Bollettino
Unione Matematica Italiana

Umberto Mosco

Analysis and Numerics of Some Fractal Boundary Value Problems

Bollettino dell'Unione Matematica Italiana, Serie 9, Vol. **6** (2013), n.1, p. 53–73.

Unione Matematica Italiana

<http://www.bdim.eu/item?id=BUMI_2013_9_6_1_53_0>

L'utilizzo e la stampa di questo documento digitale è consentito liberamente per motivi di ricerca e studio. Non è consentito l'utilizzo dello stesso per motivi commerciali. Tutte le copie di questo documento devono riportare questo avvertimento.

Articolo digitalizzato nel quadro del programma
bdim (Biblioteca Digitale Italiana di Matematica)
SIMAI & UMI
http://www.bdim.eu/

Bollettino dell'Unione Matematica Italiana, Unione Matematica Italiana, 2013.

Bollettino U. M. I.
(9) **VI** (2013), 53-73

Analysis and Numerics of Some Fractal Boundary Value Problems (*)

UMBERTO MOSCO

In Memory of Enrico Magenes

Abstract. – *We describe some recent results for boundary value problems with fractal boundaries. Our aim is to show that the numerical approach to boundary value problems, so much cherished and in many ways pioneering developed by Enrico Magenes, takes on a special relevance in the theory of boundary value problems in fractal domains and with fractal operators. In this theory, in fact, the discrete numerical analysis of the problem precedes the, and indeed give rise to, the asymptotic continuous problem, reverting in a sense the process consisting in deriving discrete approximations from the PDE itself by finite differences or finite elements. As an illustration of this point, in this note we describe some recent results on: the approximation of a fractal Laplacian by singular elliptic partial differential operators, by Vivaldi and the author; the asymptotic of degenerate Laplace equations in domains with a fractal boundary, by Capitanelli-Vivaldi; the fast heat conduction on a Koch interface, by Lancia-Vernole and co-authors.*

We point out that this paper has an illustrative purpose only and does not aim at providing a survey on the subject.

2000 Mathematics Subject Classification. 35J20, 35J70; 35B20, 35B27; 74K05, 74K15.
Keywords and phrases. Elliptic operators, singular homogenizations, fractals, composite media.
This work was supported by the NSF grant No. 1109356.

Introduction

Very simple fractals, as the Koch curve or the Sierpiński gasket, are non-differentiable sets, therefore they do not allow the explicit writing of an intrinsic differential operator. Moreover, if the fractal is the (compact) boundary of an open domain of the plane, the boundary trace spaces may be difficult to characterize. Merging classical boundary value problems – even of the simple kind as

(*) Following an agreement between Springer and Unione Matematica Italiana this article has also been published in the volume "Analysis and Numerics of Partial Differential Equations", F. Brezzi, P. Colli Franzone, U.P. Gianazza, G. Gilardi (eds.), Springer Indam Series, vol. 4, Springer, 2013.

those related to the Laplace equation or the heat equation – with the theory of fractal sets and fractal operators is a challenging task. In fact, fractal sets and operators are the result of an *asymptotic* process, the one induced by the infinite iterations of a family of contractive similarities. Such an asymptotic feature of fractals introduces an additional approximation level to those already inherent in the numerical approximation of the classical PDE at hand.

The construction of fractal sets goes back to the early years of the twentieth century. The construction of the Laplace and heat equations on a large class of fractals was achieved in the late 1980's, first by probabilistic methods, then analytically. We refer to [24] for a brief description of this early work.

A problem related to heat conduction in a planar domain relies on the idea that the insertion in the domain of a highly conductive material path connecting two point of the boundary can efficiently act as a preferential fast absorbing trail for the heat stream. An early model of the kind, with an infinitely conductive layer, was produced by Cannon and Meyer in 1971, [5], in connection with so called fractured oil wells. A related singular homogenization problem was later considered by Pam Huy-Sanchez Palencia in 1974, [31], see also [2], [24].

In the simplest version of this model, the domain is a rectangle, the infinitely conductive pattern is the segment connecting the middle points A and B of two opposite sides of of the rectangle, and the segment is approximated by thin highly conductive rectangles of transversal size ε. In the two regions above and below this thin layer, the two-dimensional heat equation, with a normalized conductivity coefficient and with a prescribed source term, is assumed to govern the slow diffusion of heat. At the same time, the fast diffusion of heat within the $\varepsilon-$layer is described by the two-dimensional heat equation, this time with a conductivity coefficient of the order $1/\varepsilon$. The boundary condition for the temperature is zero on the boundary of the rectangle. In the limit as $\varepsilon \to 0$, the thin layer shrinks to the transversal segment. In this process, the temperature converges to a limit temperature, given by the *two-dimensional* heat equation in each one of the open domains separated by the segment and by the *one-dimensional* heat equation along the segment itself. These two equations – *both* of second order – are coupled by a transmission condition across the segment. This condition stipulates that the *jump* of the external normal derivatives from each side of the segment acts as a *source term* for the one-dimensional tangential heat equation within the segment; moreover, the tangential diffusion has boundary values zero at the end points A and B.

We note, incidentally, that the natural Sobolev space taking into account this homogeneous boundary condition for the tangential equation is the so-called *Lions-Magenes* space $H^{1/2}_{0,0}$. A fractal analogue of the Lions-Magenes space occurs in the problems that we now describe.

A big innovation into the transmission model was indeed carried out in 2002 by M.R.Lancia, [16]. The segment connecting the points A and B of the boundary of the domain was replaced by a *fractal Koch curve*, connecting again A with B. The rationale for this new model is clear. By increasing the length of the preferential pattern that conveys the heat stream towards the two selected points A and B, and actually making the length of this path *infinite* in the limit, we expect that the cooling effect of the layer will be increased.

As mentioned previously, in Lancia's fractal model the transmission problem with the fractal Koch curve is obtained in the limit of a sequence of transmission problems for the approximating pre-fractal polygonal curves as the number of iteration increases to infinity. This model opens two related orders of problems. One is the rigorous analytic formulation of the second order transmission condition, in suitable fractional Besov spaces. This study was first carried out by Lancia in [16] and we refer to this paper and to [18] for the technical details. The second problem is the analytical and numerical study of the approximating pre-fractal equations. We report on this study in Section 3.

We also report on some recent results of a joint research by Vivaldi and the author, [29], [30], that is indeed related to the second order transmission problems discussed so far. The object of this study is a sequence of second order elliptic operators

$$A^n_{\varepsilon_n} u = -div\Big(a^n_{\varepsilon_n}(x, y)\nabla u\Big) \tag{0.1}$$

in divergence form in a bounded domain Ω of $\mathbb{R}^2$, with discontinuous coefficients a^n_ε. The coefficients a^n_ε develop an increasing number of singularities on an array of thin fibers Σ^n_ε obtained by the iterated action of a given family of contractive similarities. The geometry and the singularity of the conductivity coefficients are initially prescribed on an array of thin hexagonal fibers connecting the essential fixed points of the similarities. The parameter $n = 1, 2, \dots$ indicates the level of iteration of the similarities, the parameters ε_n refer to the transversal thickness of the fiber at each $nth-$iteration. The detailed geometry of the fibers and the expression of the singular coefficients a^n_ε are described in Section 1. One of the main objective of this study is to prove the convergence of the spectral measures for the elliptic operators to the spectral measure of a limit self-adjoint operator. The limit operator is the intrinsic Laplace operator of the fractal set defined by the given family of similarities. We outline our main results in Section 1.

A related topic has been the object of a recent paper by Capitanelli and Vivaldi, [7]. They consider the domain bounded by a square snowflake type domain, bounded by four Koch curves, and the approximated domains obtained by replacing each side of the pre-fractal Koch curves by a quadrilateral thin fiber of the kind mentioned before. Differently than in the previous case, the conductivity of the fiber is now assumed to vanish with the iterations. On such increasingly insulating boundary layers, a homogeneous Dirichlet condition is imposed to the solution of a Laplace equation. A characterization of the limit boundary value problem in the snow-flake domain is then given, that depends on the relative size of the thickness and the conductivity of the boundary layer. These results are outlined in Section 2.

There is no space in this note to report on some other aspects of the numerics of fractals. However, we wish to mention a new kind of interesting problems that deserve further research. The objective of this study is to approximate the two dimensional Laplace operator (and related PDEs) in an open domain of the plane, with a sequence of curvilinear one-dimensional Laplacians (and related ODEs), taken along a sequence of fractal curves homeomorphic to the segment $[0, 1]$, that asymptotically fill the whole open domain. Such a dynamical dimensional blow up is of theoretical interest in itself. It seems to be also interesting from the numerical point of view and, in the applications, as a model for the study of *invasive interfaces* that infiltrate the whole space.

In conclusion, we may observe that introducing fractal constructions into the classic theory of PDEs opens a wide new field of study, both theoretically and numerically. With the very simple examples which are the object of this note, and with other recent contributions – in particular, the works by Vacca [32], Bagnerini-Buffa-Vacca [3], Wasyk [33], Evans [10], Liang [23], [9], and the work by Achdou-Sabot-Tchou, [1] that explores different but related problems – this new field has been only scratched.

Enough however to unveil promising new directions in applied analysis and PDEs, and to point out how fundamental is to keep analysis and numerics in tight contact one each other. A point of view this one to which the author was already exposed in early years of his scientific life in Rome by his advisor Gaetano Fichera, and that he is happy to take now also as one of the most illuminating aspects of the scientific legacy of Enrico Magenes.

1. – Elliptic operators with fractal singularities

In this section we describe the recent work carried out by Vivaldi and the author in [29], [30]. In these papers, a singular elliptic operator is submitted to the iterated action of a family of similarities and the convergence of the spectrum is investigated.

We begin by introducing the similarities. We consider a family $\Psi = \{\psi_1, \ldots, \psi_N\}$ of $N \geq 2$ contractive similarities in $\mathbb{R}^2$, with distinct fixed-points, with a common contractive factor α^{-1}, $\alpha > 1$; a *similarity*, or *similitude*, in a Euclidean space is a map obtained by composition of translations, orthogonal transformations, and homotheties. The set of *essential fixed-points* of these maps will be denoted by Γ; a point $b_r \in \mathbb{R}^2$ is an *essential fixed-point* for the family Ψ if $b_r = \psi_i(b_r) = \psi_j(b_s)$ for some $i \in \{1, \ldots, N\}, j \neq i, j \in \{1, \ldots, N\}$ and b_s a fixed-point of a map of Ψ.

Here, for simplicity, we assume that Ψ is the so-called *Koch family* of similarities, that is, the family $\Psi = \{\psi_1, \ldots, \psi_4\}$ of the following $N = 4$ similitudes, each one contractive with a factor α^{-1}, $\alpha = 3$:

$$\psi_1(z) = \frac{z}{3}, \qquad \psi_2(z) = \frac{z}{3}\ e^{i\pi/3} + \frac{1}{3},$$
$$\psi_3(z) = \frac{z}{3}\ e^{-i\pi/3} + \frac{1}{2} + \frac{i \sin \pi/3}{3}, \quad \psi_4(z) = \frac{z+2}{3},$$

where $z = x + iy \in \mathbb{C}$. The set of the essential fixed-points of this family is $\Gamma = \{A, B\}$, where $A = (0, 0)$ and $B = (1, 0)$. The third vertex of the equilateral triangle based on the side A, B is the point $C = (1/2, \sqrt{3}/2)$.

We now define a reference fiber in the Cartesian plane $\mathbb{R}^2$. This fiber is a thin hexagon which has the segment connecting the points $A = (0, 0)$ and $B = (1, 0)$ as its longitudinal axis. The middle point of the segment AB is denoted by $AB/2$. The fiber is symmetric with respect to the $x - axis$ and the vertical line $x = 1/2$, therefore, it suffices to describe the geometry of the fiber in the region $y \geq 0$, $x \leq 1/2$. We consider the right triangle with vertices $A, AB/2, Q_0$ which makes the angle $\pi/12$ at A. Thus, $Q_0 = (1/2, \varepsilon_0)$, where $\varepsilon_0 = h_0/2$, $h_0 = \tan(\pi/12)$. For every $0 < \varepsilon \leq \varepsilon_0$, we consider the two points $Q_1(\varepsilon) = (\varepsilon/h_0, \varepsilon)$ and $Q_0(\varepsilon) = (1/2, \varepsilon)$ and the quadrilateral $A, AB/2, Q_0(\varepsilon), Q_1(\varepsilon)$. We then define the set $\Sigma^0_{0,2\varepsilon}$ to be the thin hexagon obtained by reflection of this quadrilateral across the $x - axis$, followed by a symmetry across the vertical axis $x = 1/2$. The vertices of $\Sigma^0_{0,2\varepsilon}$, listed clockwise, are the points $A, Q_1(\varepsilon), Q_2(\varepsilon), B, Q_3(\varepsilon), Q_4(\varepsilon)$, where now $Q_2(\varepsilon) = (1 - \varepsilon/h_0, \varepsilon)$, $Q_3(\varepsilon) = (1 - \varepsilon/h_0, -\varepsilon)$, $Q_4(\varepsilon) = (\varepsilon/h_0, -\varepsilon)$. The perimeter of the hexagon $\Sigma^0_{0,2\varepsilon}$ gives the external profile of our fiber. Inside the hexagon $\Sigma^0_{0,2\varepsilon}$, we now insert a smaller hexagon $\Sigma^0_{0,\varepsilon}$. The construction of this hexagon is similar to that of $\Sigma^0_{0,2\varepsilon}$, by replacing the triangle $A, AB/2, Q_0$ with the smaller right triangle with vertices $A, AB/2, P_0$, where $P_0 = (1/2, \varepsilon_0/2)$. The angle of this triangle at A is $\arctan(h_0/2)$. The vertices of the hexagon $\Sigma^0_{0,\varepsilon}$, again listed clockwise, are now the points $A, P_1(\varepsilon), P_2(\varepsilon), B, P_3(\varepsilon), P_4(\varepsilon)$, where now $P_1(\varepsilon) = (\varepsilon/h_0, \varepsilon/2)$, $P_2(\varepsilon) = (1 - \varepsilon/h_0, \varepsilon/2)$, $P_3(\varepsilon) = (1 - \varepsilon/h_0, -\varepsilon/2)$, $P_4 = (\varepsilon/h_0, -\varepsilon/2)$.

With the notation set before, the reference fiber is given by the two co-axial thin hexagons $\Sigma^0_{0,\varepsilon} \subseteq \Sigma^0_{0,2\varepsilon}$, of largest transversal size ε and 2ε, respectively. The two hexagons meet at the common vertices A and B, and $\Sigma^0_{0,\varepsilon} \setminus \{A, B\}$ is contained in the interior of $\Sigma^0_{0,2\varepsilon}$.

In the case at hand, the Koch family of similarities, the single reference fiber $\Sigma^0_{0,2\varepsilon}$ of our construction connects the two essential fixed-points $\Gamma = \{A, B\}$, which, in this special case, are the only essential fixed points of the family Ψ (the pair-wise connection of all essential fixed points of the family by means of a fiber is a requirement of this theory).

Our next step is to submit the fiber to the iterated action of the family Ψ. We first set a useful notation. For each integer $n \geq 0$, we consider arbitrary $n-$tuples of indices $i|n = (i_1, i_2, \ldots, i_n) \in \{1, \ldots, N\}^n$ and define $\psi_{i|n} = \psi_{i_1} \circ \psi_{i_2} \circ \cdots \circ \psi_{i_n}$ if $n > 0$, with $\psi_{i|n}$ the identity map if $n = 0$; for every set $\mathcal{O} \subseteq \mathbb{R}^2$, we define $\mathcal{O}^{i|n} = \psi_{i|n}(\mathcal{O})$. With this notation at hand, for every $n \geq 0$, we construct the array of fibers obtained by defining

$$\Sigma^n_{2\varepsilon} = \bigcup_{i|n} \Sigma^{i|n}_{2\varepsilon}, \qquad \Sigma^{i|n}_{2\varepsilon} = \psi_{i|n}(\Sigma^0_{2\varepsilon}), \tag{1.1}$$

$$\Sigma^n_{\varepsilon} = \bigcup_{i|n} \Sigma^{i|n}_{\varepsilon}, \quad \Sigma^{i|n}_{\varepsilon} = \psi_{i|n}(\Sigma^0_{\varepsilon}) = \bigcup_{b_r \neq b_s \in \Gamma} \Sigma^{i|n}_{\varepsilon}(b_r, b_s), \tag{1.2}$$

where $\Sigma^0_\varepsilon = \Sigma^0_{0,\varepsilon}$, $\Sigma^0_{2\varepsilon} = \Sigma^0_{0,2\varepsilon}$.

The space $\mathbb{R}^2$ – actually, a bounded open domain $\Omega \subset \mathbb{R}^2$, with $\overline{\Omega}$ containing the fibers $\overline{\Sigma}^0_{0,2\varepsilon}$ and $\Sigma^n_{2\varepsilon}$ for every ε and every n and which will be specified later on – is now converted into a physical *composite body*. This is done by defining a discontinuous conductivity matrix $a^n_\varepsilon(\xi, \eta)Id$, $(\xi, \eta) \in \Omega$, again by the iterated action of the similarity family. The whole iteration process is externally governed by two sequences of constants, $\zeta_n > 0$ and $\gamma_n > 0$. The limit values assigned to ζ_n and γ_n as $n \to +\infty$ affect the nature and the properties of the asymptotic effective medium.

In order to observe boundary effects, we choose Ω in a way that Γ belongs to the boundary $\partial\Omega$ of Ω, namely Ω is now the triangle with vertices $D = (1/2, -\sqrt{3}/2)$, $E = (3/2, \sqrt{3}/2)$, $F = (-1/2, \sqrt{3}/2)$. The domain Ω contains the interior of the triangle of vertices A, B, C, and the vertices A, B, C belong to $\partial\Omega$.

The matrix $a^n_\varepsilon Id$ – for given $\varepsilon > 0$ and $n \geq 0$ – is defined at every $(\xi, \eta) \in \Omega$ by

$$a^n_\varepsilon(\xi, \eta)Id = \begin{cases} \zeta_n \mathbf{1}_{\Omega \setminus \Sigma^n_{2\varepsilon}}(\xi, \eta)Id + \mathbf{1}_{\Sigma^n_{2\varepsilon} \setminus \Sigma^n_\varepsilon}(\xi, \eta)Id + \\ 1/2\, \sigma_n \sum\limits_{b_r \neq b_s \in \Gamma} w^n_\varepsilon(\xi, \eta) \mathbf{1}_{\Sigma^n_\varepsilon(b_r, b_s)}(\xi, \eta)Id. \end{cases} \tag{1.3}$$

In this expression, Id is the 2–dimensional identity matrix and $\mathbf{1}_S$ the indicatrix function of a set $S \subset \mathbb{R}^2$, that is, $\mathbf{1}_S(\xi, \eta) = 1$ if $(\xi, \eta) \in S$, $\mathbf{1}_S(\xi, \eta) = 0$ if $(\xi, \eta) \notin S$.

The constants σ_n, which will be specified later on, are scaling factors associated with Ψ. At each iteration n, $\zeta_n > 0$ is a material constant that takes into account how the conductivity of the surrounding medium evolves with n. Since in (1.3) the conductivity of the coating region $\Sigma^n_{2\varepsilon} \setminus \Sigma^n_\varepsilon$ has been normalized to 1, the constant ζ_n can be interpreted as a *viscosity* coefficient, that expresses the relative strength of the conductivity of the space that surrounds the fiber $\Sigma^n_{2\varepsilon}$ with respect to the conductivity of the fiber Σ^n_ε itself. For every $n \geq 0$ and $i|n$, the conductivity of the fiber $\Sigma^{i|n}_\varepsilon(b_r, b_s)$ is given by

$$w^n_\varepsilon(\xi, \eta) \mathbf{1}_{\Sigma^{i|n}_\varepsilon(b_r, b_s)}(\xi, \eta) = \gamma_n \alpha^n w^0_\varepsilon \circ \psi^{-1}_{i|n}(\xi, \eta) \mathbf{1}_{\Sigma^n_\varepsilon(b_r, b_s)}(\xi, \eta). \tag{1.4}$$

This expression is obtained from the conductivity $w^0_\varepsilon(x, y)$ of the reference fiber $\Sigma^0_\varepsilon(b_r, b_s)$ by applying the map $(\xi, \eta) = \psi_{i|n}(x, y)$. The function $w^0_\varepsilon(x, y)$ is defined on the inner fiber $\Sigma^0_{0,\varepsilon}$ as follows:

$$w^0_\varepsilon(x,y) = \begin{cases} \dfrac{2+h_0^2}{4|P-P^\perp|} & \text{if} \quad (x,y)\in\mathcal{T} \\ \dfrac{1}{2|P-P^\perp|} & \text{if} \quad (x,y)\in\mathcal{R} \end{cases} \tag{1.5}$$

where $\mathcal{R}$ is the central rectangle in $\Sigma^0_{0,\varepsilon}$ with vertices P_1, P_2, P_3, P_4, and $\mathcal{T}$ is the union of the two isosceles triangles A, P_1, P_4 and P_2, B, P_3. For every $(x,y)\in\Sigma^0_{0,\varepsilon}$, we consider the point $P^\perp = (x,0)$ on the longitudinal axis of $\Sigma^0_{0,\varepsilon}$ and we define $P=(x,y_P)$ to be the intersection of the vertical line through $P^\perp=(x,0)$ with the boundary $\partial\Sigma^0_{0,\varepsilon}$ of $\Sigma^0_{0,\varepsilon}$ in the half plane $y\geq 0$. This boundary is the polygonal line connecting the vertices $A, P_1(\varepsilon), P_2(\varepsilon), B$. Then $|P-P^\perp|$ is the (Euclidean) distance between P and $P^\perp$ in $\mathbb{R}^2$.

At this stage, we are confronted with two asymptotic limits. For fixed n, the limit as $\varepsilon\to 0$ gives vanishing thickness to the fibered neighborhood of the prefractal polygonal curve. The limit as $n\to+\infty$ leads to the fractal set included in Ω. We proceed diagonally, by suitably choosing for each n a value $\varepsilon_n>0$, such that $\varepsilon_n\to 0$ as $n\to+\infty$. Then, we take the single limit as $n\to+\infty$.

We consider the sequence of operators $A_n = A_{\varepsilon_n}$, where A_{ε_n} are the operators given in (0.1). The operators A_n are defined as self-adjoint operators in the space $L^2(\Omega)$ (with Neumann boundary condition on $\partial\Omega$). Our main goal is to show the convergence of the spectral measures $P_n(d\lambda)$ of the operators A_n to the spectral measure of a suitable self-adjoint asymptotic operator, as $n\to+\infty$. We rely on variational and convergence tools from [24]. In particular, by a general result in [24], we obtain the convergence of the spectral measures of the operators A_n to the spectral measure of a limit operator A as a consequence of the M-convergence of the (extended-valued) energy forms associated with these operators.

For every n, and for the specified value of $\varepsilon_n>0$, the energy form of the operator A_{ε_n} in $L^2(\Omega)$ is the functional

$$F_n[u] = F^n_{\varepsilon_n}[u] = \begin{cases} \displaystyle\int_\Omega a^n_{\varepsilon_n}(x,y)|\nabla u|^2 dxdy & \text{if} \quad u\in D[F^n_{\varepsilon_n}] \\ +\infty & \text{if} \quad u\in L^2(\Omega)\setminus D[F^n_{\varepsilon_n}] \end{cases}$$

where $a^n_{\varepsilon_n} Id$ is the coefficient matrix defined in (1.3) for $\varepsilon=\varepsilon_n$, and the domain $D[F^n_{\varepsilon_n}]\subset L^2(\Omega)$ is the completion of $C^1(\overline{\Omega})$ in the norm

$$\|u\|_{D[F^n_{\varepsilon_n}]} = \left\{\int_\Omega |u|^2 dxdy + \int_\Omega |\nabla u|^2 a^n_{\varepsilon_n} dxdy\right\}^{\frac{1}{2}}. \tag{1.6}$$

As explained before, we let, simultaneously, the iteration parameter n go to $+\infty$ and the transversal size ε of the fibers go to 0, by choosing $\varepsilon = \varepsilon_n$ to be infinitesimal as $n \to +\infty$. We must also choose the scaling constants σ_n. These scaling laws can be expressed in terms of a single parameter $\delta > 0$. The value of δ is given by the ratio

$$\delta := \frac{d_H}{d_S}$$

of the Hausdorff dimension d_H of $\mathcal{G}$ and the spectral dimension d_S of $\mathcal{G}$. The constant δ is an effective metric parameter that depends on the fractal $\mathcal{G}$. For the Koch curve, $N = 4, \alpha = 3, \delta = ln4/ln3$. For the Sierpiński gasket, $N = 3, \alpha = 2, \delta = ln5/ln4$. Note that in both cases $\delta > 1$. We then define

$$\rho = \frac{\alpha^{2\delta}}{N}$$

and take

$$\varepsilon_n = \left(\frac{\rho}{N}\right)^n \omega_n, \text{ (with } \quad \omega_n \to 0 \quad \text{as} \quad n \to +\infty), \qquad \sigma_n = \left(\frac{\rho}{\alpha}\right)^n. \tag{1.7}$$

We also assume that the material constants ζ_n, γ_n remain finite and non-vanishing through the iteration process:

$$\lim \zeta_n = \zeta^* \in (0, +\infty), \qquad \lim \gamma_n = \gamma^* \in (0, +\infty) \tag{1.8}$$

as $n \to +\infty$. In [30] the following result is obtained, which extend previous results from [27] and [28] :

THEOREM 1.1. – *With the value of $\delta > 0$ specified before, under the assumptions* (1.7) *and* (1.8) *the sequence of functionals F_n M-converges in $L^2(\Omega)$ to the functional*

$$F[u] = \begin{cases} \zeta^* \displaystyle\int_\Omega |\nabla u|^2 dxdy \; + \; \gamma^* \, \mathcal{E}_\mathcal{G}[u|_\mathcal{G}] & \text{if} \quad u \in H^1(\Omega), u|_\mathcal{G} \in D[\mathcal{E}] \\ +\infty & \text{if} \quad u \in L^2(\Omega) \setminus \{u : \, u \in H^1(\Omega), \, u|_\mathcal{G} \in D[\mathcal{E}]\}. \end{cases} \tag{1.9}$$

where $\mathcal{E}_\mathcal{G}[u|_\mathcal{G}]$ is the energy functional on the fractal $\mathcal{G}$.

In this statement, $H^1(\Omega) \subset L^2(\Omega)$ is the Sobolev space obtained as the completion of $C^1(\overline{\Omega})$ in the norm

$$\|u\|_{H^1(\Omega)} = \left\{ \int_\Omega |u|^2 dxdy + \int_\Omega |\nabla u|^2 dxdy \right\}^{\frac{1}{2}}$$

and $u|_{\mathcal{G}}$ is the trace of $u \in H^1(\Omega)$ on $\mathcal{G}$, defined, *e.g*, as in [13], [14]. The energy functional $\mathcal{E}_{\mathcal{G}}[u]$ is obtained as the increasing limit

$$\mathcal{E}_{\mathcal{G}}[u] = \lim_{n \to +\infty} \mathcal{E}^n_{\mathcal{G}}[u] \tag{1.10}$$

of the discrete energy forms

$$\mathcal{E}^n_{\mathcal{G}}[u] = \frac{1}{2} \frac{\alpha^{2n\delta}}{N^n} \sum_{i|n} \sum_{b_r \neq b_s \in \Gamma} (u(\psi_{i|n}(b_r)) - u(\psi_{i|n}(b_s)))^2 \,, \tag{1.11}$$

on the domain

$$D[\mathcal{E}_{\mathcal{G}}] = \{u \in C(\mathcal{G}) | \quad \sup_{n \geq 0} \mathcal{E}^n_{\mathcal{G}}[u|_{V^n}] < +\infty\} \,.$$

Here for every $n \geq 0$ the set V^n is obtained by iteration as

$$V^n = \bigcup_{i|n} \psi_{i|n}(\Gamma). \tag{1.12}$$

The fractal $\mathcal{G}$ is the closure in $\mathbb{R}^2$ of the set $V^\infty = \bigcup_{n=0}^{+\infty} V^n$. We note that the functional (1.9) is non trivial, because it is finite on the domain $D[F] = \{u : u \in H^1(\Omega), u|_{\mathcal{G}} \in D[\mathcal{E}]\}$ which is dense in $L^2(\Omega)$ (see, *e.g.*, [13]). The functional F defines a densely defined self-adjoint operator $A = -\Delta_{\mathcal{G}}$ in the Hilbert space $L^2(\mathcal{G}, \mu_{\mathcal{G}})$, which takes the role of intrinsic Laplace operator in $\mathcal{G}$ with Neumann condition on Γ. The measure $\mu_{\mathcal{G}}$ is the (normalized) Hausdorff measure on $\mathcal{G}$.

The case of *Dirichlet* conditions, on both $\partial\Omega$ and Γ, is covered by the next result. The functional $F_n[u]$ of the previous theorem is now replaced by the functional

$$F_n[u] = F^n_{\varepsilon_n}[u] = \begin{cases} \int_\Omega a^n_{\varepsilon_n}(x, y) |\nabla u|^2 dxdy & \text{if} \quad u \in D_0[F^n_{\varepsilon_n}] \\ +\infty & \text{if} \quad u \in L^2(\Omega) \setminus D_0[F^n_{\varepsilon_n}] \end{cases} \tag{1.13}$$

where the domain $D_0[F^n_{\varepsilon_n}] \subset L^2(\Omega)$ is now the completion of $C^1_0(\Omega)$ in the norm $\|u\|_{D[F^n_{\varepsilon_n}]}$ given in (1.6). The limit functional is defined on $L^2(\Omega)$ by

$$F[u] = \begin{cases} \zeta^* \int_\Omega |\nabla u|^2 dxdy \, + \, \gamma^* \, \mathcal{E}[u|_{\mathcal{G}}] & \text{if} \quad u \in H^1_0(\Omega), u|_{\mathcal{G}} \in D_0[\mathcal{E}] \\ +\infty \quad \text{if} \quad u \in L^2(\Omega) \setminus \{u : u \in H^1_0(\Omega), \, u|_{\mathcal{G}} \in D_0[\mathcal{E}]\}. \end{cases} \tag{1.14}$$

The functional (1.14) is finite on the domain $D_0[F] = \{u : u \in H^1_0(\Omega), u|_{\mathcal{G}} \in D_0[\mathcal{E}]\}$, where $D_0[\mathcal{E}]$ is the subspace of all functions in $D[\mathcal{E}]$ that vanish on Γ. Again, this functional is non trivial, because $D_0[F]$ is dense in $L^2(\Omega)$. The self-adjoint operator $A = -\Delta_{\mathcal{G}}$, defined now in the Hilbert space $L^2(\mathcal{G}, \mu_{\mathcal{G}})$ by the functional F, is the Laplace operator $-\Delta_G$ in the fractal $\mathcal{G}$ with Dirichlet boundary condition on Γ.

The result in [30] is

THEOREM 1.2. – *Under the same scaling assumptions as in Theorem* 1.1, *the sequence of functionals F_n, defined in* (1.13) *M-converges in $L^2(\Omega)$ to the functional F defined in* (1.14) *as $n \to +\infty$.*

The special scaling laws for the parameters imply in particular that the thickness of the fibers tends to zero while their conductivity diverges to $+\infty$, the product of them remaining bounded, as $n \to +\infty$. In the same paper some cases where this condition is not satisfied are also studied.

We point out that in both Theorems the asymptotic energy has two interacting components, the standard Dirichlet integral extended to the two dimensional domain Ω, and a lower-dimensional fractal energy term. Globally, the limit functional F defines a self-adjoint operator A in the space $L^2(\Omega)$. Formally, such operator A is given by the two-dimension Laplace operator Δ in the open set $\Omega \setminus \mathcal{G}$ – with Neumann or Dirichlet boundary condition on $\partial\Omega$ – together with the fractal-Laplacian $\Delta_{\mathcal{G}}$ on $\mathcal{G}$ – with Neumann or Dirichlet condition on $\Gamma = \mathcal{G} \cap \partial\Omega$. The two operators are coupled by a second order transmission condition on $\mathcal{G}$. The condition states that the jump of the normal derivative of the function u from Ω across $\mathcal{G}$, taken on $\mathcal{G}$, equals the Laplacian $\Delta_{\mathcal{G}}$ acting on the trace of u on $\mathcal{G}$. In the case of Dirichlet condition on Γ, a fractal analogue of the Lions-Magenes trace space, mentioned earlier, comes into play. For a rigorous definition of the transmission problems when $\mathcal{G}$ is the von Koch curve we refer to [16] and [18].

As mentioned in the introduction, the convergence of the energy functionals implies the convergence of the spectral measures and of the spectral subspaces.

THEOREM 1.3. – *In the same assumptions of Theorem* 1.1 *and Theorem* 1.2, *for every $\lambda < \mu$ which are not in the point spectrum of the operator A in $L^2(\Omega)$, the projection operator $P^n((\lambda, \mu])$ of the spectral resolution P^n of the operator A^n in $L^2(\Omega)$ converges strongly in $L^2(\Omega)$ to the projection operator $P((\lambda, \mu])$ of the spectral resolution P of the operator A in $L^2(\Omega)$.*

This result follows from the convergence of the functionals, by applying Theorem 2.4.1 and its Corollary 2.7.1 from [24].

In the problems considered so far in this Section the parameter ζ^* is positive. This is the case when the medium in which the fibers are imbedded keeps finite viscosity up to the limit. Then, as seen before, the energy is only partially absorbed into the lower dimensional fractal inclusion. The *vanishing viscosity* case, when

$$\lim \zeta_n = \zeta^* = 0 .$$

has been considered in [29]. In this case the limit functional is composed *only* by the fractal energy term.

Such a collapse of geometry and energy on a lower dimensional fractal set is an interesting feature, both in fractal and PDEs theories. It shows, in particular, that fractal Laplacians can be obtained as the (spectral) limit of singular second order elliptic operators in divergence form.

In the vanishing viscosity case, however, there is a loss of coercivity as $n \to +\infty$. In fact, the uniform H^1 estimate that plays a basic role in the previous theorems, that is

$$c\,\|\nabla u\|^2_{L^2(\Omega)} \le F^n_{\varepsilon_n}[u]$$

with $c > 0$ independent of n, now fails, due to the vanishing of the coefficient ζ_n as $n \to +\infty$. The domains $D[F_n]$ of the functionals F_n and the domain $D[F]$ of the limit functional F are no more contained in the single Hilbert space $H = L^2(\Omega)$, which is the space where the convergence of the previous theorems takes place.

This difficulty has been overcome in [29], by relying on the generalization of $M-$convergence of functionals to variable Hilbert spaces, developed by Kuwae - Shioya in [15]. Generally speaking, the convergence of the functionals takes now place in a larger Hilbert space, $\bigotimes_0^\infty H^n$.

In [29], for every $n \ge 0$ the following Hilbert space is considered:

$$H^n = L^2(\Omega, \mu^n_{\varepsilon_n}),$$

where the Borel measure $\mu^n_{\varepsilon_n} = \mu_n$ is defined in Ω by

$$\mu_n = \zeta_n \mathbf{1}_{\Omega \setminus \Sigma^n_{2\varepsilon_n}} \mathcal{L} + \mathbf{1}_{\Sigma^n_{2\varepsilon_n} \setminus \Sigma^n_{\varepsilon_n}} \mathcal{L} + \tau_n w^n_{\varepsilon_n} \mathbf{1}_{\Sigma^n_{\varepsilon_n}} \mathcal{L}.$$

Here $\mathcal{L}$ is the 2−dimensional Lebesgue measure, $0 < \zeta_n \le 1$ are the viscosity parameters and τ_n are scaling constants, depending on the fractal, that will be specified later on. The functional F_n is now defined for each n in the spaces H^n s follows:

$$F_n[u] = F^n_{\varepsilon_n}[u] = \begin{cases} \displaystyle\int_\Omega a^n_{\varepsilon_n}(x,y)|\nabla u|^2 dxdy & \text{if} \quad u \in D[F_n] \\ +\infty & \text{if} \quad u \in L^2(\Omega, \mu_n) \setminus D[F_n] \end{cases}$$

where $a^n_{\varepsilon_n} Id$ is again the coefficient matrix defined in (1.3) for $\varepsilon = \varepsilon_n$, but now the domain $D[F_n]$ – $D[F_n] \subset L^2(\Omega, \mu_n) = H^n$ – is the space of all functions $u \in L^2(\Omega, \mu_n)$ with distribution week gradient in $L^2(\Omega, \mu_n)$. The functional F_n defines a regular, closed Dirichlet form in H^n. The generator of such a form is a self-adjoint operator $-A^n$ densely defined in the space H^n. The operator A^n is the positive-definite self-adjoint realization in the space $L^2(\Omega, \mu_n)$ of the second order elliptic operator in divergence form (0.1), with natural Neumann conditions on $\partial\Omega$. The spectrum of the operator $A^n + Id_{H_n}$ is a point spectrum, with eigenvalues $\lambda_k^n \to +\infty$ as $k \to +\infty$.

The measures μ_n are the so-called *speed measures* of the Markov processes generated by $-A^n$. They replace, in the choice of the Hilbert space, the two dimensional Lebesgue measure of the non-vanishing viscosity case. The transmission condition at the interface of $\mathcal{G}$ and Ω is affected by this change.

We now summarize the assumptions in the present case. The coefficients $a_{\varepsilon_n}^n$ are defined as previously in (1.3), and they depend on the two sequences of constants ζ_n and γ_n. The constants N, α, δ and ρ – that depend on the fractal – are the same as specified before. As before, we also take

$$\varepsilon_n = \left(\frac{\rho}{N}\right)^n \omega_n, \ (\text{with} \quad \omega_n \to 0 \ \text{ as } \quad n \to +\infty), \qquad \sigma_n = \left(\frac{\rho}{\alpha}\right)^n. \tag{1.15}$$

In addition, for the Sierpiński case considered in [29], we assume that

$$\tau_n = \frac{1}{3}\left(\frac{\alpha}{N}\right)^n$$

(for other fractals, the numerical coefficient 3 may be replaced by another numerical constant depending on the cardinality of Γ). With this choice of the constants τ_n, it is proved in [29] that the measures μ_n weak* converge to the measure $\mu_{\mathcal{G}}$ as $n \to +\infty$, that is

$$\int_\Omega \phi d\mu_n \to \int_\Omega \phi d\mu_{\mathcal{G}}$$

as $n \to +\infty$, for every $\phi \in C(\bar{\Omega})$.

The result of [29] for the Sierpiński fractal is:

THEOREM 1.4. – *With the scaling constants ε_n, σ_n and τ_n specified before, let the constants $0 < \zeta_n \leq 1$ and γ_n be such that $\lim \zeta_n = 0$ and $\lim \gamma_n = \gamma^* \in (0, +\infty)$ as $n \to +\infty$. Then the sequence of functionals F_n in H^n M-converges (in the sense of Kuwae-Shioya) to the functional*

$$F[u] = \begin{cases} \gamma^* \, \mathcal{E}_{\mathcal{G}}[u] & \text{if} \quad u \in D[F] \\ +\infty & \text{if} \quad u \in L^2(\mathcal{G}, \mu_{\mathcal{G}}) \setminus D[F] \end{cases} \tag{1.16}$$

where $D[F] = \{u \in L^2(\mathcal{G}, \mu_{\mathcal{G}}) : u \in D[\mathcal{E}_{\mathcal{G}}]\}$ and $\mathcal{E}_{\mathcal{G}}[u]$ is the energy functional on the fractal $\mathcal{G}$, with domain $D[\mathcal{E}_{\mathcal{G}}] \subset L^2(\mathcal{G}, \mu_{\mathcal{G}})$.

We note that, in the present context, the M-convergence of the functionals F_n to the functional F, in the sense of Kuwae-Shioya, is defined as the usual M-convergence of functionals, provided strong and weak convergence of sequences of vectors are defined in the following way: a sequence of vectors $u_n \in H^n$ *converges strongly* to a vector $u \in H$ if there exists a sequence $\phi_m \in C(\bar{\Omega})$, such that

$\|\phi_m - u\|_H \to 0$ as $m \to +\infty$, and

$$\lim_m \limsup_n \|\phi_m - u_n\|_{H^n} \to 0, \quad \text{as} \quad n \to +\infty \quad \text{and} \quad m \to +\infty\,.$$

The sequence $u_n \in H^n$ *converges weakly* to $u \in H$, if the inner product $(u_n, v_n)_{H^n}$ converge to the inner product $(u, v)_H$ for every v_n converging strongly to v as $n \to +\infty$.

Similarly as before, from the convergence of the functionals we get the convergence of the spectral measures, see Theorem 3.4 in [15]:

THEOREM 1.5. – *In the same assumptions of Theorem* 1.4, *for every $\lambda < \mu$ not in the point spectrum of the self-adjoint operator $A = -\Delta_{\mathcal{G}}$ in $L^2(\mathcal{G}, \mu_{\mathcal{G}})$, defined by F, the projection operator $P^n((\lambda, \mu])$ of the spectral resolution P^n of the self-adjoint operators A^n in $L^2(\Omega, \mu_n)$, defined by F_n, converges strongly to the projection operator $P((\lambda, \mu])$ of the spectral resolution P of the operator A, in the Kuwae-Shioya sense.*

In this statement, the strong convergence of the spectral projectors has to be intended according to the following general definition: a sequence of bounded operators B_n in H^n converges strongly to a bounded operator B in H if for every $u_n \in H^n$ converging strongly to $u \in H$ the sequence $B_n u_n \in H^n$ converges strongly to $Bu \in H$, with the strong convergence of vectors defined as before.

2. – Elliptic operators with fractal degeneracy

In this section we report on the recent papers by Capitanelli and Vivaldi, [6], [7]. The problem studied in these papers is the boundary approximation with suitable *insulating fibers* of Laplace equations in a domains bounded by four Koch curves.

The domain Ω_0 is now the square $\{(x, y) : 0 < x < 1, -1 < y < 0\}$, with vertices $A = (0, 0), B = (1, 0), C = (1, -1)$ and $D = (0, -1)$. On each one of the four sides of the square a Koch curve $K_j, j = 1, \ldots, 4$, is constructed, moving outward from the square. At the iteration n, the domain bounded by the four pre-fractal Koch curves K_j^n is denoted by Ω^n. For each n, and for every $0 < \varepsilon \leq \varepsilon_0 < 1/2$, the open set Ω^n is enlarged to become the open set

$$\Omega^n_\varepsilon = \bar{\Omega}^n \cup \Sigma^n_{j,\varepsilon}\,,$$

where for each j the set $\Sigma^n_{j,\varepsilon}$ is the open fibered neighborhood of K_j^n constructed by similarity from the initial reference fiber $\Sigma_{0,\varepsilon}$. However, since now the prefractal is on the boundary, we cut the reference fiber in half, by keeping only the (open) half fiber that lays above the $x - axis$. The fibered set $\Sigma^n_{j,\varepsilon}$ lies then externally to the domain Ω^n and is disjoint from $\bar{\Omega}^n$.

The conductivity coefficients of the fibered set $\Sigma^n_{j,\varepsilon}$ are defined, as in the previous section, in terms of the constants γ_n and τ_n and of the functions w^n_ε. However, in the definition of w^n_ε, a substantial change is performed: in the definition of the conductivity of the reference fiber $\Sigma_{0,\varepsilon}$, the factor $|P-P^\perp|^{-1}$ is replaced by the factor $|P-P^\perp|$ (and the numerical coefficients are modified conveniently). With this change, the fibers present vanishing conductivity as ε tends to 0. The conductivity $a^n_\varepsilon(x,y)$ of the enlarged domain Ω^n_ε is then defined to be equal to $w^n_\varepsilon(x,y)$ if $(x,y)\in\Sigma^n_\varepsilon$, and equal to 1 if $(x,y)\in\bar{\Omega}^n$.

The spaces $H^1(\Omega^n_\varepsilon, w^n_\varepsilon)$ and $H^1_0(\Omega^n_\varepsilon, w^n_\varepsilon)$ are defined to be the completion of $C^1(\bar{\Omega}^n_\varepsilon)$ and $C^1_0(\Omega^n_\varepsilon)$, respectively, in the norm

$$\|u\|_{H^1(\Omega^n_\varepsilon, w^n_\varepsilon)} = \left\{ \int_{\Omega^n_\varepsilon} |u|^2 dxdy + \int_{\Omega^n_\varepsilon} |\nabla u|^2 w^n_\varepsilon dxdy \right\}^{\frac{1}{2}}.$$

By Ω^* we denote the unit disc with center at $P_0=(1/2,1/2)$, We then consider the following functionals:

$$F_n[u] = F^n_{\varepsilon_n}[u] = \begin{cases} \displaystyle\int_{\Omega^n_{\varepsilon_n}} a^n_{\varepsilon_n}(x,y)|\nabla u|^2 dxdy & \text{if } u\in L^2(\Omega^*) \text{ and } u|_{\Omega^n_\varepsilon} \in H^1_0(\Omega^n_\varepsilon, w^n_\varepsilon) \\ +\infty & \text{if } u\in L^2(\Omega^*) \text{ and } u|_{\Omega^n_\varepsilon} \notin H^1_0(\Omega^n_\varepsilon, w^n_\varepsilon) \end{cases}$$

By $\mu_{\partial\Omega}$ we denote the measure on $\partial\Omega$ such that the restriction of $\mu_{\partial\Omega}$ to each fractal component K_j of $\partial\Omega$ coincides with the Hausdorff measure μ_{K_j} of K_j, $j=1,\dots,4$.

Then the following result is given in [6]:

THEOREM 2.1. – *Let us assume that* $\gamma_n>0$, $\gamma^*>0$ *and* $\gamma_n\to\gamma^*$ *as* $n\to+\infty$. *Let* ε_n *be an arbitrary sequence such that* $\varepsilon_n\to 0$ *as* $n\to+\infty$. *Then the functional* F_n $M-$*converge to the functional*

$$F[u] = \begin{cases} \displaystyle\int_{\Omega} |\nabla u|^2 dxdy + \gamma^* \int_{\partial\Omega} |u|^2 d\mu_{\partial\Omega} & \textit{if } u\in L^2(\Omega^*) \textit{ and } u|_\Omega \in H^1(\Omega) \\ +\infty & \textit{if } u\in L^2(\Omega^*) \textit{ and } u|_\Omega \notin H^1(\Omega)\,. \end{cases}$$

We point out that the boundary value problem for the Laplace operator in Ω associated with the limit functional F implies a Robin type condition on $\partial\Omega$.

In [6], the case when $\gamma_n\to 0$ – leading to Neumann boundary condition on $\partial\Omega$ – and the case $\gamma_n\to+\infty$, with an additional assumption on the rate of convergence of $\varepsilon_n\to 0$ – leading to a Dirichlet condition on $\partial\Omega$ – are also studied, as well as the generalization to boundaries obtained from irregularly scaled Koch curves, or *Koch mixtures*, in the sense of [4] and [26].

3. – Interfacial heat transmission

Two-dimensional second order transmission problems across a highly conductive layer of Koch type have been studied by Lancia, Vernole and co-authors in a series of recent papers, [19], [20], [21], [22] and [8].

In reporting on this work in the context of this note, we confine ourselves mainly to the papers [19], [20] and [8]. In [19] the authors obtain their first results on the heat transmission problem that we already mentioned in the Introduction. In particular, they show the existence and uniqueness of the strict solution for both the fractal and the pre-fractal problem; moreover they study the regularity and the convergence of the solutions of the pre-fractal problems as the pre-fractal layer converges to the fractal set. In [8], the authors provide the finite element approximation for this kind of problems.

The pre-fractal transmission problems studied in [20] can be formally stated as follows:

$$(\overline{P_n}) \begin{cases} \dfrac{du_n(t,P)}{dt} - \Delta u_n(t,P) = f(t,P) & \text{in } [0,T]\times\Omega_n^i,\ i=1,2 & i)\\ \dfrac{du_n}{dt} - \Delta_{K_n} u_n(t,P) = \left[\dfrac{\partial u_n(t,P)}{\partial \nu}\right] + f & \text{on } [0,T]\times K_n, & ii)\\ u_n(t,P)=0 & \text{on } [0,T]\times\partial\Omega, & iii)\\ u_n^1(t,P)=u_n^2(t,P) & \text{on } [0,T]\times K_n, & iv)\\ u_n(t,P)=0 & \text{on } [0,T]\times \partial K_n & v)\\ u_n(0,P)=0 & \text{on } \Omega & vi) \end{cases}$$

In this problem Ω is a rectangular domain, for example the open rectangle with vertices $A=(0,-\sqrt{3}/2)$, $B=(1,-\sqrt{3}/2)$, $C=(1,\sqrt{3}/2)$ and $D=(0,\sqrt{3}/2)$. The source term $f(t,P)$ is a given function in $C^\delta([0,T];L^2(\Omega,m_n))$ with $\delta\in(0,1)$. For a fixed n, K_n is the pre-fractal Koch curve with endpoints $(0,0)$ and $(1,0)$. The curve K_n separates Ω into two open subsets, Ω_n^1 and Ω_n^2. The restriction of u_n to Ω_n^i is denoted by u_n^i, $i=1,2$. The piecewise-tangential Laplacian defined on the polygonal curve K_n is denoted by Δ_{K_n}. The jump of the normal derivatives across K_n is given by

$$\left[\frac{\partial u_n}{\partial \nu}\right] = \frac{\partial u_n^1}{\partial \nu_1} + \frac{\partial u_n^2}{\partial \nu_2}\,,$$

where ν_i is the inward normal vector to the boundary of Ω_n^i.

Let us introduce the Hilbert space space $L^2(\Omega,m_n)$, where

$$dm_n = dx\,dy + ds\,, \tag{3.1}$$

with inner product $(\cdot,\cdot)_{m_n}$ and norm $\|u\|_{2,m_n} = \big(\int_{\Omega_n}|u|^2 dxdy + \int_{K_n}|u|^2 ds\big)^{\frac{1}{2}}$ and the forms

$$E^{(n)}(u_n, u_n) = \int_{\Omega} |\nabla u_n|^2 dxdy + \int_{K_n} |\nabla_\tau \gamma_0 u_n|^2 ds, \tag{3.2}$$

defined on the domain

$$V(\Omega, K_n) = \{u_n \in H_0^1(\Omega) : \gamma_0 u_n \in H_0^1(K_n)\}\,. \tag{3.3}$$

In (3.3), $H_0^1(\Omega)$ denotes the usual Sobolev space in Ω, $H_0^1(K_n)$ the trace space on K_n and $\gamma_0 u_n$ is the trace of u_n on K_n (denoted simply by u_n below). Moreover, the second integral at the right-hand side of (3.2) is defined piece-wise by

$$\int_{K_n} |\nabla_\tau \gamma_0 u_n|^2 ds = \sum_{M \in K_n} \int_{M} |\nabla_\tau \gamma_0 u_n|^2 ds,$$

where the sum is taken over the segments M that compose K_n, ∇_τ is the tangential derivative along M. The measure ds is the one–dimensional arc length measure on K_n. This integral expresses the energy $E_{K_n}(\cdot,\cdot)$ of the curve K_n. The space $V(\Omega, K_n)$ given by (3.3) is a Hilbert space under the norm

$$\|u_n\|_{V(\Omega,K_n)} = \{E^{(n)}(u_n, u_n)\}^{1/2}\,. \tag{3.4}$$

Moreover for each $n \in \mathbb{N}$, $E^{(n)}(\cdot,\cdot)$, with domain $V(\Omega, K_n)$, is a regular, strongly local Dirichlet form in $L^2(\Omega)$ and in $L^2(\Omega, m_n)$, respectively.

In [20], Problem $(\overline{P_n})$ is dealt with by semigroup methods. More precisely, for every fixed n the following abstract Cauchy problem is studied

$$(P_n) \qquad \begin{cases} \dfrac{\mathrm{d}u_n(t)}{\mathrm{d}t} = A_n\, u_n(t) + f(t), & 0 \le t \le T \\ u_n(0) = 0 \end{cases} \tag{3.5}$$

where $A_n : \mathcal{D}(A_n) \subset L^2(\Omega, m_n) \to L^2(\Omega, m_n)$ is the generator associated with the energy form $E^{(n)}$,

$$E^{(n)}(u_n, v) = -\int_{\Omega} A_n u_n v\, dm_n, \quad u_n \in \mathcal{D}(A_n), \quad v \in V(\Omega, K_n),$$

The following existence and uniqueness result is then obtained

THEOREM 3.1. – *Let* $0<\delta<1$, $f \in C^\delta([0,T], L^2(\Omega, m_n))$, *and let*

$$u_n(t) = \int_0^t T_n(t-s) f(s)\,\mathrm{d}s \qquad \textit{for every } n \in \mathbb{N}, \tag{3.6}$$

where $T_n(t)$ is the analytic semigroup generated by A_n. Then u_n is the unique "strict" solution of (P_n). Moreover,

$$\|u_n\|_{C^1([0,T],L^2(\Omega,m_n))} + \|u_n\|_{C^0([0,T],\mathcal{D}(A_n))} \le c\|f\|_{C^\delta([0,T],L^2(\Omega,m_n))}. \tag{3.7}$$

where c is a constant independent of n.

The solution of the abstract Cauchy problem (P_n) is the "strong" solution of Problem $(\overline{P}_n)$, as described by this result in [8]:

THEOREM 3.2. – *Let u_n be the solution of Problem (P_n). For every fixed $t \in [0,T]$ we have*

$$\begin{cases} \dfrac{\mathrm{d}u_n(t,P)}{\mathrm{d}t} - \Delta u_n(t,P) = f(t,P), & \text{for } P \in \Omega_n^i, i = 1,2 \\ \dfrac{\partial u_n^i}{\partial \nu_i} \in L^2(K_n), & i = 1,2, \\ \dfrac{\mathrm{d}u_n}{\mathrm{d}t} - \Delta_{K_n} u_n|_{K_n} = \left[\dfrac{\partial u_n}{\partial \nu}\right] + f, & \text{in } L^2(K_n), \\ u_n(t,P) = 0, & \text{for } P \in \partial\Omega. \end{cases} \tag{3.8}$$

Moreover, $\dfrac{\partial u_n^i}{\partial \nu_i} \in C([0,T], L^2(K_n))$, $i = 1,2$.

In [19] and [20] the following regularity result is also obtained

THEOREM 3.3. – *For every fixed $t \in [0,T]$, $u_n^1 \in H^{2,\alpha_1}(\Omega_n^1)$ with $\alpha_1 > \dfrac{2}{5}$; $u_n^2 \in H^{2,\alpha_2}(\Omega_n^2)$ with $\alpha_2 > \dfrac{1}{4}$, and $u_n \in C^0(\overline{\Omega})$, $u_n|_{K_n} \in H^2(K_n)$.*

The definition of the weighted Sobolev spaces $H^{2,\alpha_i}(\Omega_n^i)$ is rather delicate. If $\mathcal{D}$ is a non-convex polygonal domain in $\mathbb{R}^2$ and $\alpha > 0$, the space $H^{2,\alpha}(\mathcal{D})$ is defined to be the space

$$H^{2,\alpha}(\mathcal{D}) = \{v \in H^1(\mathcal{D}) : \ r^\alpha \cdot D^\beta v \in L^2(\mathcal{D}),\ \beta = (\beta_1,\beta_2) \in \mathbb{N} \times \mathbb{N} \,\text{s.t.}\, |\beta| = 2\},$$

equipped with the norm

$$\| v \|_{H^{2,\alpha}(\mathcal{D})} := \left\{ \| v \|^2_{H^1(\mathcal{D})} + \sum_{|\beta|=2} \| r^\alpha \cdot D^\beta v \|^2_{L^2(\mathcal{D})} \right\}^{1/2}.$$

The delicate point in this definition is the construction of the weight function $r : \mathcal{D} \to \mathbb{R}_+$, that we now describe. Let $\{P_j, 1 \le j \le \mathcal{N}\}$ be the set of vertices of

$\mathcal{D}$. For $j = 1, \dots, \mathcal{N}$, let θ_j be the interior angle of $\mathcal{D}$ at P_j. Let $\mathcal{R}$ be the set of the indices $\left\{ j = 1, \dots, \mathcal{N} : \dfrac{\pi}{\theta_j} < 1 \right\}$ and let $\mathcal{Q} = \{P_j\}_{j \in \mathcal{R}}$ be the subset of the vertices with reentrant angles θ_j (these are the points where the solutions are singular). We set $\eta := \left\{ \dfrac{1}{4} \cdot \min |P_j - P_k|;\ j, k \in \mathcal{R}, j \neq k \right\}$ and arbitrarily choose $0 < \varepsilon < \eta$. For j in $\mathcal{R}$, we define $r_j(P) := |P - P_j|$ for all P in $\mathrm{B}_\varepsilon(P_j) = \{P \in \mathcal{D} :\ |P - P_j| < \varepsilon\}$. We then define the function $r : \mathcal{D} \to \mathbb{R}_+$ by putting $r(P) := r_j(P)$, for all $P \in \mathrm{B}_\varepsilon(P_j)$ and j in $\mathcal{R}$, and $r(P) := 1$ for all $P \in \mathcal{D} \backslash \bigcup_{j \in \mathcal{R}} \mathrm{B}_{2\varepsilon}(P_j)$.

We conclude this section with some remarks on the numerical approximation of these problems, reporting mainly on the paper [8] and [9].

The pre-fractal curve K_n induces a natural triangulation $\mathcal{T}_{n,h}$ of the domain Ω, in which the vertices of K_n belongs to the set of nodes of $\mathcal{T}_{n,h}$. Starting with this triangulation, a mesh refinement process is given, that generates a regular and conformal family of finer triangulations $\{\mathcal{T}_{n,h}\}$.

The need for such refined triangulations comes from the presence of reentrant angles in the boundaries of the domains Ω_n^1 and Ω_n^2, which were previously described. As already mentioned, the solution u_n is singular at these angles, indeed u_n is not in the Sobolev space $H^2(\Omega_n^i)$, as it is the case in a smoothly bounded domain. Instead, as seen with Theorem 3.3, $u_n^i \in H^{2,\alpha_i}(\Omega_n^i)$, $i = 1, 2$, with $\alpha_1 > \dfrac{2}{5}$ and $\alpha_2 > \dfrac{1}{4}$. In view of these singularities, in order to get optimal rate of convergence for the finite element approximations the triangulation of the domains Ω_n^i must be refined, according to the conditions introduced in this regard by Grisvard in [11].

The authors are able to implement Grisvard's conditions by satisfying at the same time an additional important property for their refinements. The refined meshes are constructed as a "nested" sequence of meshes, *i.e.*, all the nodes of $\mathcal{T}_{n,h}$ belong also to $\mathcal{T}_{n+1,h}$. This property is of course of great help when the numerical approximation is carried out at various levels of the fractal iteration. We refer to [8] and [9] for more details. We also point out that in [9] more complicated boundaries, made by suitable mixtures of Koch curves, are also considered.

With the appropriate triangulations at hand, the numerical approximation of the problem $(\overline{P_n})$ is carried out in two steps. In the first step the semi-discrete problem, obtained by discretizing with a Galerkin method only the space variable, is considered. The following a priori error estimates of the order of convergence is then obtained

THEOREM 3.4. – *Let $u_n(t)$ be the solution of $(\overline{P_n})$, $u_n^i(t)$ be the restriction to Ω_n^i of $u_n(t)$, for $i = 1, 2$, and $u_{n,h}(t)$ be the semi-discrete solution. Then for every*

$t \in [0, T]$ *we have*

$$\| u_n(t) - u_{n,h}(t) \|^2_{L^2(\Omega,m_n)} + \int_0^t \| u_n(\tau) - u_{n,h}(\tau) \|^2_{V(\Omega,K_n)} d\tau \leq ch^2 \Big(\int_0^t \| f(\tau) \|^2_{L^2(\Omega,m_n)} d\tau \Big)$$

where c is a suitable constant independent of h.

In the second step the fully discretized problem is considered. By applying a finite difference scheme on the time variable, the so-called θ method, an error estimate between the semi-discrete solution $u_{n,h}(t_l)$ and the fully discrete solution $u^l_{n,h}$ is obtained.

From this estimate and from Theorem 3.4, they finally get

THEOREM 3.5. – *Assume that* $f \in C^\delta([0,T]; L^2(\Omega, m_n))$ *and* $\dfrac{\partial f}{\partial t} \in L^2([0,T] \times \Omega, dt \times dm_n)$. *Let* n *be fixed and let* $u_n(t)$ *be the solution of problem* $(\overline{P_n})$, $u^l_{n,h}$ *be the fully discretized solution, as given by the* $\theta-$*method with* $\frac{1}{2} \leq \theta \leq 1$. *Then,*

$$\|u_n(t_l) - u^l_{n,h}\|^2_{L^2(\Omega,m_n)} \leq ch^2 \Big(\int_0^T \|f(\tau)\|^2_{L^2(\Omega,m_n)} d\tau \Big) + C_\theta \triangle t^2 \cdot \left(\|f(0)\|^2_{L^2(\Omega,m_n)} + \int_0^T \left\| \frac{\partial f}{\partial \tau}(\tau) \right\|^2_{L^2(\Omega,m_n)} d\tau \right).$$

A final remark about future research. In all the problems discussed in this paper an important question remains to be investigated, namely, to obtain some quantitative estimate for the asymptotic fractal limit. Such estimates should reflect the stability properties of the problem at hand in presence of the wild changes in the geometry. The very nature of the estimates – whether they can be stated in suitable function spaces or they are just of scalar energy kind – must be better understood, in each one of the special cases described before.

Acknowledgments. The author wishes to thank the authors of the papers presented in this note for their kindness in providing him with the preprints of their work and for their useful comments. The author wishes also to thank Professors Ugo Gianazza, Giuseppe Savare' and Carlo Sbordone who gave him the opportunity of dedicating this paper to the Memory of Enrico Magenes.

REFERENCES

[1] Y. ACHDOU - C. SABOT - N. TCHOU, *A multiscale numerical method for Poisson problems in some ramified domains with fractal boundary*, Multiscale Model. Simul. **5** (3) (2006), 828-860.

[2] H. ATTOUCH, *Variational Convergence for Functions and Operators*, Pitman Advanced Publishing Program, London 1984.

[3] P.BAGNERINI - A. BUFFA - E. VACCA, *Finite elements for a prefractal transmission problem*, C.R.A.S. Math, **342** (3) (2006), 211-214.

[4] M. T. BARLOW - B. M. HAMBLY, *Transition density estimates for Brownian motion on scale irregular Sierpiński gaskets*, Ann. Inst. Henri Poincaré, **33** (5) (1997), 531-557.

[5] J. R. CANNON - G. H. MEYER, *On a diffusion in a fractured medium*, SIAM J. Appl. Math., **3** (1971), 434-448.

[6] R. CAPITANELLI - M. A. VIVALDI, *Insulating layers and Robin problems on Koch mixtures*, J. Diff. Eqn., **251** (4-5) (2011), 1332-1353.

[7] R. CAPITANELLI - M. A. VIVALDI, *On the Laplacean transfer across a fractal mixture*, Asymptotic Analysis, to appear. DOI 10.3233/ASY-2012-1149.

[8] M. CEFALO - G. DELLA'ACQUA - M.R. LANCIA, *Numerical approximation of transmission problems across Koch-type highly conductive layers*, Appl. Math. Comp., **218** (2012), 5453-5473.

[9] M. CEFALO - M.R. LANCIA - H. LIANG, *Heat flow problems across fractal mixtures: regularity results of the solutions and numerical approximation*, Differential and Integral Equations, in print.

[10] E. EVANS, *Extension Operators and Finite Elements for Fractal Boundary value Problems*, PhD thesis, Worcester Polytecnic Institute, USA, 2011.

[11] P. GRISVARD, *Elliptic problems in nonsmooth domains*, Advanced Publishing Program, London 1985.

[12] J. E. HUTCHINSON, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713-747.

[13] A. JONSSON, *Dirichlet forms and Brownian motion penetrating fractals*, Potential Analysis, **13** (2000), 69-80.

[14] A. JONSSON - H. WALLIN, *Function Spaces on Subsets of* $\mathbb{R}^n$, Part 1 (Math. Reports) Vol. 2. London: Harwood Acad. Publ. 1984.

[15] K. KUWAE - T. SHIOYA, *Convergence of spectral structures: a functional analytic theory and its applications to spectral geometry*, Comm. in Anal. and Geom., **11** (4) (2003), 599-673.

[16] M. R. LANCIA, *A transmission problem with a fractal interface*, AZ. Anal. und Ihre Anwend., **21** (2002), 113-133.

[17] M. R. LANCIA - E. VACCA, *Numerical approximation of heat flow problems across highly conductive layers*, "Quaderni di Matematica", Dept. Me. Mo. Mat., U. Roma La Sapienza, 2007.

[18] M. R. LANCIA - M. A. VIVALDI, *Asymptotic convergence for energy forms*, Adv. Math. Sc. Appl., **13** (2003), 315-341.

[19] M. R. LANCIA - P. VERNOLE, *Convergence results for parabolic transmission problems across highly conductive layers with small capacity*, AMSA, **16** (2006), 411-445.

[20] M. R. LANCIA - P. VERNOLE, *Irregular heat flow problems*, SIAM J. Math. Anal., **42** (4), (2010), 1539-1567.

[21] M. R. LANCIA - P. VERNOLE, *Semilinear evolution transmission problems across fractal layers*, Nonlinear Anal. Th. and Appl., **75** (2012), 4222-4240.

[22] M. R. LANCIA - P. VERNOLE, *Semilinear fractal problems: approximation and regularity results*, Nonlinear Anal. Th. and Appl., to appear. DOI: 10.1016/j.na.2012.08.020.
[23] H. LIANG, *On the construction of certain fractal mixtures*, MS Thesis, WPI, 2009.
[24] U. MOSCO, *Composite media and asymptotic Dirichlet forms*, J. Funct. Anal., **123**, 2 (1994), 368-421.
[25] U. MOSCO, *Variational fractals*, Ann. Scuola Norm. Sup. Pisa, Cl. Sci. (4) Vol. XXV (1997), 683-712.
[26] U. MOSCO, *Harnack inequalities on scale irregular Sierpiński gaskets*, Nonlinear Problems in Math. Physics and Related Topics II, Edited by Birman et al., Kluwer Acad./Plenum Publ. (New York, 2002), 305-328.
[27] U.MOSCO - M. A. VIVALDI, *An example of fractal singular homogenization*, Georgian Math. J. **14** (1) (2007), 169-194.
[28] U. MOSCO - M. A. VIVALDI, *Fractal Reinforcement of Elastic Membranes*, Arch. Rational Mech. Anal., **194** (2009), 49-74.
[29] U. MOSCO - M. A. VIVALDI, *Vanishing viscosity for fractal sets*, Discrete and Continuous Dynamical Systems. **28** (3) (2011), 1207-1235.
[30] U. MOSCO - M. A. VIVALDI, *Thin fractal fibers*, Math. Methods in the Applied Sciences, to appear. DOI: 10.1002/mma.1621.
[31] H. PHAM HUY - E. SANCHEZ PALENCIA, *Phénomènes des transmission à travers des couches minces de conductivité élevée*, J. Math Anal. Appl., **47** (1974), 284-309.
[32] E. VACCA, *Galerkin Approximation for Highly Conductive Layers*, PhD Thesis, Dept. Me. Mo. Mat., U. Roma La Sapienza, 2005.
[33] R. WASYK, *Numerical solution of a transmission problem with pre fractal interface*, PhD Thesis, WPI, 2007.

Department of Mathematical Sciences, Worcester Polytechnic Institute,
100 Institute Road, Worcester MA 01609-2280, USA
E-mail: mosco@wpi.edu

Received July 16, 2012 and in revised form September 12, 2012