y_1),\dots,g(y_{2^d})), \qquad y_j \in \Omega(x), \qquad j = 1,\dots,2^d \tag{18}$$

and its Euclidean norm is $\|g[x]\|_2^2 = \sum_{j=1}^{2^d} g(y_j)^2 = \sum_{y\in\Omega(x)} g(y)^2$.

We denote by S the fundamental set, which is $[0,\pi]^d$ for τ algebra and $[0,2\pi]^d$ for circulant and Hartley algebra. To ensure the validity of the key assumptions (8c) we define p_i, the generating function of the projector, according to the following conditions.

PROJECTOR CONDITIONS. – *Let x_0 be the unique zero of f_i in S, $\forall x \in S$ we choose p_i such that*

$$\limsup_{x \to x_0} \left| \frac{p_i(y)}{f_i(x)} \right|^{\theta} < +\infty, \qquad y \in \mathcal{M}(x),\ i = 0, \ldots, m-1, \tag{19}$$

$$0 < \sum_{y \in \Omega(x)} p_i^2(y), \qquad i = 0, \ldots, m-1, \tag{20}$$

where $\theta = 2$ for TGM optimality and $\theta = 1$ for V-cycle optimality.

The TGM optimality was proved in [23] for $d = 1$, in [24] for $d = 2$ and in [36] for all $d \geq 1$. The V-cycle optimality was proved in [2] for $d = 1$ and in [1] for $d > 1$.

From (20) the projector R_i is full rank. In the following proposition we prove that with the conditions (19) and (20) the assumption (8c) is verified and therefore the AMG defined in Section 4.1 is convergent.

PROPOSITION 8 ([2, 1]). – *Let $A = \mathcal{A}_{\boldsymbol{n}}(f)$ with $\mathcal{A} \in \{\mathcal{C}, \mathcal{H}, \tau\}$ and f be a d-variate nonnegative trigonometric polynomial with a single zero in the fundamental set. Let $R = K_{\boldsymbol{n}} \cdot \mathcal{A}_{\boldsymbol{n}}(p)$ as in Table (3) and define $CGC = I_{N(\boldsymbol{n})} - R^H (RAR^H)^{-1} RA$ as in (4). If $p(x)$ fulfils (19) and (20) then there exists a real and positive value γ such that*

$$\|CGC\, \boldsymbol{x}\|_A^2 \leqslant \gamma \|\boldsymbol{x}\|_{A^2}^2, \qquad \boldsymbol{x} \in \mathbb{C}^{N(\boldsymbol{n})}. \tag{21}$$

5.2 – *Optimality*

In order to prove (10), in Propositions 6 and 8 we showed that values α_i, β_i and γ_i exist in $(0, +\infty)$ such that they ensure (8) and then (6) (i.e., the AMG (2) is convergent); such values depend on the function f_i (γ_i depends on p_i too) but not on the dimensions $\boldsymbol{n}_i$ neither on the number of grids m used in algorithm (2). Therefore, the (10) is ensured if $\{f_i\}$ converges uniformly to a function f_* and the constants α_*, β_* and γ_* related to f_* are positive. In this paragraph we will use the symbol $\xrightarrow{u}$ to represent uniform function convergence (with respect to the usual sup norm), e.g. $f_i \xrightarrow{u} f_*$.

In the following we will consider generating functions as

$$f(\boldsymbol{x}) = \mu \cdot \chi_{2\pi\mathbb{Z}^d}(\boldsymbol{x}) + \sum_{r=1}^{d} \left[1 - \cos(x_r)\right]^q \cdot \psi^{(r)}(\boldsymbol{x}) \tag{22}$$

being $q \in \mathbb{N} \setminus \{0\}$, $\mu \geqslant 0$, $\psi^{(r)} \in \mathcal{F}_d$ and f positive in $[-\pi, \pi)^d \setminus \{\boldsymbol{0}\}$ and vanishing with order $2q$ around $\boldsymbol{0}$, i.e., $\psi^{(r)}(\boldsymbol{0}) > 0$, $r = 1, \ldots, d$. We take a particular choice

for polynomials p_i:

$$p_i(\boldsymbol{x}) = \zeta_i \cdot \chi_{2\pi\mathbb{Z}^d}(\boldsymbol{x}) + c \cdot \prod_{r=1}^{d} \left[1 + \cos(x_r)\right]^q \qquad (\zeta \in \mathbb{R}). \tag{23}$$

We emphasize that this choice (23) implies that all the functions f_i share the structure (22).

LEMMA 9. – *Assume that f_0 takes the form* (22):

$$f_0(\boldsymbol{x}) = \mu_0 \cdot \chi_{2\pi\mathbb{Z}^d}(\boldsymbol{x}) + \sum_{r=1}^{d} \left[1 - \cos(x_r)\right]^q \cdot \psi_0^{(r)}(\boldsymbol{x}), \tag{24}$$

with $\mu_0 \in \mathbb{R}$ and $\psi_0^{(r)} \in \mathcal{F}_d$ for $r = 1, \ldots, d$, and let p_i and $\{f_i\}_{i\in\mathbb{N}}$ be defined as in (23) *and* (12) *respectively ($f_{i+1} = \Psi_d(p_i^2 f_i)$).*

Then it holds that also f_i takes the form (22) *for all i, in detail*

$$f_i(\boldsymbol{x}) = \mu_i \cdot \chi_{2\pi\mathbb{Z}^d}(\boldsymbol{x}) + \sum_{r=1}^{d} \left[1 - \cos(x_r)\right]^q \cdot \psi_i^{(r)}(\boldsymbol{x}), \qquad i \in \mathbb{N} \tag{25}$$

with $\{\mu_i\}_{i\in\mathbb{N}}$ and $\{\psi_i^{(r)}\}_{i\in\mathbb{N}}$, $r = 1, \ldots, d$, defined as

$$\begin{cases} \mu_{i+1} & = 2^{-d}\left(\zeta_i + 2^{qd}c\right)^2 \mu_i \\ \psi_{i+1}^{(r)}(\boldsymbol{x}) & = c^2\, \Psi_d\left[(\phi_{\boldsymbol{q}^{[r]}} \psi_i^{(r)})(\boldsymbol{x})\right] \quad \stackrel{def}{=} c^2\, \Phi_{\boldsymbol{q}^{[r]}}\left[\psi_i^{(r)}(\boldsymbol{x})\right] \end{cases} \qquad i \in \mathbb{N}, \tag{26}$$

where

$$\phi_{\boldsymbol{q}^{[r]}}(\boldsymbol{x}) = \left[\frac{1 + \cos(x_r)}{2}\right]^q \prod_{\substack{j=1,\ldots,d \\ j\neq r}} \left[1 + \cos(x_j)\right]^{2q} \tag{27}$$

and $\boldsymbol{q}^{[r]} = 2q\boldsymbol{e} - q\boldsymbol{e}_r$ is the degree of $\phi_{\boldsymbol{q}^{[r]}}$, where $\boldsymbol{e}_r$ is the r-th vector of the canonical basis of $\mathbb{R}^d$ and $\boldsymbol{e} = (1, \ldots, 1) \in \mathbb{N}^d$.

REMARK 10. – Choice (23) for p_i is fundamental to get uniform convergence of $\{f_i\}$ since it shows that the structure (22) is kept at each level, it is then enough to show $\psi_i^{(r)} \xrightarrow{u} \psi_*^{(r)}$. Moreover, choice (23) satisfies (19) and (20) (refer to the following Lemma 11) and therefore the Proposition 8. Of course a different choice for p_i could still satisfy (19) and (20) (see e.g. [24, 40]) but no longer to preserve the structure (25) for $\{f_i\}$.

LEMMA 11. – *Let f be defined as in* (22) *and p_i as in* (23) *for $i = 0, \ldots, m-1$. Then* (19) *and* (20) *hold true.*

Therefore, under the same assumptions of Lemma 11, by Proposition 8 it exists $\gamma_i > 0$ such that (8c) holds true.

From Lemma 9 and Remark 10 we obtain the main tools in order to show that $f_i \xrightarrow{u} f_*$: it simply follows from $\psi_i^{(r)} \xrightarrow{u} \psi_*^{(r)}$, but we still have to prove that the latter is true. The key is equation (26), which defines the d sequences $\{\psi_i^{(r)}\}_{i\in\mathbb{N}}$, $r = 1, \ldots, d$.

The proof will act as follows: from Proposition 4 we have that $\{\partial\psi_i^{(r)}\}_i$ is bounded by $\boldsymbol{q}^{[r]}$ definitely, and by equations (26) and (13) we have that each step $\psi_i^{(r)} \to \psi_{i+1}^{(r)}$ is linear. Convergence $\psi_i^{(r)} \xrightarrow{u} \psi_*^{(r)}$ can be shown in the finite dimension vector space $\mathbb{R}_{\boldsymbol{q}^{[r]}}[\boldsymbol{x}]$ by using linear algebra tools (mainly resorting to the Perron-Frobenius theorem [47] applied to the matrix of the transformation having dominant eigenvalue equal to 1), and then $f_i \xrightarrow{u} f_*$ holds true with

$$f_*(\boldsymbol{x}) = \mu_* \cdot \chi_{2\pi\mathbb{Z}^d}(\boldsymbol{x}) + \sum_{r=1}^{d} \left[1 - \cos(x_r)\right]^q \cdot \psi_*^{(r)}(\boldsymbol{x})$$

whenever $\mu_i \to \mu_*$ holds in $\mathbb{R}$.

From a technical point of view, it is easier to work with $\mathbb{C}_{\boldsymbol{q}^{[r]}}[\boldsymbol{x}]$ than with $\mathbb{R}_{\boldsymbol{q}^{[r]}}[\boldsymbol{x}]$, being $\mathbb{C}_{\boldsymbol{\eta}}[\boldsymbol{x}]$ the vector space of d-variate trigonometric polynomials with complex coefficients and degree up to $\boldsymbol{\eta} \in \mathbb{N}^d$

$$\mathbb{C}_{\boldsymbol{\eta}}[\boldsymbol{x}] = \left\{ \sum_{|\boldsymbol{k}|\leqslant\boldsymbol{\eta}} a_{\boldsymbol{k}} \mathrm{e}^{\boldsymbol{i}\langle\boldsymbol{k}|\boldsymbol{x}\rangle} \ \text{ s.t. } \ a_{\boldsymbol{k}} \in \mathbb{C} \right\}$$

since it is possible to use its canonical basis $\mathcal{B}_{\eta}[\boldsymbol{x}] = \bigcup_{|\boldsymbol{k}|\leqslant\eta} \{e^{\boldsymbol{i}\langle\boldsymbol{k}|\boldsymbol{x}\rangle}\}$. Of course we need (see (26)) the Fourier coefficients of $\phi_{\boldsymbol{q}^{[r]}}(\boldsymbol{x})$ to get the entries of the matrix $c^2 M(\Phi_{\boldsymbol{q}^{[r]}})$ that represent the transform $\psi_i^{(r)} \to \psi_{i+1}^{(r)} = c^2 \Phi_{\boldsymbol{q}^{[r]}}(\psi_i^{(r)})$ with respect to $\mathcal{B}_{\boldsymbol{q}^{[r]}}[\boldsymbol{x}]$.

LEMMA 12 ([2, 1]). – *For $d = 1$ the Fourier coefficients $b_k^{(q)}$ of $\phi_{\boldsymbol{q}^{[1]}} \equiv \phi_q$ are given by*

$$b_k^{(q)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left[\frac{1 + \cos(x)}{2}\right]^q \mathrm{e}^{-\boldsymbol{i}kx}\, \mathrm{d}x = \begin{cases} \dfrac{1}{4^q}\binom{2q}{q+k} & \text{if } |k| \leqslant q \\ 0 & \text{if } |k| > q \end{cases}. \tag{28}$$

For $d > 1$ the Fourier coefficients $b_{\boldsymbol{k}}^{(\boldsymbol{q}^{[r]})}$ of $\phi_{\boldsymbol{q}^{[r]}}$ (given by (27)) are

$$b_{\boldsymbol{k}}^{(\boldsymbol{q}^{[r]})} = 2^{2q(d-1)} \prod_{s=1}^{d} b_{k_s}^{(q_s^{[r]})} \in \begin{cases} (0; +\infty) & \text{if } |\boldsymbol{k}| \leqslant \boldsymbol{q}^{[r]}, \\ \{0\} & \text{otherwise}, \end{cases} \qquad \boldsymbol{k} \in \mathbb{Z}^d$$

thus

$$\phi_{\boldsymbol{q}^{[r]}}(\boldsymbol{x}) = \sum_{|\boldsymbol{k}| \leqslant \boldsymbol{q}^{[r]}} b_{\boldsymbol{k}}^{(\boldsymbol{q}^{[r]})} e^{\boldsymbol{i}\langle \boldsymbol{k}|\boldsymbol{x}\rangle}$$

holds true.

The linear dependence of $\psi_{i+1}^{[1]}$ on $\psi_i^{[1]}$ is exploited by the following

PROPOSITION 13 ([2, 1]). – *Assume $d \in \mathbb{N} \setminus \{0\}$, $r \in \{1, \ldots, d\}$ and let $M(\Phi_{\boldsymbol{q}^{[r]}})$ be the matrix related to the linear function $\Phi_{\boldsymbol{q}^{[r]}} : \mathbb{C}_{\boldsymbol{q}^{[r]}}[\boldsymbol{x}] \longrightarrow \mathbb{C}_{\boldsymbol{q}^{[r]}}[\boldsymbol{x}]$ with respect to the basis $\mathcal{B}_{\boldsymbol{q}^{[r]}}[\boldsymbol{x}] = \bigotimes_{s=1}^{d} \mathcal{B}_{(\boldsymbol{q}^{[r]})_s}[x_s]$. The following three properties hold:*

1. $[M(\Phi_{\boldsymbol{q}^{[r]}})]_{\boldsymbol{i},\boldsymbol{j}} \geqslant 0$ *and*

$$M(\Phi_{\boldsymbol{q}^{[r]}}) = 2^{2q(d-1)} \cdot \bigotimes_{s=1}^{d} M\big(\Phi_{(\boldsymbol{q}^{[r]})_s}\big), \qquad r \in \{1, \ldots, d\} \tag{29}$$

2. *the dominant eigenvalue of $M(\Phi_{\boldsymbol{q}^{[r]}})$ is $2^{2q(d-1)-d}$ and it is simple;*
3. *there exists a dominant eigenvector $\boldsymbol{a}^{(\boldsymbol{q}^{[r]})} \in \bigotimes_{s=1}^{d} \mathbb{R}^{2(\boldsymbol{q}^{[r]})_s+1}$ (to which we refer with the usual d-index notation, assuming the s-th index to range in $\{-(\boldsymbol{q}^{[r]})_s, \ldots, (\boldsymbol{q}^{[r]})_s\}$) related to the dominant eigenvalue such that*
   (a) $\boldsymbol{a}_{\boldsymbol{j}}^{(\boldsymbol{q}^{[r]})} = 0$ *if* $|\boldsymbol{j}_s| = (\boldsymbol{q}^{[r]})_s$ *al least for an* $s \in \{1, \ldots, d\}$;
   (b) $\boldsymbol{a}_{\boldsymbol{j}}^{(\boldsymbol{q}^{[r]})} > 0$ *if* $|\boldsymbol{j}| < \boldsymbol{q}^{[r]}$;
   (c) $\sum_{|\boldsymbol{j}| < \boldsymbol{q}^{[r]}} \boldsymbol{a}_{\boldsymbol{j}}^{(\boldsymbol{q}^{[r]})} = 1$.

Moreover the polynomial $\mathcal{B}_{\boldsymbol{q}^{[r]}}[\boldsymbol{x}] \cdot \boldsymbol{a}^{(\boldsymbol{q}^{[r]})} \in \mathbb{C}_{\boldsymbol{q}^{[r]}}[\boldsymbol{x}]$, whose components with respect to $\mathcal{B}_{\boldsymbol{q}^{[r]}}[\boldsymbol{x}]$ are $\boldsymbol{a}^{(\boldsymbol{q}^{[r]})}$, is equal to $\prod_{s=1}^{d} \Big(\mathcal{B}_{(\boldsymbol{q}^{[r]})_s}[x_s] \cdot \boldsymbol{a}^{((\boldsymbol{q}^{[r]})_s)}\Big)$, and it is real and positive in $[-\pi, \pi)^d$.

REMARK 14. – In the following we fix the restrictor parameter in (23) as $c = 2^{\frac{d}{2}+q(1-d)}$. Therefore from Proposition 13 the maximum eigenvalue of the restriction of $c^2 \Phi_{\boldsymbol{q}^{[r]}}$ to $\mathbb{C}_{\boldsymbol{q}^{[r]}}[\boldsymbol{x}]$ is 1, and it is simple. Furthermore, a related eigenvector is $\mathcal{B}_{\boldsymbol{q}^{[r]}}[\boldsymbol{x}] \cdot \boldsymbol{a}^{(\boldsymbol{q}^{[r]})}$ and

$$\psi_i^{(r)} \xrightarrow{u} \psi_*^{(r)} = \psi^{(r)}(\boldsymbol{0})\, \mathcal{B}_{\boldsymbol{q}^{[r]}}[\boldsymbol{x}] \cdot \boldsymbol{a}^{(\boldsymbol{q}^{[r]})},$$

since $c^2 \Phi_{\boldsymbol{q}^{[r]}}$ does not change the value at the origin ($c^2 \Phi_{\boldsymbol{q}^{[r]}}(g(\boldsymbol{0})) = g(\boldsymbol{0})$ for each $g \in \mathcal{F}_d$) and $\mathcal{B}_{\boldsymbol{q}^{[r]}}[\boldsymbol{x}] \cdot \boldsymbol{a}^{(\boldsymbol{q}^{[r]})}$ is the only eigenvector of $c^2 \Phi_{\boldsymbol{q}^{[r]}}$ (apart rescaling) related to the eigenvalue 1 that is dominant and simple.

Finally, summarizing all the proposed results we obtain the following property of optimality for the algorithm $\mathcal{AMG}$ in (2).

THEOREM 15 ($\mathcal{AMG}$ optimality, [2, 1]). – *Let $t, m, q \in \mathbb{N} \setminus \{0\}$ with $t > m$ and assume $\boldsymbol{z} \in (\mathbb{N} \setminus \{0\})^d$ and $f \in \mathbb{R}_{\boldsymbol{z}}$ as in* (22):

$$f(\boldsymbol{x}) = \mu \cdot \chi_{2\pi\mathbb{Z}^d}(\boldsymbol{x}) + \sum_{r=1}^{d} \left[1 - \cos(x_r)\right]^q \cdot \psi^{(r)}(\boldsymbol{x})$$

such that f is positive valued in $[-\pi, \pi)^d \setminus \{\mathbf{0}\}$ and vanishes around $\mathbf{0}$ with order $2q$.

To solve the linear system $\mathcal{A}_{\boldsymbol{n}}(f)\boldsymbol{x} = \boldsymbol{b}$, $\mathcal{A} \in \{\mathcal{C}, \tau, \mathcal{H}\}$ ($\mu = 0$ if $\mathcal{A} = \tau$) consider the algorithm $\mathcal{AMG}$ (2) *with the assumptions of Table* 3 *and with the following choices for $i = 0, \ldots, m-1$:*

1. *the projectors p_i are as in* (23):

$$p_i(\boldsymbol{x}) = \zeta_i \cdot \chi_{2\pi\mathbb{Z}^d}(\boldsymbol{x}) + c \cdot \prod_{r=1}^{d} \left[1 + \cos(x_r)\right]^q, \qquad c = 2^{\frac{d}{2}+q(1-d)},$$

   ($\zeta_i = 0$ if $\mathcal{A} = \tau$);

2. *the smoothers $\mathcal{S}_i$, $\widetilde{\mathcal{S}}_i$ are the Richardson method with $\nu_i + \vartheta_i = 1$ and the relaxation parameter chosen according to Proposition* 6.

Then at least one in (10) *is satisfied and thanks to Theorem 1 there exists a constant* $\mathrm{const}(f) < 1$ *such that*

$$\|AMG_0\|_A \leqslant \mathrm{const}(f) < 1,$$

with AMG_0 defined in (5). *In particular* $\mathrm{const}(f)$ *depends only on f (i.e. on q, μ and $\{\psi_r\}_r$) but not on $\boldsymbol{n}$. Moreover each step of $\mathcal{AMG}$ has linear computational cost (i.e. $O(N(\boldsymbol{n}))$) and therefore the algorithm $\mathcal{AMG}$ is optimal in the sense of* [3].

5.3 – *Generalizations*

In this short subsection we mention some issues that could be considered, together with useful generalizations of the previous results obtained for circulant, tau, and Hartley linear systems.

- With regard to the convergence theory, it is worth observing that Theorem 15 ensures $\mathcal{AMG}$ optimality if just one smoother iteration is performed: if we choose $\nu_i + \vartheta_i$ larger on each grid i, we improve the convergence factor

and optimality holds whenever $\nu_i + \vartheta_i \leqslant \text{const}(2^d - 1)^i$ according to Lemma 5. Furthermore, the optimality results of Theorem 15 can be extended to the linear system $\mathcal{A}_{\boldsymbol{n}}(\tilde{f})\boldsymbol{x} = \boldsymbol{b}$ generated from a function $\tilde{f}$ with a zero shifted in $\bar{\boldsymbol{x}} \in \mathbb{R}^d$.

- In the Toeplitz setting a careful modification of the cutting matrix allows to preserve the (multilevel) Toeplitz structure, with an associated reduction of the computational cost.

- So far multigrid convergence and optimality have been completely investigated in the matrix algebra case. The Toeplitz case is of higher interest and challenging, both for the definition of a practical multigrid technique and for the theoretical convergence analysis. Many different projector strategies are proposed in literature for design multigrid methods for Toeplitz matrices [23, 24, 36, 30, 9, 2]. Nevertheless, the optimality is proved only for the TGM case in [36], while in [9] a level independency property is proved for multiple zeros of order up to two. In [36] the optimality of TGM for matrix algebra is extended also to finite difference discretizations of elliptic partial differential equations (PDE).

- The use of order relationships in the space of Hermitian matrices, which can be proved using the symbol, are of paramount importance for extending the proof of convergence from the pure shift invariant case (Toeplitz and matrix algebra multilevel structures) to locally variant structures as those appearing in the approximation of partial differential equations (PDEs) via local methods, i.e., Finite Differences and Finite Element techniques. In that setting the symbol is available for very large classes of hidden structures involving virtually all approximations by local methods of integro-differential operators and the associated algebra [43, 37, 39, 4].

- With regard to the relations with the PDE setting, it is interesting to remind the link between our matrix algebra approach and the Local Fourier Analysis approach (see [17]). In particular, in [17] it is shown that when f comes from an elliptic PDE, conditions (19) and (20) on the projector are equivalent to the classical 'order' conditions in [27, 50]. More in general, the simple message is that our approach is more abstract and can be considered a matrix theoretic extension of the Local Fourier analysis; as a specific case, we can also deal with integral equations, as those appearing in the image restoration setting [16].

- The results of the previous section can be easily adapted to the structures belonging to the multilevel discrete cosine algebra (DCT), by employing a proper cutting matrix for preserving the structure; see [11, 12, 42]. Also in

this case, the interest is related to the use of the cosine algebra in imaging when dealing with Neuman or reflecting boundary conditions and symmetric operators: it is useful to remind that the more accurate anti-reflective boundary conditions [38] are strongly related with the same sine transform associated with the multilevel Tau algebra.

- The size-reduction strategy in the recursive multigrid process can be generalized, allowing a very general approach as shown in [21] for $d = 1$. The general case of $d > 1$ is not difficult to extend, with potential application to aggregation methods and to special signal restoration problems.
- The approach based on the knowledge of the symbol is very powerful because it allows to select the subspaces in which the convergence is enhanced: this property is very welcome in the image restoration setting, where the convergence has to be avoided in the high frequency space where the noise lives, see [19]. Some details on this specific application, taken from [19, 20], are reported in the next § 6.

6. – Multigrid regularization

In this section we consider the classical de-blurring problem of noisy and blurred signals or images, which is usually modeled by a first kind integral equation. By using a proper approximation scheme and by imposing suitable boundary conditions (BCs), a linear system of the form

$$A\boldsymbol{x} + \boldsymbol{\xi} = \boldsymbol{b} \tag{30}$$

is obtained, where the vector $\boldsymbol{x}$ represents the unknown true object, $\boldsymbol{\xi}$ the noise, $\boldsymbol{b}$ the observed object (the blurred noisy version of $\boldsymbol{x}$) and A models the blurring phenomena, via the point spread function (PSF) that we assume spatially invariant. Moreover, for the sake of notational simplicity, we suppose that every involved object has the same size in each direction, and hence $\boldsymbol{x},\ \boldsymbol{b},\ \boldsymbol{\xi} \in \mathbb{R}^N$, while $A \in \mathbb{R}^{N\times N}$. The matrix A has a special d-level structure depending on the imposed BCs (see [26]), e.g., for zero Dirichlet BCs it is a d-level Toeplitz matrix, where in the case of standard two-dimensional images we have $d = 2$.

In the first part of the paper, multigrid methods have been defined in order to obtain a fast convergence and these strategies have been adapted in [8, 30, 15, 16], in order to deal with the regularized linear system arising from Tikhonov or Total Variation regularization techniques. In this section we discuss the regularizing multigrid introduced in [19] which combines numerical linear algebra requests (low complexity, exploitation of the structured matrices etc.) and regularization issues (a good precision at the optimal iteration). A theoretical analysis of the regularizing feature of such multigrid is provided in [20], while a

recent improvement by a wavelet soft-thresholding is proposed in [18]. After the seminal work [19], other researchers have investigate regularizing multigrid methods in cascadic or wavelets frameworks (see e.g. [22, 33]).

6.1 – *Low frequencies projection*

We give some arguments to explain the reasons why a multigrid approach can improve the regularization property of iterative methods like conjugate gradient (CG), Richardson, CG for normal equations (CGNE), or Landweber. When the PSF is space invariant and we impose BCs the coefficient matrix is generated by a function that is zero or close to zero in a (possibly large) neighborhood of $\{\pi\}^d$ and reaches the maximum value (which is 1 thanks to the normalization condition) at the origin. Therefore, since the ill-conditioned subspace is associated with small eigenvalues, this degenerating subspace has very large dimension (this characterizes the discretized ill-posed problems) and it essentially contains the high frequencies subspace where usually the noise lives (all these crucial spectral features are deduced from the study of the associated spectral symbol). As a consequence, the restoration error has the usual semi-convergence property: it decreases while we are working in the low frequencies subspace (at the beginning iterations), reaches a minimum, and then it increases, i.e, the approximations change to the worse, when for large values of the iteration we arrive to work in the (unfortunately large) ill-conditioned subspace, corresponding to the high frequencies in the context of blurring models.

In this section we do not use post-smoothing (step 7 in (2)) and so the presmoother is simply called smoother, that is chosen to be an iterative regularization method (CG, Landweber, etc.). In order to obtain an effective and fast method according to the conditions (19) and (20), we have to project the system into the high frequencies subspace because the latter is the space where the smoother is ineffective and where we would like to obtain a better approximation. Unfortunately, the high frequencies not only contain fundamental parts of the image (e.g. the non negligible high frequency portion of the edges) but also a substantial part of the noise. Therefore, as shown in [16], we obtain the noise amplification already after a few iterations and, consequently, we must resort to the Tikhonov regularization and to apply the algebraic multigrid to the regularized system. The approach proposed in [19], that we discuss here, employs a specialized multigrid directly as a regularizer. Instead of projecting into the high frequencies subspace, the idea is to project into a subspace where we can discriminate between the noise contribution and the details of the image. The latter important feature can be obtained via projection techniques employed in the geometric multigrid, i.e., we use low past filters, as proved in [20]. Therefore, the projection into the low frequencies subspace can be obtained as $R_i = K_{N_i}\ \mathcal{A}_{N_i}(p_i)$ with $p_i(x) = 1 + \cos(x)$ where $\mathcal{A}$ depends on the BCs. In this way

we force the smoother to solve better the problem in the subspace where there is less noise. We remark that by projecting into the low frequencies subspace we lose the optimality property of the algebraic multigrid, but now the method is used as a regularizer and not as a fast solver for algebraic systems. Therefore, the algorithm (2) is a regularizing multigrid taking

- $R_i = K_{N_i}\ \mathcal{A}_{N_i}(p_i)$ with $p_i(x) = 1 + \cos(x)$,
- $v_0 = 0$ and $v_i = 1$ for $i = 1, \ldots, m-1$,
- $\vartheta_i = 0$, for all $i = 0, \ldots, m-1$,
- $m = \log_2\big(\min_{r=1,\ldots,d}(n_r)\big) - 3$.

We denote by $\mathsf{MGM}(S, \lambda)$ one multigrid iteration with smoother S and λ recursive calls: $\lambda = 1$ is the V-cycle in (2), while $\lambda > 1$ means that the step 5 of algorithm (2) is applied λ times. Let $W(n)$ be the computational cost of one smoother iteration S, then the computational cost of one multigrid iteration is estimated as follows

$$C(\mathsf{MGM}(S,\lambda)) \approx \begin{cases} \frac{1}{3}W(N), & \lambda = 1, \\ W(N), & \lambda = 2, \\ 3W(N), & \lambda = 3. \end{cases} \tag{31}$$

See [19] for details.

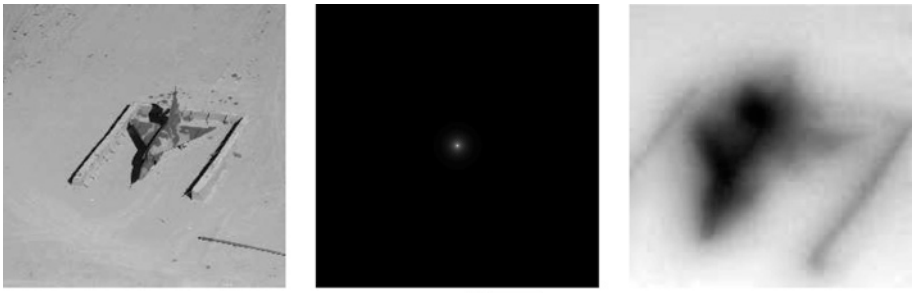

Fig. 1. – Images of the airplane and of the PSF with SNR = 100.

6.2 – *A numerical example*

We assume that the true image $\boldsymbol{x}$ is known so that, at each iteration j, we can evaluate and plot the relative error norm $e_j = \|\boldsymbol{x} - \boldsymbol{x}^{(j)}\|_2 / \|\boldsymbol{x}\|_2$ for every iterative regularization method. The algorithms are implemented in Fortran 90 using double precision, while the images and the graphs are made using Matlab. The PSF is created as a uniform sampling of 51 points of $e^{-\sqrt[4]{x^2+y^2}}$ in $[-20, 20] \times [-20, 20]$,

while for the noise we add a random vector ξ with uniform distribution and signal-noise-ratio (SNR) equal to 100. Figure 1 shows the original airplane image and its blurred and noisy version (the original picture is a portion of a larger image from which the blurred one is obtained). We apply periodic BCs and hence the matrix A in (30) is block circulant with circulant blocks. The smallest eigenvalue of the coefficient matrix is of the order of 10^{-3} and the matrix is positive definite. As a matter of fact, it is not strictly necessary to apply CG or Richardson to the normal equations, but it is recommended in order to obtain a good quality of the de-blurred image. It is interesting to observe that the proposed multigrid, with the simple Richardson (Rich) method as smoother, leads to a restoration error lower than the one obtained by CGNE or Landweber (see Table 4 and Figure 2). The two-level method (TL) corresponds to set $m = 1$. In this example the regularizing multigrid with a smoother for normal equations does not improve the quality of the restored image, because the value 0.112 is about the minimum error norm from a modelistic point of view for a least square regularization method.

TABLE 4. – Minimum error and the corresponding iteration number.

Method	$\min_{j=1,\dots}(e_j)$	$\arg\min_{j=1,\dots}(e_j)$
CG	0.1215	4
Rich	0.1218	8
TL(CG)	0.1132	8
TL(Rich)	0.1134	16
MGM(Rich,1)	0.1127	12
MGM(Rich,2)	0.1129	5
CGNE	0.1135	178
RichNE	0.1135	352

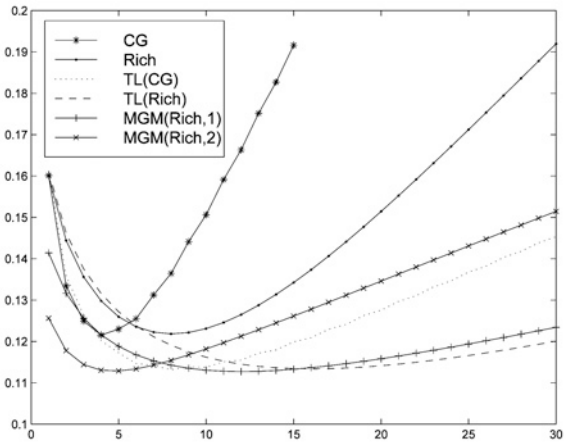

Fig. 2. – Relative error norm vs number of iterations.

7. – Conclusion

In this paper we have reviewed the V-cycle optimality of the proposed AMG for coefficient matrices generated by a real and nonnegative multivariate polynomial f and belonging to multilevel matrix algebras like circulant, tau or Hartley. The AMG considered here, which was introduced in [1, 2], is an extension of that proposed in [24]: now the projector has to satisfy the stricter conditions (19) and (20). Concerning the future work, the main point to investigate is the extension of this proof to multidimensional Toeplitz matrices. Preliminary results in this direction can be found in [9, 13] for the level independency in the case of generating function with zeros of order at most 2, and in [36] for the TGM algorithm and implicitly for the level independency.

In the second part we have presented a class of regularizing multigrid algorithms proposed in [19] whose features are the following: if it is compared with other regularizing procedures (CG, Richardson, Riley) applied directly to the system $A\boldsymbol{f} = \boldsymbol{g}$, then the curve of relative errors is much flatter, the quality of the reconstruction is higher, and the total arithmetic costs are similar; if it is compared with the best regularizing methods for the normal equations $A^TA\boldsymbol{f} = A^T\boldsymbol{g}$ (CGNE, Landweber, Tikhonov), then the accuracy of the restored image is similar (at most slightly better), the structure of the error curve is essentially the same, but the cost is greatly reduced. In every case, our multigrid can use normal equation methods only for the smoother, while the projection to a coarser grid is done always on the original coefficient matrix; this usually allows us to obtain a slightly better reconstruction and a reduced computational time compared with the best regularizing methods for the normal equations. For a theoretical proof of the regularizing feature of our method see [20]. Furthermore, we stress that the presented approach can be looked at as a general framework which has the potential of leading to several extensions and improvements. For instance, the parameter λ (the number of recursive calls) can be interpreted as a regularization parameter in order to obtain a direct (one-step!) multigrid regularization (see [19]).

Finally, in future we would like to investigate how our multigrid proposal can be used in connection with not least square methods such as total variation and Bayesian methods (see, e.g., [48]). Indeed, the nonconvex optimization (which characterizes all these quite expensive techniques) should be solved by some kind of iterative method which uses linearization, and our multigrid procedure can be applied at this level (instead of using preconditioning), not only for accelerating the procedures, but also for regularizing purposes.

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