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The Shape of a Glacier

R. J. Knops - Piero Villaggio

*Dedicated to Enrico Magenes who,
besides mathematics, loved mountains*

Abstract. – *During the last hundred years, several theories have been proposed aimed at describing the shape adopted by glaciers and their rate of downhill flow. Geophysicists, however, still cannot agree on the precise explanation of the phenomenon, and the dominant cause controlling the sliding of glaciers. This might be due to pressure arising during melting, or alternatively it might be due to stress concentration . We here derive a simple mathematical model that combines both effects.*

1. – Introduction

Glaciers may be regarded as vast rivers of ice slowly flowing downhill at a speed dependent upon the inclination of the slope, the thickness of the ice, and the nature of the rocky bed. In contrast to rivers, however, the ice gouges and scours the ground over which it flows, eroding immense amounts of soil, dislodging embedded boulders, and drastically fashioning the surrounding landscape. (c.p., Fiffe and Peter [2, p. 256]). Observation of these eroded regions inspired the Swiss geologist Agassiz to propose that the earth's climate experiences interglacial interludes. (c.p. Burrough [1, p. 23]).

Among many problems related to the creation, propagation, and fracture of glaciers, the most relevant for our study is that of explaining how the thickness of the glacier varies from head to base, where the base has a typical profile, called the "snout". The variable height of the longitudinal section in the direction of flow is accepted as the main effect that determines the glide speed of the glacier, its stress rate, the shape and formation of crevasses, and the onset of avalanches.

An early mathematical description of the evolving shape of a glacier, proposed by Finsterwalder [3], involved a nonlinear first order partial differential equation for the local height measured as a function both of the distance from the top and of time. The model, however, was criticised by Nye [7] who claimed that plasticity theory should be employed in the determination of the glacier's varying shape.

Weertman [11] objected to both approaches because the boundary condition in each assumes that the glacier's speed vanishes at the bed, and ignores the bed's supposed irregularity. Sliding is mainly caused by pressure melting (regelation) of the ice, and by stress concentrations in the vicinity of any perturbances on the bed. See also Hutter [4] for a derivation of other models and further references.

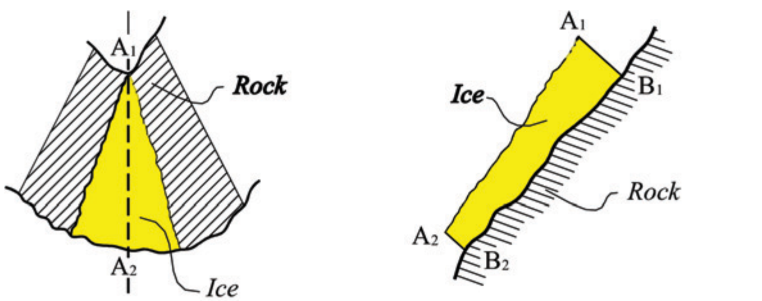
We here develop and analyse a simple one-dimensional mathematical model that predicts the geometric shape of the glacier's longitudinal cross-section due to the combined effects of the ice's mechanical resistance and the bed's roughness. The glacier's flow rate is supposed sufficiently slow to justify neglect of changes in the configuration with respect to time.

2. – Basic assumptions

Consider a large glacier occupying the valley between two summits (see Fig. 1a), and let the vertical plane intersecting the upper and lower edges at points A_1, A_2, respectively, cut the glacier longitudinally far from the banks.

The cross-section of the ice cut by the plane is depicted in Fig. 1b, where the curve A_1A_2 corresponds to the upper free surface of the glacier, and the curve B_1B_2 to that of the bed.

The aim is to determine the curve A_1A_2 given the mechanical properties of the ice (specific weight, and resistance to tensile stress), and the slope and roughness of the bed. The resistance of the ice and roughness of the bed, here both included in the discussion, are usually treated separately; (c.p., Scheidigger [8, § 7.3.3]).



(a) ***The frontal picture of a glacier*** ***(b)*** ***The longitudinal section of a glacier***

Fig. 1.

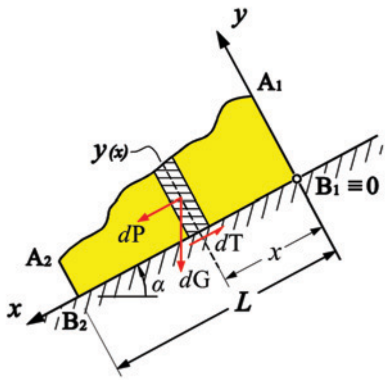


Fig. 2. – The section of a longitudinal slice.

We adopt the following assumptions:

a. The bed is taken to be a corrugated plane of constant slope α to the horizontal (Fig. 2), where $0 < \alpha < \pi/2$. The case $\alpha = 0$ corresponds to an ice sheet on a horizontal foundation that requires a different description not attempted here.

b. The bed's roughness, that opposes the sliding of the ice mass, is mathematically described by a coefficient of resistance $\mu(x)$ that may vary with respect to distance x along the glacier.

c. The progressive melting of the ice from the head of the glacier, where the ice is porous, to the base, where the ice is wet, is modelled by a corresponding increase in the ice's specific weight.

d. The ice flow is mainly one-dimensional and, consequently, the resistance of the ice is completely defined by a limiting value of the tensile stress that varies with respect to distance from the head. The limiting value also is assumed to either steadily decrease or increase depending partly upon the glacier's length.

We consider a slice of the glacier of unit width taken along its entire length L. The mean section is sketched in Fig. 1b. Assumption (b) implies that the lower edge, or bed, B_1B_2 may be taken as rectilinear (Fig. 2), while the upper edge A_1A_2 is assumed to have the equation

$$y = y(x), \qquad y(0) = H \geq 0, \tag{2.1}$$

with respect to the Cartesian system of x, y axes shown in Fig. 2. We seek to determine the unknown curve $y(x)$, subject to a positive specific weight $\gamma(x)(> 0)$ and positive coefficient of resistance $\mu(x)(> 0)$ being specified as bounded functions of the coordinate x.

A volume element $dV = y(x)\,dx$ of the slice, distance x from the origin, is maintained in equilibrium by the weight dG and resistant force dT acting on the bed. The weight, given by

$$dG = \gamma(x)y(x), \tag{2.2}$$

acts vertically and may be decomposed into components of magnitude dP and dQ respectively along the positive $x-$axis and negative $y-$axis, where

$$dP = \gamma(x)y(x)\sin\alpha, \tag{2.3}$$

$$dQ = \gamma(x)y(x)\cos\alpha. \tag{2.4}$$

In consequence, the tangential force dT retarding motion acts along the negative $x-$ direction. Its magnitude is assumed proportional to the normal force so that

$$dT = \mu(x)\,dQ, \tag{2.5}$$

where $\mu(x)$ is the coefficient of resistance.

Consider a cross-section of the slice parallel to the y-axis at a distance x from the origin. The resultant force acting over this cross-section has components $(-N(x), 0)$ where

$$N(x) = \int_0^x (dT - dP)\,d\xi + N(0) \tag{2.6}$$

$$= \int_0^x (\mu(\xi)\cos\alpha - \sin\alpha)\gamma(\xi)y(\xi)\,d\xi + N(0) \tag{2.7}$$

$$= \int_0^x f(\xi)y(\xi)\,d\xi + N(0), \tag{2.8}$$

where $N(0)$ is the total force bonding the glacier to the upper rock face, and

$$f(x) = \gamma(x)(\mu(x)\cos\alpha - \sin\alpha), \qquad 0 \le x \le L, \tag{2.9}$$

$$\ge 0, \qquad \mu(x) \ge \tan\alpha, \quad 0 \le x \le L, \tag{2.10}$$

$$< 0, \qquad 0 \le \mu(x) < \tan\alpha, \quad 0 \le x \le L. \tag{2.11}$$

Equation (2.7) is dimensionally correct, since $\gamma(x)$ has dimension kg/m^2. Observe also that the sign of $f(x)$ depends upon both the variable coefficient of resistance $\mu(x)$ and the constant slope α. According, however, to Assumption (d) on the behaviour of $N(x)$, the function $f(x)$ must be either non-negative or negative throughout the interval $[0, L]$. We consider both possibilities in the following discussion.

3. – Derivation of the governing equation

We regard the glacier to be in quasi-static equilibrium, which implies that the normal force $N(x)$ is tensile ($N(x) \geq 0$) for $0 \leq x \leq L$, and consequently generates a tensile stress $N(x)/y(x)$ in the transverse cross-section of the slice at a distance x from the origin. The stress, which should not exceed a certain variable critical positive value $\sigma(x)(>0)$, can either increase or decreases with respect to increasing x.

In the slice of unit thickness, the stress $\sigma(x)$ has dimension kg/m, and for limiting equilibrium, we have

$$\frac{N(x)}{y(x)} = \sigma(x), \tag{3.1}$$

and

$$N(0) = \sigma(0)H, \tag{3.2}$$

which, in principle, are the equation and initial condition for the determination of $y(x)$.

To solve this equation, we equate the derivatives with respect to x of $N(x)$ obtained from (2.8) and (3.1) which yields

$$N'(x) = (y(x)\sigma(x))' = f(x)y(x) = \frac{f(x)}{\sigma(x)}\sigma(x)y(x), \tag{3.3}$$

where a superposed prime denotes differentiation with respect to x. Integration of the last expression between x_1 and $x > x_1$ leads to the required form of the profile $y(x)$:

$$y(x) = \frac{\sigma(x_1)}{\sigma(x)} y(x_1) \exp\left(\int_{x_1}^{x} f(\xi)/\sigma(\xi)\, d\xi \right). \tag{3.4}$$

Verna and Keller [10] derive a similar expression for the optimum solution to the problem of a heavy string. Properties of the solution (3.4) are discussed in the next section.

We note that the above derivation permits $f(x)$ to vanish on $[0, L]$.

4. – General analysis

We explore in this section some general properties of the profile curve $y(x)$ subject to the conditions that the stress $\sigma(x) > 0$ is either monotonically increasing or decreasing with respect to x, and that the coefficient of resistance $\mu(x)$ either exceeds or is less than $\tan\alpha$ for all $x \in [0, L]$. Violation of these conditions for a particular model is discussed in Section 5. The coefficient of resistance as defined here is similar to, but distinct from, the standard coefficient of friction, a typical value of which for dry ice is 0.6, and for wet ice is 0.005. We conjecture that the coefficient $\mu(x)$ is determined from the pressure melting (regelation) of the ice, which in turn is affected by the weight of ice, and consequently, the slope. Furthermore, at high altitudes, the presumably lower temperature, in the absence of geothermal warming, depresses regelation and leads to an increased coefficient of resistance and reduced influence of the slope. At lower altitudes, the slope and weight have a greater affect on the pressure which increases the ice melt and leads to a lower coefficient of resistance. We consider separately various possibilities.

4.1 – *Decreasing stress*

This condition, usually observed in long glaciers of comparatively small slope, may be explained by the weakening of the cohesive stress due to progressive melting of the ice. We suppose that

$$\sigma(x_1) > \sigma(x_2) > 0, \qquad 0 \leq x_1 < x_2 \leq L. \tag{4.1}$$

The subsequent analysis depends upon the sign of $f(x)$.

4.1.1 – Decreasing stress. Large coefficient of resistance

We suppose that for $0 \leq x \leq L$, the coefficient of resistance satisfies $\mu(x) \geq \tan\alpha$ so that from (2.10) we have $f(x) \geq 0$. Consequently, together with (4.1), we conclude that

$$\int_{x_1}^{x} f(\xi)/\sigma(\xi)\, d\xi \geq 0, \qquad 0 \leq x_1 < x \leq L. \tag{4.2}$$

We therefore obtain from (3.4), the bound

$$y(x) = \frac{\sigma(x_1)}{\sigma(x)} y(x_1) \exp\left(\int_{x_1}^{x} f(\xi)/\sigma(\xi)\, d\xi \right) \tag{4.3}$$

$$\geq \frac{\sigma(x_1)}{\sigma(x)} y(x_1) \tag{4.4}$$

$$\geq y(x_1), \qquad 0 \leq x \leq L, \tag{4.5}$$

which demonstrates that under the stipulated conditions, $y(x)$ is non-decreasing with respect to x on $0 \leq x \leq L$. In particular, when $x_1 = 0$, we may use the initial condition $(2.1)_2$ to show that (3.4) reduces to

$$y(x) = \frac{\sigma(0)}{\sigma(x)} H \exp\left(\int_0^x f(\xi)/\sigma(\xi)\, d\xi \right), \tag{4.6}$$

and inequality (4.5) becomes

$$y(x) \geq H, \qquad 0 \leq x \leq L. \tag{4.7}$$

Again, from (4.6), we have

$$y'(x) = \frac{\sigma(0)H}{\sigma^2(x)} [-\sigma'(x) + f(x)] \exp\left(\int_0^x f(\xi)/\sigma(\xi)\, d\xi \right), \tag{4.8}$$

and because $y(x)$ is non-decreasing, so that $y'(x) \geq 0, 0 \leq x \leq L$, we are led necessarily to the condition

$$\sigma'(x) \leq f(x), \qquad 0 \leq x \leq L, \tag{4.9}$$

which on integration gives

$$\sigma(x) \leq \int_0^x f(\xi)\, d\xi + \sigma(0), \qquad 0 \leq x \leq L. \tag{4.10}$$

We are now able to derive an improved lower bound for $y(x)$. On using the bound (4.10) in the exponential (4.6), we obtain

$$\exp\left(\int_0^x f(\xi)\sigma^{-1}(\xi)\, d\xi \right) \geq \exp\left(\int_0^x \frac{f(\xi)}{\left(\int_0^\xi f(\eta)\, d\eta + \sigma(0) \right)} d\xi \right) \tag{4.11}$$

$$= \exp\left(\ln \left[\frac{\int_0^x f(\xi)\, d\xi + \sigma(0)}{\sigma(0)} \right] \right) \tag{4.12}$$

$$= \left[\frac{\int_0^x f(\xi)\, d\xi + \sigma(0)}{\sigma(0)} \right]. \tag{4.13}$$

Substitution in (4.6) and appeal to (4.1) leads to the lower bound

$$y(x) \geq \frac{\sigma(0)}{\sigma(x)} H \left[\frac{\int_0^x f(\xi)\, d\xi + \sigma(0)}{\sigma(0)} \right] \tag{4.14}$$

$$\geq H \left[\frac{\int_0^x f(\xi)\, d\xi + \sigma(0)}{\sigma(0)} \right], \qquad 0 \leq x \leq L, \tag{4.15}$$

in terms of measurable data.

An upper bound for $y(x)$ follows from the boundedness conditions assumed for the functions γ and μ and the tacit assumption that $\sigma(0)$ is also bounded.We derive from (4.1) and (4.4) for $0 \leq x \leq L$ the estimate

$$y(x) = \frac{\sigma(0)}{\sigma(x)} H \exp\left(\int_{x_1}^x f(\xi)/\sigma(\xi)\, d\xi \right) \tag{4.16}$$

$$\leq \frac{\sigma(0)}{\sigma(L)} H \exp\big(L\bar{f}/\sigma(L)\big), \tag{4.17}$$

where

$$\bar{f} = \bar{\gamma}(\bar{\mu} \cos \alpha - \sin \alpha), \tag{4.18}$$

$$\bar{\gamma} = \max_{[0,L]} \gamma(x), \qquad \bar{\mu} = \max_{[0,L]} \mu(x). \tag{4.19}$$

The upper bound is strictly less than H only when $f(x)$ is non-positive which is the case next discussed.

4.1.2 – Decreasing stress. Small coefficient of resistance

We next suppose that the stress continues to satisfy (4.1), but that the coefficient of resistance now satisfies $0 \leq \mu(x) < \tan \alpha$ for $0 \leq x \leq L$. As already stated, the condition implies $f(x) < 0$, and consequently

$$\exp\left(\int_0^x f(\xi)/\sigma(\xi)\, d\xi \right) \leq 1. \tag{4.20}$$

For $0 \le x_1 < x \le L$, we derive from (4.20) and (3.4) the relation

$$y(x) = \frac{\sigma(x_1)}{\sigma(x)} y(x_1) \exp\left(\int_{x_1}^{x} f(\xi)/\sigma(\xi)\, d\xi \right) \tag{4.21}$$

$$\le \frac{\sigma(x_1)}{\sigma(x)} y(x_1), \tag{4.22}$$

which together with (4.1) implies that we cannot deduce that $y(x)$ is monotonic on $[0, L]$. Indeed, it follows from (4.1) that $\sigma'(x) < 0$ and accordingly the sign of the expression

$$w(x) = -\sigma'(x) + f(x) \tag{4.23}$$

depends upon the relative magnitudes of the terms involved. Upon recalling (4.8), we conclude that the monotonicity of $y(x)$ requires additional restrictions. For example, on supposing that

$$w(x) = -\sigma'(x) + f(x) < 0, \qquad 0 \le x \le L, \tag{4.24}$$

we have

$$\int_0^x \frac{f(\xi)}{\sigma(\xi)}\, d\xi < \int_0^x \frac{\sigma'(\xi)}{\sigma(\xi)}\, d\xi = \ln \frac{\sigma(x)}{\sigma(0)}, \tag{4.25}$$

which with (4.6) leads to

$$y(x) \le H, \qquad 0 \le x \le L. \tag{4.26}$$

Alternatively, let us require that

$$\ln \frac{\sigma(0)}{\sigma(L)} < x\Big(-\underline{\gamma}(\bar{\mu} \cos \alpha - \sin \alpha)/\sigma(0) \Big), \tag{4.27}$$

where now $\bar{\mu} < \tan \alpha$, and

$$\underline{\gamma} = \min_{[0,L]} \gamma(x). \tag{4.28}$$

Condition (4.27) implies

$$\frac{\sigma(0)}{\sigma(L)} \exp\left(\int_0^x f(\xi)/\sigma(\xi)\, d\xi \right) < 1, \tag{4.29}$$

and upon insertion into (4.6) we recover the bound (4.26).

REMARK 4.1. – It follows from the condition (4.20) that when the coefficient of resistance is sufficiently small, the height $y(x) \to 0$ as $x \to \infty$; that is, the snout diminishes in height as the overall length of the glacier becomes very large.

REMARK 4.2. – We infer from (4.8) that irrespective of the sign of $f(x)$, the function $y(x)$ is increasing, constant, or decreasing on those sub-intervals of $[0, L]$ for which respectively $\sigma'(x)$ is less than, equal to, or greater than $f(x)$. Equality occurs only when $f(x)$ is negative.

4.2 – *Increasing stress*

In short glaciers at high altitude, the stress may be monotonically increasing with respect to length. In these circumstances we assume that

$$0 < \sigma(x_1) < \sigma(x_2) \leq m, \qquad 0 \leq x_1 < x_2 \leq L, \tag{4.30}$$

where m is a specified positive bounded constant. We again consider large and small coefficients of resistance.

4.2.1 – Large coefficient of resistance

We suppose the coefficient of resistance is sufficiently large such that $\mu(x) > \tan\alpha$, $0 \leq x \leq L$, and consequently $f(x) > 0$, $0 \leq x \leq L$, from which we conclude as before that

$$\exp\left(\int_0^x f(\xi)/\sigma(\xi)\, d\xi\right) \geq 1, \qquad x \geq 0. \tag{4.31}$$

First, we seek sufficient conditions under which $y(x) \geq H$. For this purpose, we introduce the positive constant $\underline{\mu}$ defined by

$$\mu(x) \geq \underline{\mu} > \tan\alpha, \qquad 0 \leq x \leq L, \tag{4.32}$$

and let

$$\underline{\gamma} = \min_{[0,L]} \gamma(x), \tag{4.33}$$

$$\underline{f} = \underline{\gamma}\Big(\underline{\mu} \cos\alpha - \sin\alpha\Big). \tag{4.34}$$

An argument similar to that employed in Section 4.1.2, shows that from (3.4) by imposing the condition

$$\ln\frac{\sigma(x)}{\sigma(0)} < \frac{x\underline{f}}{\sigma(L)}, \qquad 0 \leq x \leq L, \tag{4.35}$$

we are led to $y(x) \geq H, 0 \leq x \leq L$.

On the other hand, on restricting the stress to satisfy

$$\sigma(x) \geq \sigma(0)\exp\big(x\bar{f}/\sigma(0)\big), \qquad 0 \leq x \leq L, \tag{4.36}$$

we conclude that $y(x) \leq H, 0 \leq x \leq L$.

Note that both (4.35) and (4.36) are sufficient conditions only, and there may be other conditions under which $y(x)$ becomes larger or smaller than its initial value $y(0) = H$.

4.2.2 – Small coefficient of resistance

We examine properties of $y(x)$ subject to a sufficiently small coefficient of resistance that satisfies $0 < \mu(x) < \tan\alpha$ so that $f(x) < 0$ and (4.20) holds.

In consequence, the assumption of increasing stress ($\sigma'(x) > 0$) implies that

$$-\sigma'(x) + f(x) < 0, \qquad 0 \leq x \leq L, \tag{4.37}$$

and we immediately conclude from (4.8) that $y(x)$ is monotonically decreasing on $[0, L]$, and therefore

$$y(x) \leq H, \qquad 0 \leq x \leq L. \tag{4.38}$$

4.3 – *Constant stress*

When conditions are such that the stress $\sigma(x)$ is constant,we have that $w(x) = f(x)$, where $w(x)$ is defined by (4.23). It follows easily from (4.8) that the sign of $y'(x)$ is that of $f(x)$, so that from (2.10) and (2.11) we have

$$\begin{aligned} y(x) &\geq H, & \mu(x) &> \tan\alpha, & 0 &\leq x \leq L, \\ &= H, & \mu(x) &= \tan\alpha, & 0 &\leq x \leq L, \\ &\leq H, & \mu(x) &< \tan\alpha, & 0 &\leq x \leq L. \end{aligned}$$

5. – Particular analysis

The discussion of the previous section assumes that the coefficient of resistance at each point $x \in [0, L]$ satisfies either $\mu(x) \leq \tan\alpha$, or $\mu(x) > \tan\alpha$. We wish to explore consequences when these assumptions are relaxed, while still retaining the assumption of a monotonically decreasing or increasing stress. For this purpose, we investigate properties of the profile $y(x)$ for particular choices of the monotonic variables $\gamma(x), \mu(x)$ that are deemed physically plausible. In principle, these variables may be represented by rational functions for which the integral in (3.4) may be evaluated. Here, we simply suppose the functions are linear. Since $\gamma(x)$ is increasing and $\mu(x)$ is decreasing, we take

$$\gamma(x) = \gamma_0\Big(1 + \beta_\gamma x/L\Big), \qquad 0 \le x \le L, \tag{5.1}$$

$$\mu(x) = \mu_0\Big(1 - \beta_\mu x/L\Big), \qquad 0 \le x \le L, \tag{5.2}$$

where $\gamma_0, \mu_0, \beta_\gamma, \beta_\mu$ are positive constants. For the case of decreasing or increasing stress, we adopt the respective linear laws

$$\sigma(x) = \sigma_0(1 - \beta_\sigma x/L), \qquad 0 \le x \le L, \tag{5.3}$$

$$\sigma(x) = \sigma_0(1 + \beta_\sigma x/L), \qquad 0 \le x \le L, \tag{5.4}$$

where σ_0 and β_σ are positive constants. In fact, we discuss only the case of decreasing stress, since results for increasing stress may be deduced by simply reversing the sign of β_σ.

To ensure that $y(x)$ remains bounded, the parameters $\beta_\gamma, \beta_\mu, \beta_\sigma$ are chosen to satisfy

$$0 \le \beta_\gamma, \beta_\mu, \beta_\sigma < 1. \tag{5.5}$$

Further constraints are introduced below.

Note that for the choice (5.2), the condition $\mu(x) > \tan\alpha$ is implied by

$$\mu_0(1 - \beta_\mu) > \tan\alpha, \tag{5.6}$$

while $\mu(x) < \tan\alpha$ is implied by

$$\mu_0 < \tan\alpha. \tag{5.7}$$

In order to negate these assumptions, it is sufficient to suppose that

$$\mu_0 > \tan\alpha, \tag{5.8}$$

and

$$\mu_0\Big(1 - \beta_\mu\Big) < \tan\alpha. \tag{5.9}$$

These conditions imply that there exists $x_\mu \in [0, L]$ such that

$$\mu_0\Big(1 - \beta_\mu \frac{x_\mu}{L}\Big) = \tan\alpha. \tag{5.10}$$

Irrespective of further constraints on the parameters, we evaluate explicitly, subject to assumptions (5.1), (5.2), and (5.3), the expression (4.6) in preparation for later treatment.

We suppose that $\beta_\sigma \neq 0$ and postpone consideration of the case $\beta_\sigma = 0$. We have

$$\frac{f(x)}{\sigma(x)} = \frac{\gamma_0}{\sigma_0} \frac{\left(1+\beta_\gamma x/L\right)\left[\mu_0\left(1-\beta_\mu x/L\right)\cos\alpha - \sin\alpha\right]}{(1-\beta_\sigma x/L)} \tag{5.11}$$

$$= \frac{\gamma_0\mu_0}{\sigma_0} \frac{\left[a(1-\beta_\sigma x/L)^2 + b(1-\beta_\sigma x/L) + c\right]\cos\alpha}{(1-\beta_\sigma x/L)}$$

$$-\frac{\gamma_0}{\sigma_0} \frac{[d + e(1-\beta_\sigma x/L]\sin\alpha}{(1-\beta_\sigma x/L)}, \tag{5.12}$$

where

$$a = -\frac{\beta_\gamma\beta_\mu}{\beta_\sigma^2}, \tag{5.13}$$

$$b = \frac{\left[\beta_\sigma(\beta_\mu - \beta_\gamma) + 2\beta_\gamma\beta_\mu\right]}{\beta_\sigma^2}, \tag{5.14}$$

$$a + b = \frac{\left[\beta_\sigma(\beta_\mu - \beta_\gamma) + \beta_\gamma\beta_\mu\right]}{\beta_\sigma^2}, \tag{5.15}$$

$$c = \frac{\left[\beta_\sigma^2 - \beta_\sigma(\beta_\mu - \beta_\gamma) - \beta_\gamma\beta_\mu\right]}{\beta_\sigma^2} \tag{5.16}$$

$$= \frac{(\beta_\gamma + \beta_\sigma)(\beta_\sigma - \beta_\mu)}{\beta_\sigma^2}, \tag{5.17}$$

$$d = \frac{\left(\beta_\gamma + \beta_\sigma\right)}{\beta_\sigma}, \tag{5.18}$$

$$e = -\frac{\beta_\gamma}{\beta_\sigma}. \tag{5.19}$$

Accordingly, we have for $0 \leq x \leq L$,

(5.20)

$$\exp\left(\int_0^x f(\xi)/\sigma(\xi)\,d\xi\right) = \exp\left[\left(ax(2 - \frac{\beta_\sigma x}{L}\right)\frac{\gamma_0\mu_0\cos\alpha}{2\sigma_0} + \left(\frac{\gamma_0 b_1 x}{\sigma_0}\right)\right]\left(1 - \frac{\beta_\sigma x}{L}\right)^{-\frac{c_1 L\gamma_0}{\beta_\sigma\sigma_0}},$$

where

$$b_1 = (\mu_0 b \cos\alpha - e\sin\alpha), \tag{5.21}$$

$$c_1 = (\mu_0 c \cos\alpha - d\sin\alpha) \tag{5.22}$$

$$= \frac{(\beta_\sigma + \beta_\gamma)}{\beta_\sigma^2}\Big[\mu_0 \cos\alpha(\beta_\sigma - \beta_\mu) - \beta_\sigma \sin\alpha\Big]. \tag{5.23}$$

Insertion of the expression (5.20) into (4.6) after some rearrangement leads to

$$y(x) = H \exp\left(\frac{\gamma_0 L}{2\sigma_0 \beta_\sigma^2} z(x/L)\right)\left(1 - \frac{\beta_\sigma x}{L}\right)^s, \qquad 0 \le x \le L, \tag{5.24}$$

where

$$A = \beta_\gamma \beta_\sigma \beta_\mu, \tag{5.25}$$

$$B = \Big[\beta_\sigma \beta_\mu - \beta_\gamma(\beta_\sigma - \beta_\mu)\Big]\mu_0 \cos\alpha + \beta_\gamma \beta_\sigma \sin\alpha, \tag{5.26}$$

$$z(x/L) = \left[A\mu_0 \cos\alpha \left(\frac{x}{L}\right)^2 + \frac{2x}{L}B\right], \tag{5.27}$$

$$s = -\left(\frac{c_1 \gamma_0 L}{\beta_\sigma \sigma_0} + 1\right). \tag{5.28}$$

Subject to the constraints (5.8) and (5.9) on the coefficient of resistance, the expression (5.24) admits various shapes for the profile $y(x)$ determined by corresponding relations between the parameters $\beta_\gamma, \beta_\sigma$ and β_μ. We illustrate possibilities when the stress is decreasing.

5.1 – *Decreasing stress*

We consider the shape of $y(x)$ when the stress is given by (5.3) and the coefficient of resistance satisfies conditions (5.8) and (5.9). It follows that the function $f(x)$, explicitly given by

$$f(x) = \gamma_0\left(1 + \beta_\gamma \frac{x}{L}\right)\left[\mu_0\left(1 - \beta_\mu \frac{x}{L}\right)\cos\alpha - \sin\alpha\right], \tag{5.29}$$

satisfies

$$f(x) > 0, \qquad 0 \le x < x_\mu, \tag{5.30}$$

$$= 0, \qquad x = x_\mu, \tag{5.31}$$

$$< 0, \qquad x_\mu < x \le L, \tag{5.32}$$

where x_μ is defined by (5.10).

To determine the slope of $y(x)$, we appeal to (4.8), and in this respect note that

$$w(x) = -\sigma'(x) + f(x) \tag{5.33}$$

$$= \frac{\sigma_0\beta_\sigma}{L} + f(x) \tag{5.34}$$

$$> 0, \qquad 0 \le x \le x_\mu. \tag{5.35}$$

In consequence,

$$y(x) \ge H, \qquad 0 \le x \le x_\mu. \tag{5.36}$$

Whether or not $y(x)$ increases throughout the interval $[0, L]$, depends upon the detailed behaviour of $w(x)$ which we express as

$$w(x) = E_1 - E_2\frac{x}{L} - E_3\left(\frac{x}{L}\right)^2, \tag{5.37}$$

where

$$E_1 = \frac{\sigma_0\beta_\sigma}{L} + G > 0, \tag{5.38}$$

$$E_2 = G\left(1 - \beta_\gamma\right) > 0, \tag{5.39}$$

$$E_3 = \gamma_0\mu_0\beta_\gamma\beta_\mu > 0, \tag{5.40}$$

and

$$G = \gamma_0(\mu_0 \cos\alpha - \sin\alpha) > 0. \tag{5.41}$$

The profile $y(x)$ achieves its maximum at $\bar{x}$ where $w(\bar{x}) = 0$, and

$$\frac{\bar{x}}{L} = \frac{\left(\sqrt{(E_2^2 + 4E_1E_3)} - E_2\right)}{2E_3}, \tag{5.42}$$

so that $\bar{x} < L$ when

$$E_1 < E_2 + E_3, \tag{5.43}$$

or explicitly,

$$\frac{\sigma_0\beta_\sigma}{L} < \beta_\gamma\gamma_0\left(\beta_\mu\mu_0 - [\mu_0 \cos\alpha - \sin\alpha]\right). \tag{5.44}$$

When condition (5.44) is satisfied, then $y(x)$ decreases on $(\bar{x}, L]$. Conditions under which $y(L)$ may be less or greater than H may be easily extracted from (5.24). Details are omitted. Violation of condition (5.44) implies $y(x)$ increases for $x \in [0, L]$ and, indeed, $y(x) \ge H, x \in [0, L]$.

It is of interest to investigate how the profile $y(x)$ is additionally influenced by the comparative magnitudes of the parameters β_σ, β_μ, subject to the previous assumptions (5.8)-(5.9), and (5.3).

5.2 – $\beta_\sigma > \beta_\mu$

Let us first suppose that

$$\beta_\sigma > \beta_\mu. \tag{5.45}$$

Then, after appeal to (5.9) we have

$$B = \Big[\beta_\sigma\beta_\mu - \beta_\gamma(\beta_\sigma - \beta_\mu)\Big]\mu_0 \cos\alpha + \beta_\gamma\beta_\sigma \sin\alpha \tag{5.46}$$

$$> \beta_\mu\mu_0 \cos\alpha\Big[\beta_\sigma + \beta_\gamma(1-\beta_\sigma)\Big]$$

$$> 0. \tag{5.47}$$

On the other hand, we have from (5.23) that

$$c_1 = \frac{(\beta_\sigma + \beta_\mu)}{\beta_\sigma^2}\Big[\mu_0 \cos\alpha(\beta_\sigma - \beta_\mu) - \beta_\sigma \sin\alpha\Big] \tag{5.48}$$

$$< \frac{(\beta_\sigma + \beta_\mu)}{\beta_\sigma}\Big[\mu_0 \cos\alpha(1 - \beta_\mu) - \sin\alpha\Big] \tag{5.49}$$

$$< 0, \tag{5.50}$$

where we have employed the inequality

$$(1-\beta_\mu) > \frac{(\beta_\sigma - \beta_\mu)}{\beta_\sigma}. \tag{5.51}$$

Let us further suppose that

$$-c_1 \leq \frac{\beta_\sigma\sigma_0}{\gamma_0 L}, \tag{5.52}$$

which implies $s \leq 0$ where s is given by (5.28). We deduce from (5.24) that

$$y(x) \geq H, \qquad 0 \leq x \leq L. \tag{5.53}$$

Alternatively, suppose that

$$-c_1 > \frac{\beta_\sigma\sigma_0}{\gamma_0 L}, \tag{5.54}$$

which implies that $s > 0$. Now, from (5.24) we have

$$y(L) = H \exp\left(\frac{\gamma_0 L}{2\sigma_0\beta_\sigma^2} D\right)(1-\beta_\sigma)^s, \tag{5.55}$$

where

$$D = A\mu_0 \cos\alpha + 2B \tag{5.56}$$

$$= \Big[\beta_\gamma\beta_\sigma\beta_\mu + 2\beta_\sigma\beta_\mu + 2\beta_\gamma(\beta_\mu - \beta_\sigma)\Big]\mu_0 \cos\alpha + 2\beta_\gamma\beta_\sigma \sin\alpha \tag{5.57}$$

$$= \beta_\sigma\beta_\mu\Big[3\beta_\gamma + 2\Big]\mu_0 \cos\alpha + 2\beta_\gamma\beta_\sigma[\sin\alpha - \mu_0 \cos\alpha] \tag{5.58}$$

$$> 0, \tag{5.59}$$

by virtue of (5.8).

In consequence, whenever the parameters and the length L satisfy the condition

$$(1-\beta_\sigma)^s < \exp\left(-\frac{\gamma_0 LD}{2\sigma_0\beta_\sigma^2}\right) < 1, \tag{5.60}$$

then

$$y(L) < H. \tag{5.61}$$

5.3 – $\beta_\sigma \leq \beta_\mu$

We next examine the consequences of supposing that

$$\beta_\sigma \leq \beta_\mu. \tag{5.62}$$

It immediately follows from (5.46) that $B > 0$, while from (5.48) we conclude that $c_1 < 0$. Accordingly, the discussion proceeds as in the previous section. Note, however, that the calculations do not require the coefficient of resistance to satisfy either assumption (5.8) or assumption (5.9). Indeed, the conclusions remain valid when the coefficient of resistance is unrestricted.

5.4 – $\beta_\sigma = 0$

Results when the stress $\sigma(x)$ is everywhere constant and equal to σ_0, may be recovered as a special case of the conclusions established in Section 5.1. We need set only $\beta_\sigma = 0$ to find that the main features are unaffected and are easily derived from the previous calculations.

5.5 – *Further comment*

The various expressions obtained for the profile $y(x)$ in this Section and Section 4 are derived under the assumption that each cross-section of the glacier is subject to a tensile stress, $\sigma(x)$. The assumption, however, is invalidated whenever a local temperature increase causes a reduction in the ice's resistance. Mutual detachment of the parts within the affected section may lead to the formation of transverse fractures, which explains the onset of crevasses. In general, detachment first occurs at the initial section $x = 0$, where the crevasse thus formed is called the "bergschrund".

6. – Conclusion

The surface of a glacier flowing over an inclined plane bed may have its shape predicted on the basis of purely mechanical considerations. The glacier is regarded as a heavy slab resting on a rough inclined bed subject to a tensile stress equal to the limiting resistance of the ice. The instantaneous equilibrium configuration of the ice layer in one dimension is determined from the specific weight of the ice, its resistance to the bed, and its limit stress. Broad features, established under general monotonic constitutive functions and physically plausible assumptions on the resistance, demonstrate that there are conditions for which the glacier may either increase or decrease in height from its head. The model may also be examined when the constitutive functions are linear and the assumptions on the resistance are contravened. Apart from the height monotonically increasing or decreasing, there are conditions for which the height achieves a maximum value at a point intermediate between the top and bottom of the glacier.

The belief is widespread that glaciers tend to sharpen towards the base to form a typical protuberance known as the "snout" (cf., Scheidegger [8, pp. 377-378]), but this is observed only in glaciers of relatively short length such as are found, for example, in the Alps. A snout-shaped profile also occurs in the ice sheet covering Greenland (cp., Nadai [6, pp. 309-310]). The phenomenon, however, is not seen in the long glaciers of the Himalayas and polar regions (cf., Isserman and Weaver [5]).

The present analysis investigates values of the constitutive functions for which the glacier sharpens or thickens during the slow flow from the bergschrund to its base. In addition, localised losses in the ice's resistance can create crevasses that interrupt the continuity of the ice slab and, consequently, its shape (cp., Sturm and Zintl [9]).

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